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QUASI BI-SLANT SUBMANIFOLDS OF PARA-KAEHLER MANIFOLDS

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Abstract. In this paper, we introduce the notion of quasi bi-slant submanifolds of para-Hermitian manifolds. We investigate necessary and sufficient conditions for integrability of distributions and find the conditions for these submanifolds to be totally geodesic when ambient manifold is a para-Kaehler manifold. Some examples have also been constructed.

1. Introduction

The submanifolds of an almost Hermitian manifold can be classified into three classes according to the behavior of the tangent bundle of a submanifold with respect to the action of J, the almost complex structure of the ambient manifold. The first one is the class of holomorphic submanifolds in which J maps each tangent space into itself; second class contains submanifolds whose tangent space is maped into normal space by J, known as the class of totally real submanifolds. The third class was introduced by B. Y. Chen [**8**] in 1990, and submanifolds belonging to this class are generalization of both holomorphic and totally real submanifolds, called slant submanifolds. The author of [**8**] studied various properties of slant submanifolds in complex geometry.

A. Lotta [**13**] introduced the notion of slant submanifolds in contact geometry in 1996. B. Y. Chen and O. Garay [**9**] initiated the study of slant submanifolds in a semi-Riemannian manifold in 2009. Slant submanifolds of para Hermitian manifolds were studied by P. Alegre and A. Carriazo [**2**]. S. K. Chanyal [**7**] introduced and studied slant submanifolds in an almost para compact manifold.

Slant submanifolds were studied by many researchers [**1, 5, 6, 10, 14, 18**] in differentiable manifolds equipped with various structures. Several geometers studied various generalizations of slant submanifolds like quasi slant [**11**], semi-slant [**4, 6, 19**], hemi (pseudo)-slant [**6, 12, 20**], bi-slant [**3, 21**] and quasi-bi-slant [**15**] in certain manifolds. Quasi-hemi-slant submanifolds of Sasakian and cosymplectic manifolds were characterized by R. Prasad and others [**16**]. Recently,

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R. Prasad et al. [**17**] studied Quasi Bi-Slant Submanifolds of Kaehler Manifolds.

The present paper is organised as follows: Section 2 contains some basic definitions and properties of almost para complex manifolds. In Section 3, we define quasi bi-slant submanifolds and some basic properties of submanifolds. In Sections 4 and 5, we find respectively the necessary and sufficient conditions for integrability of distributions and conditions for the submanifolds to be totally geodesic. Finally, some examples of quasi bi slant submanifolds are constructed in Section 6.

2. Preliminaries

Let $\bar{M}$ be a $2n$-dimensional manifold. A $(1,1)$-tensor field J on $\bar{M}$ is said to be an almost product structure if

$$J^2 = I. \tag{2.1}$$

In this case, the pair $(\bar{M}, J)$ is called an almost product manifold. If in addition the two eigenbundles T^+M and T^-M associated with the two eigenvalues ± 1 of J have the same rank, $(\bar{M}, J)$ is called an almost para-complex manifold. Let g be a semi-Riemannian metric on $\bar{M}$, satisfying

$$g(JX, Y) + g(X, JY) = 0, \tag{2.2}$$

or equivalently, $$g(JX, JY) = -g(X, Y),$$

for any vector fields X, Y on $\bar{M}$, $(\bar{M}, J, g)$ is called an almost para-Hermitian manifold. It is said to be para-Kaehler manifold if

$$(\bar{\nabla}_X J)Y = 0 \iff \bar{\nabla}_X JY = J\bar{\nabla}_X Y \tag{2.3}$$

for every $X, Y \in T\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$.

***Example* 2.1.** Consider $\mathbb{R}^{10}$ with coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5)$. We define the structure J as

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \quad \text{and} \quad J\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i}, \quad i = 1, 2, \dots, 5.$$

Let g be the pseudo-Riemannian metric on $\mathbb{R}^{10}$ given by

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = 1, \quad g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i}\right) = -1, \quad g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = 0;\ i, j = 1, 2, \dots, 5.$$

It can be easily shown that $(\mathbb{R}^{10}, J, g)$ is a para-Kaehler manifold.

Now, let M be a submanifold isometrically immersed in $\bar{M}$, the Lie algebra of vector fields in M is denoted by TM. We denote by $T^\perp M$ the set of all vector fields normal to M and by g the induced pseudo-Riemannian metric on M. Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.5}$$

for all vector fields $X, Y \in TM$ and $V \in T^{\perp}M$, where h is the second fundamental form of M, A_V is the Weingarten endomorphism associated with V, ∇ is induced Riemannian connection on M, and $\nabla^{\perp}$ is connection in the normal bundle. The shape operator A and the second fundamental form h are related by

$$g(A_V X, Y) = g(h(X, Y), V).$$

For any $X \in TM$, we put

$$JX = \phi X + \omega X, \tag{2.6}$$

and for any $V \in T^{\perp}M$, we write

$$JV = BV + CV, \tag{2.7}$$

where ϕX (resp. BV) and ωX (resp. CV), respectively, are the tangential and normal components of JX (resp. JV).

The covariant derivative of projection morphisms in (2.6) and (2.7) are defined as

$$\begin{aligned}
(\bar{\nabla}_X \phi)Y &= \nabla_X \phi Y - \phi \nabla_X Y,\\
(\bar{\nabla}_X \omega)Y &= \nabla^{\perp}_X \omega Y - \omega \nabla_X Y,\\
(\bar{\nabla}_X B)V &= \nabla_X BV - B\nabla^{\perp}_X V,\\
(\bar{\nabla}_X C)V &= \nabla^{\perp}_X CV - C\nabla^{\perp}_X V,
\end{aligned} \tag{2.8}$$

where $X, Y \in TM$ and $V \in T^{\perp}M$.
Using equations (2.3), (2.4), (2.6) and (2.7) in (2.3) and collecting tangent and normal components, we have the following values of covariant derivative of projection morphisms given in equation (2.8):

$$(\bar{\nabla}_X \phi)Y = A_{\omega Y} X + Bh(X, Y), \tag{2.9}$$

$$(\bar{\nabla}_X \omega)Y = Ch(X, Y) - h(X, \phi Y), \tag{2.10}$$

$$(\bar{\nabla}_X B)V = A_{CV} X - \phi A_V X, \tag{2.11}$$

$$(\bar{\nabla}_X C)V = -h(X, BV) - \omega A_V X. \tag{2.12}$$

Alegre [**2**] gave the following definition.

Definition 2.1 ([**2**]). A semi-Riemannian submanifold M of a para-Hermitian manifold $(\bar{M}, J, g)$ is called a slant submanifold if for every space-like or time-like vector field X, the quotient $\frac{g(\phi X, \phi X)}{g(JX, JX)}$ is constant.

It is clear if M is invariant, i.e., $J = \phi$, then this quotient is equal to 1, and for anti-invariant submanifolds this quotient is zero. A slant submanifold is called a proper slant submanifold if it is neither invariant nor anti-invariant submanifold.

Three different types of proper slant submanifolds are distinguished as follows:

Definition 2.2 ([**2**]). Let M be a proper slant semi-Riemannian submanifold of a para-Hermitian manifold $(\bar{M}, J, g)$. Then it is of:

(1) Type 1 if for any space-like (time-like) vector field X, ϕX is time-like (space-like) and $\frac{|\phi X|}{|JX|} > 1$,
(2) Type 2 if for any space-like (time-like) vector field X, ϕX is time-like (space-like) and $\frac{|\phi X|}{|JX|} < 1$,
(3) Type 3 if for any space-like (time-like) vector field X, ϕX is space-like (time-like).

The following theorem is a characterization of three types of proper slant submanifolds.

Theorem 2.1 ([**2**]). *Let M be a semi-Riemannian submanifold of a para-Hermitian manifold $(\bar{M}, J, g)$. Then,*

(1) *M is a slant of type 1 if and only if for any space-like (time-like) vector field X, ϕX is time-like (space-like), and there exists a constant $\lambda \in (1, +\infty)$ such that $\phi^2 = \lambda I$, where $\lambda = \cosh^2 \theta$ with $\theta > 0$.*
(2) *M is a slant of type 2 if and only if for any space-like (time-like) vector field X, ϕX is time-like (space-like), and there exists a constant $\lambda \in (0, 1)$ such that $\phi^2 = \lambda I$, where $\lambda = \cos^2 \theta$ with $0 < \theta < \frac{\pi}{2}$.*
(3) *M is a slant of type 3 if and only if for any space-like (time-like) vector field X, ϕX is space-like (time-like), and there exists a constant $\lambda \in (-\infty, 0)$ such that $\phi^2 = \lambda I$, where $\lambda = -\sinh^2 \theta$ with $\theta > 0$.*

In each case, θ is called a slant angle.

Slant distributions are defined and characterized as follows [**3**]:

Definition 2.3 ([**3**]). A differentiable distribution D on a para-Hermitian manifold $(\bar{M}, J, g)$ is called a slant distribution if for every non-light-like $X \in D$, the quotient $\frac{g(P_D X, P_D X)}{g(JX, JX)}$ is constant, where $P_D X$ is the projection of JX over D.

A distribution is called invariant if it is a slant with slant angle zero, that is, if $\frac{g(P_D X, P_D X)}{g(JX, JX)} = 1$ for all non-light-like $X \in D$. It is called anti-invariant if $P_D X = 0$ for all $X \in D$. D is called a proper slant distribution if it is neither invariant nor anti-invariant.

Definition 2.4 ([**3**]). Let D be a proper slant distribution of a para-Hermitian manifold $(\bar{M}, J, g)$. Then it is of:

(1) Type 1 if for any space-like (time-like) vector field X, $P_D X$ is time-like (space-like) and $\frac{|P_D X|}{|JX|} > 1$,
(2) Type 2 if for any space-like (time-like) vector field X, $P_D X$ is time-like (space-like) and $\frac{|P_D X|}{|JX|} < 1$,
(3) Type 3 if for any space-like (time-like) vector field X, $P_D X$ is space-like (time-like).

Theorem 2.2 ([**3**]). *Let D be a distribution of a para-Hermitian manifold $(\bar{M}, J, g)$. Then,*

(1) *D is a slant distribution of type 1 if and only if for any space-like (time-like) vector field X, $P_D X$ is time-like (space-like), and there exists a constant $\lambda \in (1, +\infty)$ such that $P_D^2 = \lambda I$, where $\lambda = \cosh^2 \theta$.*

(2) *D is a slant distribution of type 2 if and only if for any space-like (time-like) vector field X, $P_D X$ is time-like (space-like), and there exists a constant $\lambda \in (0, 1)$ such that $P_D^2 = \lambda I$, where $\lambda = \cos^2 \theta$.*

(3) *D is a slant distribution of type 3 if and only if for any space-like (time-like) vector field X, $P_D X$ is space-like (time-like), and there exists a constant $\lambda \in (0, +\infty)$ such that $P_D^2 = -\lambda I$, where $\lambda = \sinh^2 \theta$.*

θ is called a slant angle of the distribution D.

The following definition of bi-slant submanifolds of a para-Hermitian manifold is given by Alegre [**3**].

Definition 2.5. A semi-Riemannian submanifold M of a para-Hermitian manifold $(\bar{M}, J, g)$ is called a bi-slant submanifold if the tangent space admits a decomposition $TM = D_1 \oplus D_2$ with both D_1 and D_2 are slant distributions.

A bi-slant submanifold M is called a semi-slant submanifold if D_1 is an invariant distribution and D_2 is a proper slant distribution, and it is called a hemi-slant submanifold if D_1 is an anti-invariant distribution and D_2 is a proper slant distribution.

3. Quasi Bi-Slant submanifolds of para-Kaehler manifolds

R. Prasad et al. [**17**] defined quasi bi-slant sunmanifolds of an almost Hermitian manifold. Analogously, we give the following definition.

Definition 3.1. A submanifold M of a para-Hermitian manifold $(\bar{M}, J, g)$ is called a quasi bi-slant submanifold if the tangent space admits the orthogonal direct decomposition $TM = D \oplus D_1 \oplus D_2$ with D is invariant, and both D_1 and D_2 are slant distributions.

It is clear from the Definition 3.1 that if dimension of D is equal to zero, then M is a bi-slant submanifold.

Let M be a quasi bi-slant submanifold of an para-Hermitian manifold $\bar{M}$. We write

$$X = PX + QX + RX, \tag{3.1}$$

where P, Q and R, respectively, are the projections of $X \in TM$ on the distributions D, D_1 and D_2. Using equation (2.6), we have

$$JX = \phi PX + \phi QX + \phi RX + \omega QX + \omega RX. \tag{3.2}$$

This means for any $X \in TM$, we have

$$\phi X = \phi PX + \phi QX + \phi RX$$
$$\text{and} \qquad \omega X = \omega QX + \omega RX.$$

Thus, we have the following decomposition:

$$J(TM) \subset D \oplus \phi D_1 \oplus \omega D_1 \oplus \phi D_2 \oplus \omega D_2 \tag{3.3}$$

and

$$T^{\perp}M = \omega D_1 \oplus \omega D_2 \oplus \mu, \tag{3.4}$$

where μ is the orthogonal complement of $\omega D_1 \oplus \omega D_2$ in $T^{\perp}M$, and it is invariant with respect to J.

Now, we give the following properties of projection morphisms.

Lemma 3.1. *Let M be a quasi bi-slant submanifold of a para-Hermitian manifold $\bar{M}$. Then the endomorphism ϕ and projection morphisms ω, B and C on the tangent bundle of M, satisfy the following identities:*

$$\phi^2 + B\omega = I \ \text{ on } \ TM, \qquad \omega\phi X + c\omega X = 0 \ \text{ on } \ TM,$$
$$\omega B + C^2 = I \ \text{ on } \ T^{\perp}M, \qquad \phi B + BC = 0 \ \text{ on } \ T^{\perp}M,$$

where I is the identity operator.

Proof. The proof is straightforward from the equations (2.1), (2.6), and (2.7). □

Lemma 3.2. *Let M be a quasi bi-slant submanifold of a para Hermitian manifold $\bar{M}$. We have the folowing cases:*

(1) *if D_1 is a slant distribution of type 1, then $\phi^2 X = (\cosh^2\theta_1)X$, $g(\phi X, \phi Y) = -(\cosh^2\theta_1)g(X,Y)$, and $g(\omega X, \omega Y) = (\sinh^2\theta_1)g(X,Y)$;*

(2) *if D_1 is a slant distribution of type 2, then $\phi^2 X = (\cos^2\theta_1)X$, $g(\phi X, \phi Y) = -(\cos^2\theta_1)g(X,Y)$, and $g(\omega X, \omega Y) = -(\sin^2\theta_1)g(X,Y)$;*

(3) *if D_1 is a slant distribution of type 1, then $\phi^2 X = -(\sinh^2\theta_1)X$, $g(\phi X, \phi Y) = (\sinh^2\theta_1)g(X,Y)$, and $g(\omega X, \omega Y) = -(\cosh^2\theta_1)g(X,Y)$,*

where $X, Y \in \Gamma(D_1)$ and θ_1 is the slant angle of D_1.

Proof. Consider D_1 is a slant distribution of type 1 and $X, Y \in \Gamma(D_1)$. From Theorem 2.1, there exists a constant $\lambda \in (1, +\infty)$ such that

$$\phi^2 X = \lambda X \quad \text{or} \quad \phi^2 X = (\cosh^2\theta_1)X.$$

Using (2.6) and (2.7) in (2.2), we have

$$g(\phi X, \phi Y) + g(\omega X, \omega Y) = -g(X,Y),$$

which implies

$$g(\phi X, \phi Y) = -(\cosh^2\theta_1)g(X,Y)$$

and

$$g(\omega X, \omega Y) = (\sinh^2\theta_1)g(X,Y).$$

The proof can be obtained in a similar manner when D_1 is slant distribution of type 2 or type 3. □

Remark 3.1. The Lemma 3.2 is true for another slant distribution D_2 (with a slant angle θ_2) of the quasi bi-slant submanifold of a para Hermitian manifold.

4. Integrability of distributions

Now, we investigate the conditions for the integrability of various distributions involved in the definition of quasi bi-slant submanifolds. We begin with the following theorem.

Theorem 4.1. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$, then the invariant distribution D is integrable if and only if*

$$g(\nabla_W \phi Z - \nabla_Z \phi W, \phi QX + \phi RX) = g(h(Z, \phi W) - h(W, \phi Z), \omega QX + \omega RX)$$

for any $Z, W \in \Gamma(D)$ and $X \in \Gamma(D_1 \oplus D_2)$.

Proof. Suppose $Z, W \in \Gamma(D)$ and $X \in \Gamma(D_1 + D_2)$. From equations (2.2), (2.3), (2.4), and (3.1), we have

$$\begin{aligned} g([Z, W], X) &= g(\bar{\nabla}_W JZ, JX) - g(\bar{\nabla}_Z JW, JX) \\ &= g(\nabla_W JZ + h(W, JZ), JX) - g(\nabla_Z JW + h(Z, JW), JX). \end{aligned}$$

Using (2.6) in last equation, we find

$$\begin{aligned} g([Z, W], X) = g(\nabla_W \phi Z - \nabla_Z \phi W, \phi QX + \phi RX) + g(h(W, \phi Z) \\ - h(Z, \phi W), \omega QX + \omega RX). \end{aligned}$$

This completes the proof. □

Now we consider integrability of the distribution D_1. Let $Z, W \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$. Using (2.1), (2.2), (2.3), and (2.6), we have

$$\begin{aligned} g([Z, W], X) = g(\bar{\nabla}_W \phi Z, JX) + g(\bar{\nabla}_W \omega Z, JX) \\ - g(\bar{\nabla}_Z \phi W, JX) - g(\bar{\nabla}_Z \omega W, JX). \end{aligned}$$

With the help of (2.4), (2.5), and (3.2), we find

$$\begin{aligned} g([Z, W], X) = g(-A_{\omega Z} W + \bar{\nabla}^{\perp}_W \omega Z, JX) - g(-A_{\omega W} Z + \bar{\nabla}^{\perp}_Z \omega W, JX) \\ - g(\bar{\nabla}_W \phi^2 Z + \bar{\nabla}_W \omega \phi Z, X) + g(\bar{\nabla}_Z \phi^2 W + \bar{\nabla}_Z \omega \phi W, X). \end{aligned}$$

Suppose D_1 is a slant distribution of type 1. Using of Lemma 3.2, we get

$$\begin{aligned} (\sinh^2 \theta_1) g([Z, W], X) = -g(A_{\omega W} Z - A_{\omega Z} W, \phi X) + g(A_{\omega \phi W} Z - A_{\omega \phi Z} W, X) \\ + g(\nabla^{\perp}_Z \omega w - \nabla^{\perp}_W \omega Z, \omega RX). \end{aligned}$$

If D_1 is a slant distribution of type 2, using Lemma 3.2, we find

$$\begin{aligned} (-\sin^2 \theta_1) g([Z, W], X) = -g(A_{\omega W} Z - A_{\omega Z} W, \phi X) + g(A_{\omega \phi W} Z - A_{\omega \phi Z} W, X) \\ + g(\nabla^{\perp}_Z \omega w - \nabla^{\perp}_W \omega Z, \omega RX). \end{aligned}$$

If D_1 is a slant distribution of type 3, again from Lemma 3.2, we have

$$(-\cosh^2\theta_1)g([Z,W],X) = -g(A_{\omega W}Z - A_{\omega Z}W, \phi X) + g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^{\perp}_{Z}\omega w - \nabla^{\perp}_{W}\omega Z, \omega RX).$$

Now we are able to state the following theorem.

Theorem 4.2. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$. Then the slant distribution D_1 is integrable if and only if*

$$g(A_{\omega W}Z - A_{\omega Z}W, \phi X) = g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^{\perp}_{Z}\omega w - \nabla^{\perp}_{W}\omega Z, \omega RX)$$

for any $Z, W \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$.

Corollary 4.1. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$. If*

$\nabla^{\perp}_{Z}\omega w - \nabla^{\perp}_{W}\omega Z \in \omega D_1 \oplus \mu, A_{\omega\phi W}Z - A_{\omega\phi Z}W \in D_1$ *and* $A_{\omega W}Z - A_{\omega Z}W \in D_1$ *for any $Z, W \in \Gamma(D_1)$, then the slant distribution D_1 is integrable.*

Next we state the following theorem.

Theorem 4.3. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$. Then the slant distribution D_2 is integrable if and only if*

$$g(A_{\omega W}Z - A_{\omega Z}W, \phi X) = g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^{\perp}_{Z}\omega w - \nabla^{\perp}_{W}\omega Z, \omega QX)$$

for any $Z, W \in \Gamma(D_2)$ and $X \in \Gamma(D \oplus D_1)$.

Proof. The proof is similar to the proof of Theorem 4.2. □

Corollary 4.2. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$. If*

$\nabla^{\perp}_{Z}\omega w - \nabla^{\perp}_{W}\omega Z \in \omega D_2 \oplus \mu, A_{\omega\phi W}Z - A_{\omega\phi Z}W \in D_2,$ *and* $A_{\omega W}Z - A_{\omega Z}W \in D_2$ *for any $Z, W \in \Gamma(D_2)$, then the slant distribution D_2 is integrable.*

5. Totally geodesic quasi bi-slant submanifold

In this section, we obtain a necessary and sufficient condition for a quasi bi-slant submanifold to be totally geodesic.

Suppose $X, Y \in TM$ and $U \in T^{\perp}M$. Using (2.1), (2.2), (2.3), (2.6), and (3.1), we have

$$g(\bar{\nabla}_X Y, U) = g(\bar{\nabla}_X J\phi QY, U) - g(\bar{\nabla}_X \omega QY, JU) + g(\bar{\nabla}_X J\phi RY, U) - g(\bar{\nabla}_X J\omega RY, JU).$$

Using (2.4), (2.5), (2.9), and (2.10), we find

$$g(\bar{\nabla}_X Y, U) = g(h(X,PY) + h(X,\phi^2 QY) + h(X,\phi^2 RY), U) + g(\nabla^{\perp}_{X}\omega\phi QY - \nabla^{\perp}_{X}\omega\phi RY, U) + g(A_{\omega QY}X + A_{\omega RY}X, JU)$$

$$- g(\nabla^{\perp}_X \omega QY + \nabla^{\perp}_X RY, JU).$$

Since $QY \in \Gamma(D_1)$ and $RY \in \Gamma(D_2)$, using Lemma 3.2 in last equation, we obtain

$$\begin{aligned} g(\bar{\nabla}_X Y, U) &= g(h(X, PY) + \lambda_1 h(X, QY) + \lambda_2 h(X, RY), U) \\ &\quad + g(\nabla^{\perp}_X \omega\phi QY - \nabla^{\perp}_X \omega\phi RY, U) + g(A_{\omega Y} X, BU) - g(\nabla^{\perp}_X \omega Y, CU), \end{aligned}$$

where λ_1 is equal to $\cosh^2\theta_1$, $\cos^2\theta_1$, or $-(\sinh^2\theta_1)$ according to the fact that D_1 is a slant distribution of type 1, type 2, or type 3, and λ_2 is equal to $\cosh^2\theta_2$, $\cos^2\theta_2$, and $-(\sinh^2\theta_2)$, respectively, for the slant distribution D_2 is of type 1, type 2, and type 3. We summarise this in the form of the following theorem.

Theorem 5.1. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$. Then M is totally geodesic if and only if*

$$\begin{aligned} g(h(X, PY) + \lambda_1 h(X, QY) + \lambda_2 h(X, RY), U) + g(\nabla^{\perp}_X \omega\phi QY - \nabla^{\perp}_X \omega\phi RY, U) \\ + g(A_{\omega Y} X, BU) - g(\nabla^{\perp}_X \omega Y, CU) = 0 \end{aligned}$$

for any $X, Y \in TM$, $U \in T^{\perp}M$, and

$$\lambda_i = \begin{cases} \cosh^2\theta_i & \textit{if } D_i \textit{ is a slant distribution of type 1}, \\ \cos^2\theta_i & \textit{if } D_i \textit{ is a slant distribution of type 2}, \\ -\sinh^2\theta_i & \textit{if } D_i \textit{ is slant a distribution of type 3}, \quad i = 1, 2. \end{cases}$$

Now, let $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D_1 \oplus D_2)$. Form equation (2.3), we have

$$g(\bar{\nabla}_X Y, Z) = -g(\bar{\nabla}_X JY, JZ).$$

Using(2.4) and (2.6), last equation implies

$$g(\bar{\nabla}_X Y, Z) = -g(\nabla_X \phi Y, \phi Z) - g(h(X, \phi Y), \omega Z). \tag{5.1}$$

In a similar way, for any $X, Y \in \Gamma(D)$ and $U \in T^{\perp}M$, we find

$$g(\bar{\nabla}_X Y, U) = -g(\nabla_X \phi Y, BU) - g(h(X, \phi Y), CU). \tag{5.2}$$

Thus we have the following theorem.

Theorem 5.2. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$. Then the invariant distribution D defines a totally geodesic foliation on M if and only if*

$$g(\nabla_X \phi Y, \phi Z) + g(h(X, \phi Y), \omega Z) = 0$$
$$\textit{and} \qquad g(\nabla_X \phi Y, BU) + g(h(X, \phi Y), CU) = 0$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D_1 \oplus D_2)$ and $U \in T^{\perp}M$.

Proof. The proof is straight forward from the equations (5.1) and (5.2). □

Next, we consider $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D \oplus D_2)$. From equations (2.2), (2.3), and (2.6), we have

$$g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X \phi^2 Y, Z) + g(\bar{\nabla}_X \omega\phi Y, Z) - g(\bar{\nabla}_X \omega Y, \phi Z + \omega RZ). \tag{5.3}$$

Similarly, for any $X, Y \in \Gamma(D_1)$ and $U \in T^{\perp}M$, we find

$$g(\bar{\nabla}_X Y, U) = g(\bar{\nabla}_X \phi^2 Y, U) + g(\bar{\nabla}_X \omega\phi Y, U) - g(\bar{\nabla}_X \omega Y, BU + CU). \tag{5.4}$$

If D_1 is a slant distribution of type 1, using Lemma 3.2 and equation (2.5), equations (5.3) and (5.4) yield

$$\begin{aligned} g(\bar{\nabla}_X Y, Z) &= \cosh^2\theta_1 g(\bar{\nabla}_X Y, Z) - g(A_{\omega\phi Y}X, Z) \\ &\quad + g(A_{\omega Y}X, \phi Z) - g(\nabla^{\perp}_X \omega Y, \omega RZ), \end{aligned}$$

$$\begin{aligned} g(\bar{\nabla}_X Y, U) &= \cosh^2\theta_1 g(\bar{\nabla}_X Y, U) + g(\nabla^{\perp}_X \omega\phi Y, U) \\ &\quad + g(A_{\omega Y}X, BU) - g(\nabla^{\perp}_X \omega Y, CU). \end{aligned}$$

These equations imply

$$\begin{aligned} \sinh^2\theta_1 g(\bar{\nabla}_X Y, Z) &= g(A_{\omega\phi Y}X, Z) - g(A_{\omega Y}X, \phi Z) + g(\nabla^{\perp}_X \omega Y, \omega RZ), \\ \sinh^2\theta_1 g(\bar{\nabla}_X Y, U) &= g(\nabla^{\perp}_X \omega Y, CU) - g(\nabla^{\perp}_X \omega\phi Y, U) - g(A_{\omega Y}X, BU). \end{aligned} \tag{5.5}$$

If D_1 is a slant distribution of type 2 and type 3, we can find equations

$$\begin{aligned} \sin^2\theta_1 g(\bar{\nabla}_X Y, Z) &= g(A_{\omega Y}X, \phi Z) - g(A_{\omega\phi Y}X, Z) - g(\nabla^{\perp}_X \omega Y, \omega RZ), \\ \sin^2\theta_1 g(\bar{\nabla}_X Y, U) &= g(A_{\omega Y}X, BU) - g(\nabla^{\perp}_X \omega Y, CU) + g(\nabla^{\perp}_X \omega\phi Y, U), \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \cosh^2\theta_1 g(\bar{\nabla}_X Y, Z) &= g(A_{\omega\phi Y}X, Z) - g(A_{\omega Y}X, \phi Z) + g(\nabla^{\perp}_X \omega Y, \omega RZ), \\ \cosh^2\theta_1 g(\bar{\nabla}_X Y, U) &= g(\nabla^{\perp}_X \omega Y, CU) - g(\nabla^{\perp}_X \omega\phi Y, U) - g(A_{\omega Y}X, BU), \end{aligned} \tag{5.7}$$

respectively, from (5.3) and (5.4). Thus, we have the following theorem.

Theorem 5.3. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$. Then the slant distribution D_1 defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g(A_{\omega\phi Y}X, Z) - g(A_{\omega Y}X, \phi Z) + g(\nabla^{\perp}_X \omega Y, \omega RZ) = 0, \\ g(\nabla^{\perp}_X \omega Y, CU) - g(\nabla^{\perp}_X \omega\phi Y, U) - g(A_{\omega Y}X, BU) = 0 \end{aligned}$$

for any $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D \oplus D_2)$ and $U \in T^{\perp}M$.

Proof. The proof follows from the equations (5.5), (5.6), and (5.7). □

Theorem 5.4. *Let M be a proper quasi bi-slant submanifold of a para-Kaehler manifold $\bar{M}$. Then the slant distribution D_2 defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g(A_{\omega\phi Y}X, Z) - g(A_{\omega Y}X, \phi Z) + g(\nabla^{\perp}_X \omega Y, \omega QZ) = 0, \\ g(\nabla^{\perp}_X \omega Y, CU) - g(\nabla^{\perp}_X \omega\phi Y, U) - g(A_{\omega Y}X, BU) = 0, \end{aligned}$$

for any $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D \oplus D_1)$, and $U \in T^{\perp}M$.

Proof. The proof is similar to the proof of the Theorem 5.3. □

6. Example

***Example* 6.1.** Consider a 6-dimensional submanifold M of $\bar{M}$, the para-Kaehler manifold given in Example 2.1, defined as

$$(u_1, v_1, u_2, v_2, u_3, v_3) \longrightarrow (au_1, v_1, bu_1, u_1, cu_2, v_2, du_2, u_2, u_3, v_3),$$

where a, b, c, d are real numbers satisfying $a^2+b^2 \neq 1$ and $c^2+d^2 \neq 1$. The tangent bundle TM of M is spanned by the following set of linearly independent vectors

$$X_1 = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = c\frac{\partial}{\partial x_3} + d\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_4},$$
$$X_4 = \frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial x_5}, \quad X_6 = \frac{\partial}{\partial y_5}.$$

If we take

$$D = \operatorname{span}(X_5, X_6), \quad D_1 = \operatorname{span}(X_1, X_2), \quad D_2 = \operatorname{span}(X_3, X_4),$$

then

$$TM = D \oplus D_1 \oplus D_2.$$

It is easy to verify that D is an invariant distribution and D_1, D_2 are slant distributions. Hence, M is a quasi bi-slant submanifold of $\bar{M}$. Furthermore, we have the following different types of slant distributions:

- D_1 is a slant distribution with $\phi^2 X = \Big(\dfrac{a^2}{a^2+b^2-1}\Big)X$ for any $X \in D_1$, and
  - a. D_1 is of type 1 if $a^2+b^2 > 1$, $b^2 < 1$,
  - b. D_1 is of type 2 if $a^2+b^2 > 1$, $b^2 > 1$,
  - c. D_1 is of type 3 if $a^2+b^2 < 1$.
- D_2 is a slant distribution with $\phi^2 X = \Big(\dfrac{c^2}{c^2+d^2-1}\Big)X$ for any $X \in D_2$, and
  - a. D_2 is of type 1 if $c^2+d^2 > 1$, $d^2 < 1$,
  - b. D_2 is of type 2 if $c^2+d^2 > 1$, $d^2 > 1$,
  - c. D_2 is of type 3 if $c^2+d^2 < 1$.

Remark 6.1. Note that a slant distribution of type 1 or type 2 is always neutral. In the previous example, the distribution D_1 (resp. D_2) is a time-like distribution if it is a slant distribution of type 3.

The next example gives quasi bi-slant submanifolds with a space-like slant distribution of type 3.

***Example* 6.2.** The 6-dimensional submanifold M of $\bar{M}$ endowed with the para-Kaehler structure given in Example 2.1, defined by

$$(u_1, v_1, u_2, v_2, u_3, v_3) \longrightarrow (v_1, bu_1, au_1, u_1, v_2, du_2, cu_2, u_2, u_3, v_3),$$

where a, b, c, d are real numbers satisfying $a^2 - b^2 \neq 1$ and $c^2 - d^2 \neq 1$, is a quasi-bi-slant submanifold. The tangent bundle TM is spanned by

$$X_1 = b\frac{\partial}{\partial y_1} + a\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}, \quad X_2 = \frac{\partial}{\partial x_1}, \quad X_3 = d\frac{\partial}{\partial y_3} + c\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_4},$$
$$X_4 = \frac{\partial}{\partial x_3}, \quad X_5 = \frac{\partial}{\partial x_5}, \quad X_6 = \frac{\partial}{\partial y_5}.$$

We write

$$TM = D \oplus D_1 \oplus D_2,$$

where

$$D = \operatorname{span}(X_5, X_6), \quad D_1 = \operatorname{span}(X_1, X_2), \quad D_2 = \operatorname{span}(X_3, X_4).$$

It is easy to see that D is an invariant distribution and D_1, D_2 are slant distributions. Furthermore, we have the following different types of slant distributions:

- D_1 is slant distribution with $\phi^2 X = \Big(\frac{b^2}{b^2 - a^2 + 1}\Big)X$ for any $X \in D_1$, and
  a. D_1 is of type 1 if $a^2 - b^2 < 1$, $a^2 > 1$,
  b. D_1 is of type 2 if $a^2 - b^2 < 1$, $a^2 < 1$,
  c. D_1 is of type 3 if $a^2 - b^2 > 1$.
- D_2 is slant distribution with $\phi^2 X = \Big(\frac{d^2}{d^2 - c^2 + 1}\Big)X$ for any $X \in D_2$, and
  a. D_2 is of type 1 if $c^2 - d^2 < 1$, $c^2 > 1$,
  b. D_2 is of type 2 if $c^2 - d^2 < 1$, $c^2 < 1$,
  c. D_2 is of type 3 if $c^2 - d^2 > 1$.

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