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PERFECTLY ANTIMAGIC TOTAL LABELING OF STAR-LIKE TREES AND THE CORONA PRODUCT OF TWO GRAPHS

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ABSTRACT. A totally antimagic total labeling whose vertex and edge weights are also pairwise distinct is said to be a perfectly antimagic total labeling. A graph with this labeling is known as a perfectly antimagic total (PAT) graph. This paper discusses the possibility of the perfectly antimagic total labeling of a star-like trees. We also developed perfectly antimagic total labeling for the Corona product of path with nK_1 and cycle with nK_1 for all $n \geq 1$. Finally, we provide a particular solution to the open problem under total labeling on Antimagic labelings of caterpillars and Caterpillars are antimagic (Antoni Lozano et al.).

1. Introduction

All graphs used in this study are regarded as finite, undirected, and without loops and multiple edges. A mapping function on a graph G called labeling converts a collection of graph elements into a collection of positive integers. A function is referred to as a vertex or edge labeling of G if its domain is the vertex or edge set of the graph. If the domain of a one-one correspondence function is the union of the edge and vertex set of G, the labeling is referred to as total labeling l of G. The weight of a vertex u is determined by this total labeling as the sum of the labels of u and its incident edges, whereas the weight of an edge uv is defined by the sum of the labels of uv and its end vertices for all $u, v \in V(G)$ and $uv \in E(G)$, where $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively.

All graphs have vertex antimagic total labelings, according to M. Miller et al. [**5**]. For more information on graph labelings, see [**2**]. The concept of the totally antimagic total (TAT) graph was created by M. Bača et al. [**1**]. If a total labeling is antimagic on both the edge and the vertex, it is referred to as TAT labeling. A graph with TAT labels is referred to as a "TAT graph". A perfectly antimagic

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total (PAT) graph was an idea proposed by P. Swathi et al. [**6**]. A PAT labeling of G is a TAT labeling of G with pairwise distinct vertex and edge weights. A graph admits PAT labeling is called PAT graphs. Some trees that permit PAT labeling are listed in the following section.

2. Main results

The existence of PAT labeling of several types of star-like trees is shown below. The fact that lobsters are antimagic under total labeling (which is the specific solution to the open problem under total labeling in [**4**]) is further shown here.

A broom graph $Br_{n,d}$ is formed by connecting d pendant edges to any one of the pendant vertices of the path P_n. If $d = n$, then $Br_{n,n}$ is called a regular broom.

Theorem 2.1. *A regular broom $Br_{n,n}$ is PAT for all $n \geq 1$.*

Proof. Let v_i be the vertices of the path graph P_n and u_i be the pendant vertices which are incident with the vertex v_n for all $1 \leq i \leq n$. Define a total labeling l by

$$\begin{aligned} l(u_i) &= i && \text{for all } 1 \leq i \leq n, \\ l(v_i) &= 3n + 1 - i && \text{for all } 1 \leq i \leq n, \\ l(u_i v_n) &= n + i && \text{for all } 1 \leq i \leq n, \\ l(v_i v_{i+1}) &= 3n + i && \text{for all } 1 \leq i \leq n - 1. \end{aligned}$$

Under the above total labeling l, we obtain the following weights:

$$\begin{aligned} w(u_i) &= l(u_i) + l(u_i v_n) = n + 2i && \text{for all } 1 \leq i \leq n, \\ w(v_1) &= l(v_1) + l(v_1 v_2) = 6n + 1, \\ w(v_i) &= l(v_i) + l(v_i v_{i+1}) + l(v_{i-1} v_i) = 9n + i && \text{for all } 2 \leq i \leq n - 1, \\ w(v_n) &= l(v_n) + l(v_{n-1} v_n) + \sum_{i=1}^{n} l(u_i v_n) = \frac{1}{2}[3n^2 + 13n], \\ w(u_i v_n) &= l(u_i) + l(u_i v_n) + l(v_n) = 3n + 2i + 1 && \text{for all } 1 \leq i \leq n, \\ w(v_i v_{i+1}) &= l(v_i) + l(v_i v_{i+1}) + l(v_{i+1}) = 9n + 1 - i && \text{for all } 1 \leq i \leq n - 1. \end{aligned}$$

The weights of the vertices and edges are compared, and the result is that regular broom is a PAT graph. □

A bistar-like tree is a tree with two stars joined by a path at their central vertices. It becomes a regular bistar-like tree if the two stars have the same number of vertices.

Theorem 2.2. *A regular Bistar-like tree is a PAT graph.*

Proof. Let w_j represent the path graph's vertices for all $1 \leq j \leq m$, and let u_i and v_i represent the pendant vertices that are adjacent to w_1 and w_m, respectively, for all $1 \le i \le n$. Total labeling l is defined by

$$\begin{aligned} l(u_i) &= i && \text{for all } i = 1, 2, \dots, n,\\ l(v_i) &= n + i && \text{for all } i = 1, 2, \dots, n,\\ l(w_j) &= 4n + j && \text{for all } j = 1, 2, \dots, m,\\ l(u_i w_1) &= 2n + i && \text{for all } i = 1, 2, \dots, n,\\ l(v_i w_n) &= 3n + i && \text{for all } i = 1, 2, \dots, n,\\ l(w_j w_{j+1}) &= 4n + 2m - j && \text{for all } j = 1, 2, \dots, m - 1. \end{aligned}$$

We can easily prove that the regular bistar-like tree is PAT under the above total labeling l. □

A tree with only one vertex of degree more than two is referred to as a spider graph. A regular spider is the one vertex union of paths of equal length.

Theorem 2.3. *A regular spider is a PAT graph.*

Proof. Let us consider a regular spider graph with m legs and n vertices on each leg. Let v be the central vertex and u_{ij} be the ith vertex of the jth leg. Define a total labeling by

$$\begin{aligned} l(v) &= 2mn + 1,\\ l(u_{ij}) &= j + 2(i - 1)n && \text{for all } i = 1, 2, \dots, m \text{ and } 1 \le j \le n,\\ l(u_{ij} u_{(i+1)j}) &= j + (2i - 1)n && \text{for all } i = 1, 2, \dots, m - 1 \text{ and } 1 \le j \le n,\\ l(u_{mj} v) &= j + (2m - 1)n && \text{for } 1 \le j \le n. \end{aligned}$$

Under this labeling l, it is simple to determine that regular spider is PAT. □

Lobsters are trees that generate caterpillars when their leaves are removed (see [**2**]). The regular lobster is a specific class of lobsters with the same number of leaves at each vertex.

Theorem 2.4. *A regular lobster is a PAT graph.*

Proof. Let v be the central vertex of a regular lobster graph, v_i be the vertices which are adjacent to v, and v_{ij} be the pendant vertices which are adjacent to v_i for all $1 \le i \le n$ and $1 \le j \le n$. Let l be the total labeling with

$$\begin{aligned} l(v) &= 2n^2 + 2n + 1,\\ l(v_i) &= 2n^2 + i && \text{for all } 1 \le i \le n,\\ l(v_{ij}) &= n(i - 1) + j && \text{for all } 1 \le i \le n \text{ and } 1 \le j \le n,\\ l(v_i v_{ij}) &= n^2 + i && \text{for all } 1 \le i \le n,\\ l(v v_i) &= 2n^2 + n + i && \text{for all } 1 \le i \le n. \end{aligned}$$

We acquire the following weights under the above total labeling l:

$$\begin{aligned} w(v) &= \frac{1}{2}[4n^3 + 7n^2 + 5n + 2], \\ w(v_i) &= \frac{1}{2}[2n^3 + 9n^2 + 3n + 4i] && \text{for all } 1 \le i \le n, \\ w(v_{ij}) &= n^2 + n(i-1) + (i+j) && \text{for all } 1 \le i \le n \text{ and } 1 \le j \le n, \\ w(v_i v_{ij}) &= 3n^2 + n(i-1) + (2i+j) && \text{for all } 1 \le i \le n \text{ and } 1 \le j \le n, \\ w(vv_i) &= 6n^2 + 3n + 2i + 1. \end{aligned}$$

We come to the conclusion that a regular lobster is a PAT graph after comparing the weights of the vertices and edges given above. ☐

The above theorem indicates that regular lobsters are antimagic, which is the particular solution to the open problem in [**4**] under total labeling.

3. Corona product of graphs

The corona product of G_1 with a graph G_2, represented by $G_1 \odot G_2$, if G_1 has order n, is the graph formed by taking one copy of G_1 and n copies of G_2 and attaching the ith vertex of G_1 with an edge to each vertex in the ith copy of G_2.

A comb is a graph obtained by joining a single pendant edge to each vertex of a path graph P_n, i.e., the comb graph is represented by $P_n \odot K_1$.

Theorem 3.1. *A comb $P_n \odot K_1$ is a PAT graph for all $n \geq 1$.*

Proof. Let v_i represent the path graph's vertices and u_i represent the pendant vertices that are adjacent to v_i for all $1 \le i \le n$. A total labeling can be defined as

$$\begin{aligned} l(u_i) &= i && \text{for all } i = 1, 2, \dots, n, \\ l(v_i) &= \begin{cases} 2n + i & \text{for all } i = 1, 2, \dots, n \text{ and } 1 \le n \le 2, \\ 3n + 1 - i & \text{for all } i = 1, 2, \dots, n \text{ and } n \ge 3, \end{cases} \\ l(u_i v_i) &= n + i && \text{for all } i = 1, 2, \dots, n, \\ l(v_i v_{i+1}) &= 3n + i && \text{for all } i = 1, 2, \dots, n-1. \end{aligned}$$

Then the vertex weights are

$$\begin{aligned} w(u_i) &= n + 2i && \text{for all } i = 1, 2, \dots, n \text{ and } n \ge 1, \\ w(v_1) &= \begin{cases} 3n + 2 & \text{for } n = 1, \\ 7n + 1 & \text{for } n = 2, \\ 7n + 2 & \text{for } n \ge 3, \end{cases} \\ w(v_i) &= 10n + 2i && \text{for all } i = 2, 3, \dots, n-1 \text{ and } n \ge 3, \end{aligned}$$

$$w(v_n) = \begin{cases} 8n+1 & \text{for } n = 2, \\ 8n & \text{for } n \geq 3, \end{cases}$$

and the edge weights are

$$w(u_i v_i) = \begin{cases} 3(n+i) & \text{for all } i = 1, 2, \ldots, n \text{ and } 1 \leq n \leq 2, \\ 4n+i+1 & \text{for all } i = 1, 2, \ldots, n \text{ and } n \geq 3, \end{cases}$$

$$w(v_i v_{i+1}) = \begin{cases} 7n+3i+1 & \text{for all } i = 1, 2, \ldots, n-1 \text{ and } n = 2, \\ 9n+i+2 & \text{for all } i = 1, 2, \ldots, n-1 \text{ and } n \geq 3. \end{cases}$$

When we compare the weights of the vertices and edges, we conclude that a comb is a PAT graph. □

A comb is a specific example of a caterpillar with a maximum degree of 3, hence Theorem 3.1 demonstrates that a particular family of caterpillar graphs with a maximum degree of 3 is antimagic under total labeling. As a result, the open problem in [**3**] under total labeling is specifically solved by Theorem 3.1.

A caterpillar is a tree in which all the vertices are within distance 1 of a central path. A regular caterpillar is a caterpillar whose number of vertices adjacent to each edge of the path is the same, i.e., the regular caterpillar is defined as the corona of a path graph of order n with mK_1, represented by $P_n \odot mK_1$.

Theorem 3.2. *The regular caterpillar $P_n \odot mK_1$ is a PAT graph for all $n \geq 2$, $m > 1$.*

Proof. Let v_i be the vertices of the path graph P_n and v_{ij} be the pendant vertices which are adjacent to the vertices v_i for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Consider a total labeling l with

$$\begin{aligned} l(v_i) &= 2mn + i && \text{for all } 1 \leq i \leq n, \\ l(v_{ij}) &= m(i-1) + j && \text{for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq m, \\ l(v_i v_{ij}) &= mn + m(i-1) + j && \text{for all } 1 \leq j \leq m, \\ l(v_i v_{i+1}) &= 2mn + 2n - i && \text{for all } 1 \leq i \leq n-1. \end{aligned}$$

Then this labeling produces the caterpillar PAT. □

A sun graph is a graph formed from a cycle on m vertices by adding a pendant edge to each vertex (i.e, $C_m \odot K_1$).

Theorem 3.3. *A sun graph $C_m \odot K_1$ is PAT for all $m \geq 3$ and m is odd.*

Proof. Let v_i be the vertices of the cycle C_m and u_i be its pendant vertices for all $1 \leq i \leq m$ and m is odd. Now we generate a total labeling l by

$$\begin{aligned} l(u_i) &= i && \text{for all } 1 \leq i \leq m, \\ l(v_i) &= 3m + 1 - i && \text{for all } 1 \leq i \leq m, \\ l(u_i v_i) &= m + i && \text{for all } 1 \leq i \leq m, \\ l(v_i v_{i+1}) &= 3m + 1 + i && \text{for all } 1 \leq i \leq m-1, \end{aligned}$$

$$l(v_1v_m) = 3m + 1.$$

According to this labeling, the sun graph is PAT. □

The corona of a cycle of order m with nK_1, denoted by $C_m \odot nK_1$, is defined as an n-crown graph.

Theorem 3.4. *Every n-crown graph is PAT for all $n > 1$.*

Proof. Assume that the cycle's vertices are v_i and that its pendant vertices are v_{ij} for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Then we have

$$\begin{aligned} l(v_i) &= 2mn + i && \text{for all } 1 \leq i \leq m,\\ l(v_{ij}) &= n(i-1) + j && \text{for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n,\\ l(v_iv_{ij}) &= mn + n(i-1) + j && \text{for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n,\\ l(v_iv_{i+1}) &= 2mn + 2m + 1 - i && \text{for all } 1 \leq i \leq m-1,\\ l(v_1v_m) &= 2mn + m + 1. \end{aligned}$$

According to this labeling, we can conclude that the n-crown graph is PAT. □

4. Conclusion

We have confirmed the existence of PAT labeling for several star-like trees. We have further shown that for any $n \geq 1$, the corona product of the path graph with nK_1 and the cycle graph with nK_1 is PAT. Finally, we have provided the specific solution to the unsolved problem in [**3**] and [**4**] under total labeling.

For further investigation, we have an open problem.

Open problem: Does every tree admit PAT labeling?

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