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THE GENERALISED DOUBLE HAHN SEQUENCE SPACE H_ϑ^d

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ABSTRACT. Suppose that $\vartheta \in \{bp, r\}$. In this study, we introduce the generalised double Hahn sequence space H_ϑ^d as an extension of generalised Hahn sequence space h_d defined by Goes [J. Math. Anal. Appl. **39** (1972), 477–494]. We give some topological properties of this space. Then, we characterize the classes $\left(H_\vartheta^d : W\right)$ of four dimensional infinite matrices, where $W \in \left\{\mathcal{C}_\vartheta, \mathcal{M}_u, \mathcal{L}_u, H_\vartheta^d\right\}$. Finally, we determine the α-dual of the space H_ϑ^d and $\beta(bp)$- and γ-duals of the space H_r^d.

1. Introduction

Let Ω denote the vector space of all double sequences with the co-ordinatewise addition and scalar multiplication of double sequences. Vector subspaces of Ω are called double sequence spaces. A double sequence $x = (x_{kl})$ of complex numbers is called *bounded* if $\|x\|_\infty = \sup_{k,l\in\mathbb{N}} \left|x_{kl}\right| < \infty$, where $\mathbb{N}$ denotes the set of positive integers. The space of all bounded double sequences is denoted by $\mathcal{M}_u$, which is a Banach space with the norm $\|\cdot\|_\infty$. A double sequence $x = (x_{kl})$ of complex numbers is said to *converge to the limit a in Pringsheim's sense* (shortly p-converge to a) if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\left|x_{kl} - a\right| < \varepsilon$ for all $k, l > N$. By $\mathcal{C}_p$, we denote the *space of all convergent double sequences in the Pringsheim's sense.* Furthermore, we can consider the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e.,

$$\mathcal{C}_{bp} := \left\{ x = (x_{kl}) \in \mathcal{C}_p : \|x\|_\infty = \sup_{k,l\in\mathbb{N}} \left|x_{kl}\right| < \infty \right\} = \mathcal{C}_p \cap \mathcal{M}_u.$$

The main drawback of the Pringsheim's convergence is that a p-convergent double sequence need not be bounded. Hardy [**9**] lacked this disadvantage by giving the definition of regular convergence, as follows: A sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is an ordinary convergent sequence with respect to each index and denote the space of all such sequences by $\mathcal{C}_r$.

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For any notion of convergence $\vartheta \in \{p, bp, r\}$, the space of all ϑ-convergent double sequences will be denoted by $\mathcal{C}_\vartheta$ and the limit of a ϑ-convergent double sequence x by $\vartheta-\lim_{k,l} x_{kl}$. By $\mathcal{C}_{\vartheta 0}$, we denote the spaces of all double sequences which ϑ-converge to 0. Móricz [**12**] proved that the spaces $\mathcal{C}_\vartheta$ and $\mathcal{C}_{\vartheta 0}$ are Banach spaces with the norm $\|\cdot\|_\infty$, where $\vartheta \in \{bp, r\}$. Also, by $\mathcal{L}_u$ and $\mathcal{BV}$, we denote the space of all absolutely convergent double series and the space of all double sequences of bounded variation, respectively, that is,

$$\mathcal{L}_u := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} \left| x_{kl} \right| < \infty \right\},$$

$$\mathcal{BV} := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} \left| x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1} \right| < \infty \right\}.$$

A double sequence $x = (x_{kl})$ is called monotonically increasing if and only if $x_{k_2,l_2} \geq x_{k_1,l_1}$ when $k_2 \geq k_1$ and $l_2 \geq l_1$ for all $k, l \in \mathbb{N}$. Also, the double series $\sum_{k,l} a_{kl}$ of nonnegative real numbers is convergent if and only if (s_{mn}) is bounded above, where $s_{mn} = \sum_{k,l=1}^{m,n} a_{kl}$ (see [**10**]).

We write $\Phi = \operatorname{span}\left\{ \mathbf{e^{kl}} : k, l \in \mathbb{N} \right\}$, where the double sequence $\mathbf{e^{kl}} = \left(\mathbf{e}_{ij}^{\mathbf{kl}} \right)$ is defined for all $k, l, i, j \in \mathbb{N}$ as follows:

$$\mathbf{e}_{ij}^{\mathbf{kl}} := \begin{cases} 1, & (k,l) = (i,j), \\ 0, & \text{otherwise}, \end{cases}$$

for all $i, j, k, l \in \mathbb{N}$.

For a double sequence $x = (x_{kl})$, its *sections* $x^{[m,n]}$ are defined by

$$x^{[m,n]} = \sum_{k=1}^{m} \sum_{l=1}^{n} x_{kl} \mathbf{e^{kl}},$$

which are the elements of Φ, the space of all finitely non-zero double sequences, for each $m, n \in \mathbb{N}$.

For a double sequence space V, its α-, $\beta(\vartheta)$- and γ-duals V^α, $V^{\beta(\vartheta)}$ and V^γ are defined by

$$V^\alpha := \left\{ (a_{kl}) \in \Omega : \sum_{k,l} \left| a_{kl} x_{kl} \right| < \infty \text{ for all } \left(x_{kl} \right) \in V \right\},$$

$$V^{\beta(\vartheta)} := \left\{ \left(a_{kl} \right) \in \Omega : \vartheta - \sum_{k,l} a_{kl} x_{kl} \text{ exists for all } \left(x_{kl} \right) \in V \right\},$$

$$V^\gamma := \left\{ \left(a_{kl} \right) \in \Omega : \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=0}^{m,n} a_{kl} x_{kl} \right| < \infty \text{ for all } \left(x_{kl} \right) \in V \right\},$$

where ϑ denotes any kind of convergence for double sequences. It is easy to see for any two spaces V and W of double sequences that $W^\alpha \subset V^\alpha$ whenever $V \subset W$ and $V^\alpha \subset V^\gamma$, and it is known that the inclusion $V^\alpha \subset V^{\beta(\vartheta)}$ holds.

A double sequence space V is said to be *solid* [**2**, p. 153] if and only if

$$\widetilde{V} := \Big\{(u_{kl}) \in \Omega : \exists (x_{kl}) \in V \text{ such that } |u_{kl}| \leq |x_{kl}| \text{ for all } k,l \in \mathbb{N}\Big\} \subset V.$$

A double sequence space V is said to be *monotone* if $xu = (x_{kl}u_{kl}) \in V$ for every $x = (x_{kl}) \in V$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ denotes the set of all double sequences of zeros and ones. If V is monotone, then $V^{\alpha} = V^{\beta(\vartheta)}$ [**19**, p. 36] and V is monotone whenever V is solid.

A locally convex double sequence space V is called a *DK-space*, if all of the seminorms $r_{kl}\colon V \to \mathbb{R}$, $x = (x_{ij}) \mapsto |x_{kl}|$ for all $k,l \in \mathbb{N}$, are continuous. A DK-space with a Fréchet topology is called an *FDK-space*. A normed FDK-space is called a *BDK-space*. Note that $\mathcal{M}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$ endowed with the norm $\|\cdot\|_{\infty}$ are BDK-spaces (see Zeltser [**19**, p. 37]).

Let (V, τ_V) be a DK-space and ϑ be a convergence notion for double sequences. Recall that V is supposed to contain Φ. We consider the distinguished subspace

$$S_V^{(\vartheta)} := \Big\{x = (x_{kl}) \in V : x = \vartheta - \sum_{k,l} x_{kl} \quad (V, \tau_V)\Big\}$$

of V. We say that an element $x \in V$ has the $AK(\vartheta)$-property in V if $x \in S_V^{(\vartheta)}$. The space V is called an $AK(\vartheta)$-space if every its element has $AK(\vartheta)$ in V, or equivalently, $V = S_V^{(\vartheta)}$, Zeltser [**19**].

Following Rao [**13**], we define the differentiated and integrated spaces of a double sequence space V, respectively, as

$$\mathrm{d}V := \Big\{(x_{kl}) \in \Omega : \Big(\frac{1}{kl}x_{kl}\Big)_{k,l\in\mathbb{N}} \in V\Big\},$$

$$\int V := \Big\{(x_{kl}) \in \Omega : (klx_{kl})_{k,l\in\mathbb{N}} \in V\Big\}.$$

Let V and W be two double sequence spaces and $A = (a_{mnkl})$ be any four-dimensional complex infinite matrix. Then, we say that A defines a *matrix mapping* from V into W and we write $A\colon V \to W$ if for every sequence $x = (x_{kl}) \in V$, the A-transform $Ax = \{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$ of x exists and is in W, where

$$(Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnnk}x_{kl} \text{ for each } m,n \in \mathbb{N}.$$

We define the *ϑ-summability domain* $V_A^{(\vartheta)}$ of A in a space V of double sequences by

$$V_A^{(\vartheta)} := \Big\{x = (x_{kl}) \in \Omega : Ax = \Big(\vartheta - \sum_{k,l} a_{mnkl}x_{kl}\Big)_{m,n\in\mathbb{N}} \text{exists and is in } V\Big\}.$$

The reader may refer for some details of the double sequence spaces to Boos et al. [**5**], Zeltser [**19**], and Başar and Yeşilkayagil Savaşcı [**3**].

Here and after, unless stated otherwise, we assume that ϑ denotes any of the symbols bp or r.

The sequence space h defined by

$$h := \left\{x = (x_k) \in \omega : \sum_{k=1}^{\infty} k\big|\Delta x_k\big| < \infty\right\} \bigcap c_0,$$

is called Hahn sequence space, named after its introducer Hahn [**8**], where Δ denotes the forward difference operator, i.e., $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$, and c_0 is the space of all null sequences. Rao [**13**] proved that the space h is a BK-space with AK with respect to the norm

$$\|x\|_h = \sum_{k=1}^{\infty} k\big|\Delta x_k\big| < \infty \text{ for all } x = (x_k) \in h.$$

Goes [**7**] introduced the generalised Hahn space h_d for arbitrary sequences $d = (d_k)$ with $d_k \neq 0$ for all $k \in \mathbb{N}$ by

$$h_d := \left\{x = (x_k) \in \omega : \sum_{k=1}^{\infty} |d_k|\big|\Delta x_k\big| < \infty\right\} \bigcap c_0.$$

Yeşilkayagil Savaşcı and Başar defined the double Hahn sequence space H_ϑ in [**18**]. Then, following them and Goes [**7**], we define the generalised double Hahn sequence space H_ϑ^d as follows:

$$H_\vartheta^d := \left\{x = (x_{kl}) \in \mathcal{L}_u : \sum_{k,l=1}^{\infty} d_{kl}\big|\Delta x_{kl}\big| < \infty \text{ and } \vartheta-\lim_{k,l\to\infty} x_{kl} = 0\right\}, \tag{1}$$

where $d = (d_{kl})$ is a monotonically increasing double sequence of positive real numbers such that $d_{kl} \neq 0$ for all $k, l \in \mathbb{N}$ and $\Delta x_{kl} = x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}$ for all $k, l \in \mathbb{N}$.

2. Main results

Theorem 2.1. *The space H_ϑ^d is a linear space with the co-ordinatewise addition and scalar multiplication of double sequences, and is a Banach space endowed with the norm*

$$\|x\|_d = \sum_{k,l=1}^{\infty} d_{kl}\big|\Delta x_{kl}\big| \text{ for all } x = (x_{kl}) \in H_\vartheta^d. \tag{2}$$

Proof. The first part of the theorem is a routine verification, and so, we omit details.

We show that H_ϑ^d is a Banach space with the norm $\|\cdot\|_d$ defined by (2). Let $\left(x^{(m)}\right)_{m\in\mathbb{N}}$ be any Cauchy sequence in the space H_ϑ^d, where $x^{(m)} = \left\{x_{kl}^{(m)}\right\}_{k,l\in\mathbb{N}}$ for every fixed $m \in \mathbb{N}$. Then, for a given $\varepsilon > 0$, there exists a positive integer $l_0 \in \mathbb{N}$ such that

$$\big\|x^{(m)} - x^{(n)}\big\|_d = \sum_{k,l=1}^{\infty} d_{kl}\left|\Delta\left(x_{kl}^{(m)} - x_{kl}^{(n)}\right)\right| < \varepsilon \tag{3}$$

for all $m, n > l_0$. Choosing $\varepsilon = \varepsilon_1/(d_{kl})^3$ for all $k, l \in \mathbb{N}$ and using the relation (3), we easily obtain that

$$\left|\Delta\left(x_{kl}^{(m)} - x_{kl}^{(n)}\right)\right| < \frac{\varepsilon_1}{(d_{kl})^3} < \varepsilon_1 \tag{4}$$

for each $k, l \in \mathbb{N}$ and for all $m, n > l_0$, where $\varepsilon_1 > 0$. This means that $\left(x^{(m)}\right)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{BV}$ for every fixed $k, l \in \mathbb{N}$. Since the space $\mathcal{BV}$ is complete by [**1**, Theorem 2.8], it converges, say $x^{(m)} \to x$, as $m \to \infty$. Using these infinitely many limits, we define the sequence x. Therefore, by letting $n \to \infty$ in (4), we get

$$\left|\Delta\left(x_{kl}^{(m)} - x_{kl}\right)\right| < \frac{\varepsilon_1}{(d_{kl})^3} \tag{5}$$

for all $k, l \in \mathbb{N}$. Also, we have the relation

$$\left|\Delta x_{kl}\right| \le \left|\Delta x_{kl}^{(m)}\right| + \left|\Delta\left(x_{kl}^{(m)} - x_{kl}\right)\right| \tag{6}$$

for all $k, l \in \mathbb{N}$.

By the relation (5) and the definition of $\left(d_{kl}\right)$, we obtain the inequality

$$\sum_{k,l=1}^{\infty} d_{kl}\left|\Delta\left(x_{kl}^{(m)} - x_{kl}\right)\right| < \sum_{k,l=1}^{\infty} d_{kl}\frac{\varepsilon_1}{(d_{kl})^3} < \infty.$$

Since $x^{(m)} \in H_\vartheta^d$ for each fixed $m \in \mathbb{N}$, $\sum_{k,l=1}^{\infty} d_{kl}\left|\Delta x_{kl}^{(m)}\right| < \infty$. If we multiply both sides of the inequality (6) by d_{kl} and take sum over $k, l \in \mathbb{N}$, we obtain

$$\begin{aligned}
\sum_{k,l=1}^{\infty} d_{kl}\left|\Delta x_{kl}\right| &\le \sum_{k,l=1}^{\infty} d_{kl}\left\{\left|\Delta x_{kl}^{(m)}\right| + \left|\Delta\left(x_{kl}^{(m)} - x_{kl}\right)\right|\right\} \\
&= \sum_{k,l=1}^{\infty} d_{kl}\left|\Delta x_{kl}^{(m)}\right| + \sum_{k,l=1}^{\infty} d_{kl}\left|\Delta\left(x_{kl}^{(m)} - x_{kl}\right)\right| \\
&< \infty,
\end{aligned}$$

which shows that $x \in H_\vartheta^d$. Since $\left(x^{(m)}\right)_{m\in\mathbb{N}} \in H_\vartheta^d$ is an arbitrary Cauchy sequence, the space H_ϑ^d is complete.

This step concludes the proof. □

Theorem 2.2. *The space H_ϑ^d endowed with the norm $\|\cdot\|_d$, defined by* (2)*, is a BDK-space.*

Proof. Since every norm (normed space) is a seminorm (seminormed space), we say that H_ϑ^d is a seminormed space with the seminorm (2). Also, we define new seminorms r_{kl} in the space H_ϑ^d by

$$\begin{array}{rccc}
r_{kl} : & H_\vartheta^d & \longrightarrow & \mathbb{R} \\
& x = (x_{ij}) & \longmapsto & r_{kl}(x) = \left|x_{kl}\right|
\end{array}$$

for all $k, l \in \mathbb{N}$. Now, we show that each one is continuous. To do this, we use the theorem given by Boos [**4**, Theorem 6.3.12, p. 284], that is, we must find $M > 0$

for all $x \in H_\vartheta^d$ such that

$$r_{kl}(x) = \left|x_{kl}\right| \leq M\|x\|_d \tag{7}$$

for all $k, l \in \mathbb{N}$. Suppose that the relation (7) does not hold. So, there exist some $M > 0$ and $x \in H_\vartheta^d$ such that

$$\left|x_{kl}\right| > M\|x\|_d \tag{8}$$

for all $k, l \in \mathbb{N}$. Keeping in mind $x \in \mathcal{L}_u$, if we take sum over $k, l \in \mathbb{N}$ in the both sides of the inequality (8), we get

$$\infty > \sum_{k,l=1}^{\infty} \left|x_{kl}\right| > \sum_{k,l=1}^{\infty} M\|x\|_d = M\|x\|_d \sum_{k,l=1}^{\infty} 1 = \infty,$$

which is a contradiction. Thus, the relation (7) holds for all $x \in H_\vartheta^d$, i.e., the seminorms r_{kl}'s are not continuous for all $k, l \in \mathbb{N}$. Hence, the space H_ϑ^d is a DK-space. Also since it is a Banach space by Theorem 2.1, it has Fréchet topology. Therefore, it is a BDK-space with the norm (2).

The proof is completed. □

Note. Defining the generalised double Hahn sequence space H_ϑ^d, we cannot stay connected to the definition of ordinary generalised Hahn sequence space h_d. Let us define the space H_ϑ^d for any arbitrary double sequence $d = \big(d_{kl}\big)$ with $d_{kl} \neq 0$ for all $k \in \mathbb{N}$ as

$$H_\vartheta^d := \left\{x = \big(x_{kl}\big) \in \Omega : \|x\|_{d_1} = \sum_{k,l=1}^{\infty} \left|d_{kl}\right|\left|\Delta x_{kl}\right| < \infty \text{ and } \vartheta - \lim_{k,l\to\infty} x_{kl} = 0\right\}. \tag{9}$$

Explanation 1. If we take the double sequences $d = \big(d_{kl}\big)$ as bounded with $d_{kl} \neq 0$ and for all $k, l \in \mathbb{N}$, we have the space $\mathcal{BV} \bigcap \mathcal{C}_{\vartheta 0}$ defined by Milovidov and Povolotski [**11**]. Because of that, defining the space H_ϑ^d, we take $d = \big(d_{kl}\big)$ as an increasing double sequence of positive real numbers.

Explanation 2. Take the sequence $x = \big(x_{kl}\big)$ defined by

$$x_{kl} := \begin{cases} 1, & k = 1 \text{ and } l \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

for all $k, l \in \mathbb{N}$. Then it is trivial that $x \in \mathcal{C}_{\vartheta 0}$. Also,

$$\begin{aligned} \|x\|_{d_1} = \sum_{k,l=1}^{\infty} \left|d_{kl}\right|\left|\Delta x_{kl}\right| &= \sum_{k,l=1}^{\infty} \left|d_{kl}\right|\left|x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}\right| \\ &= \sum_{l=1}^{\infty} \left|d_{1l}\right|\left|x_{1l} - x_{2l} - x_{1,l+1} + x_{2,l+1}\right| \\ &= \sum_{l=1}^{\infty} \left|d_{1l}\right|\left|x_{1l} - x_{1,l+1}\right| = 0 < \infty, \end{aligned}$$

that is, $x \in H_\vartheta^d$.

If we take $k, l = 1$ in the relation (7), we must have the inequality

$$r_{11}(x) = \left|x_{11}\right| \leq M\|x\|_{d_1}. \tag{11}$$

But, if we put the sequence x defined in (10) into the inequality (11), we obtain $1 \leq 0$. Hence, H_ϑ^d is not a DK-space with the definition (9).

For this reason, we must get the additional condition in the definition of the generalised double Hahn sequence space H_ϑ^d, as $x \in \mathcal{L}_u$.

Theorem 2.3. *The space H_ϑ^d is an $AK(\vartheta)$-space.*

Proof. We derive for a sequence $x = \big(x_{kl}\big) \in H_\vartheta^d$ that

$$\big\|x - x^{[mn]}\big\|_d = \sum_{k,l=1,n+1}^{m,\infty} d_{kl}\big|\Delta x_{kl}\big| + \sum_{k,l=m+1,1}^{\infty,n} d_{kl}\big|\Delta x_{kl}\big| + \sum_{k,l=m+1,n+1}^{\infty} d_{kl}\big|\Delta x_{kl}\big|$$

for all $m,n \in \mathbb{N}$. Then, the conclusion

$$\vartheta - \lim_{m,n\to\infty} \sum_{k,l=m+1,n+1}^{\infty} d_{kl}\big|\Delta x_{kl}\big| = 0 \tag{12}$$

is obvious. Define the double sequence $\zeta = \big(\zeta_{ml}\big)$ by $\zeta_{ml} = \sum_{k=1}^{m} d_{kl}\big|\Delta x_{kl}\big|$ for all $m, l \in \mathbb{N}$. Therefore, one can immediately see that

$$\sum_{k=1}^{m}\sum_{l=n+1}^{\infty} d_{kl}\big|\Delta x_{kl}\big| = \sum_{l=n+1}^{\infty}\sum_{k=1}^{m} d_{kl}\big|\Delta x_{kl}\big| = \sum_{l=n+1}^{\infty} \zeta_{ml}. \tag{13}$$

Since $\sum_{k,l} kl\big|\Delta x_{kl}\big|$ is convergent, from (13), it is the same for the series $\sum_{l=n+1}^{\infty} \zeta_{ml}$. So, the general term of this series tends to 0 as $m,n \to \infty$. Hence,

$$\vartheta - \lim_{m,n\to\infty} \sum_{k=1}^{m}\sum_{l=n+1}^{\infty} d_{kl}\big|\Delta x_{kl}\big| = \vartheta - \lim_{m,n\to\infty} \sum_{l=n+1}^{\infty} \zeta_{ml} = 0.$$

Let $\eta_{kn} = \sum_{l=1}^{n} d_{kl}\big|\Delta x_{kl}\big|$ for all $k, n \in \mathbb{N}$. In a similar way, we obtain

$$\vartheta - \lim_{m,n\to\infty} \sum_{k=m+1}^{\infty}\sum_{l=1}^{n} d_{kl}\big|\Delta x_{kl}\big| = \vartheta - \lim_{m,n\to\infty} \sum_{l=1}^{n}\sum_{k=m+1}^{\infty} \eta_{kn} = 0. \tag{14}$$

By (12)–(14), we see that

$$\vartheta - \lim_{m,n\to\infty} \big\|x - x^{[mn]}\big\|_d = 0,$$

i.e., the space H_ϑ^d is an $AK(\vartheta)$-space. □

Theorem 2.4. *The inclusion $H_\vartheta^d \subset \mathcal{L}_u$ strictly holds.*

Proof. From the definition of the set H_ϑ^d, the inclusion $H_\vartheta^d \subset \mathcal{L}_u$ is obvious. Now, define the sequence $y = \big(y_{kl}\big)$ by

$$y_{kl} := \begin{cases} (-1)^l/l^2, & k = 1 \text{ and } l \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

for all $k, l \in \mathbb{N}$. Then, it is trivial that $x \in \mathcal{C}_{\vartheta 0}$ and

$$\sum_{k,l=1}^{\infty} \big|y_{kl}\big| = \sum_{l=1}^{\infty} \big|y_{1l}\big| = \sum_{l=1}^{\infty} \frac{1}{l^2} < \infty,$$

that is, $y \in \mathcal{L}_u$.

Since $d_{kl} > d_{k,l-1} > \cdots > d_{k1}$ for all $k, l \in \mathbb{N}$, we have the inequality

$$\begin{aligned}
\sum_{k,l=1}^{\infty} d_{kl}\left|\Delta y_{kl}\right| &= \sum_{l=1}^{\infty} d_{1l}\left|y_{1l} - y_{2l} - y_{1,l+1} + y_{2,l+1}\right| \\
&= \sum_{l=1}^{\infty} d_{1l}\left|y_{1l} - y_{1,l+1}\right| \\
&= \sum_{l=1}^{\infty} d_{1l}\left|\frac{(-1)^l}{l^2} - \frac{(-1)^{l+1}}{(l+1)^2}\right| \\
&= \sum_{l=1}^{\infty} d_{1l}\frac{2l^2+2l+1}{l^3+2l^2+l} \\
&> d_{11}\sum_{l=1}^{\infty} \frac{2l^2+2l+1}{l^3+2l^2+l}.
\end{aligned} \tag{16}$$

Since $\frac{1}{l} < \frac{2l^2+2l+1}{l^3+2l^2+l}$ by the comparison test for positive series, we obtain the series in the left hand side of the relation (16) is not convergent. Hence, $y \in \mathcal{L}_u \setminus H_\vartheta^d$. □

Theorem 2.5. *Neither of the spaces $\int \mathcal{BV}$ and H_ϑ^d includes the other one.*

Proof. Let us consider the double sequences $x \in H_\vartheta^d$ and $y \notin H_\vartheta^d$ given in (10) and (15), respectively. Then,

$$\begin{aligned}
\sum_{k,l=1}^{\infty}\left|\Delta(klx_{kl})\right| &= \sum_{k,l=1}^{\infty}\left|klx_{kl}-(k+1)lx_{k+1,l}-k(l+1)x_{k,l+1}+(k+1)(l+1)x_{k+1,l+1}\right| \\
&= \sum_{l=1}^{\infty}\left|lx_{1l} - 2lx_{2l} - (l+1)x_{1,l+1} + 2(l+1)x_{2,l+1}\right| \\
&= \sum_{l=1}^{\infty}\left|lx_{1l} - (l+1)x_{1,l+1}\right| \\
&= \sum_{l=1}^{\infty}\left|l - (l+1)\right| \\
&= \sum_{l=1}^{\infty} 1 = \infty,
\end{aligned}$$

that is, $x \notin \int \mathcal{BV}$ and

$$\begin{aligned}
\sum_{k,l=1}^{\infty}\left|\Delta(kly_{kl})\right| &= \sum_{k,l=1}^{\infty}\left|kly_{kl}-(k+1)ly_{k+1,l}-k(l+1)y_{k,l+1}+(k+1)(l+1)y_{k+1,l+1}\right| \\
&= \sum_{l=1}^{\infty}\left|ly_{1l} - 2ly_{2l} - (l+1)y_{1,l+1} + 2(l+1)y_{2,l+1}\right|
\end{aligned}$$

$$= \sum_{l=1}^{\infty} \left| l y_{1l} - (l+1) y_{1,l+1} \right|$$
$$= \sum_{l=1}^{\infty} \left| \frac{l(-1)^l}{l^2} - \frac{(l+1)(-1)^{l+1}}{(l+1)^2} \right|$$
$$= \sum_{l=1}^{\infty} \frac{1}{l(l+1)} < \infty,$$

i.e., $y \in \int \mathcal{BV}$. This completes the proof. □

Theorem 2.6. *The space H_ϑ^d is not solid.*

Proof. Define the sequence $u = (u_{kl})$ as follows:

$$u_{kl} := \begin{cases} 1, & k = 1 \text{ and } l \text{ is odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

for all $k, l \in \mathbb{N}$, and take the sequence $x = (x_{kl}) \in H_\vartheta^d$ as in (10). Then, obviously $|u_{kl}| \leq |x_{kl}|$ for all $k, l \in \mathbb{N}$. Since $d_{kl} > d_{k1}$ for all $k, l \in \mathbb{N}$, we have

$$\sum_{k,l=1}^{\infty} d_{kl} \left| \Delta u_{kl} \right| = \sum_{l=1}^{\infty} d_{1l} \left| u_{1l} - u_{1,l+1} \right|$$
$$= \sum_{l=1}^{\infty} d_{1l} (4l-1) \left| u_{1,2l} \right|$$
$$= \sum_{l=1}^{\infty} d_{1l} (4l-1)$$
$$> d_{11} \sum_{l=1}^{\infty} (4l-1) = \infty,$$

that is, $u \notin H_\vartheta^d$. Thus, the proof is completed. □

Remark. Take the sequences x and u as in Theorem 2.6. Also, u is in $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Then, we have $|x_{kl} u_{kl}| = |u_{kl}|$ for each $k, l \in \mathbb{N}$. Hence, the space H_ϑ^d is not monotone.

Definition 2.7. Let V be a BDK-space. A subset E of Φ is called a determining set for V if $D(V)$ is the absolutely convex hull of E, where $D = D(V) = \{x \in \Phi : \|x\|_V \leq 1\}$.

Theorem 2.8. *Assume that $s_d^{\mathbf{kl}} = \frac{1}{d_{kl}} \sum_{i,j=1}^{k,l} \mathbf{e^{ij}}$ for $k, l \in \mathbb{N}$. Consider the set*

$$E = \left\{ s_d^{\mathbf{kl}} : k, l \in \mathbb{N} \right\}.$$

Then, E is the determining set for the space H_ϑ^d.

Proof. Suppose that K denotes the absolutely convex hull of E and let $x \in D(H_\vartheta^d)$. Then, $x \in \Phi$ and $\|x\|_d \leq 1$. Consequently,

$$x = \sum_{k,l=1}^{m,n} x_{kl}\mathbf{e}^{\mathbf{kl}} = \sum_{k,l=1}^{m,n} t_{kl} s_d^{\mathbf{kl}},$$

where $t_{kl} = d_{kl}\Delta x_{kl}$ for $k, l \in \mathbb{N}$. Also, $\sum_{k,l=1}^{m,n} |t_{kl}| \leq \|x\|_d \leq 1$. Thus, $x \in K$. Therefore, the inclusion $D(H_\vartheta^d) \subset K$ holds.

Conversely, assume that $x \in K$. Then, we can write

$$x = \sum_{k,l=1}^{m,n} t_{kl} s_d^{\mathbf{kl}} \tag{18}$$

with $\sum_{k,l=1}^{m,n} |t_{kl}| \leq 1$. By the relation (18), we observe that

$$\begin{aligned}
x_{11} &= \sum_{k=1}^{m}\sum_{l=1}^{n} \frac{t_{kl}}{d_{kl}}, \quad x_{12} = \sum_{k=1}^{m}\sum_{l=2}^{n} \frac{t_{kl}}{d_{kl}}, \quad \ldots, \quad x_{1n} = \sum_{k=1}^{m} \frac{t_{kn}}{d_{kn}};\\
x_{21} &= \sum_{k=2}^{m}\sum_{l=1}^{n} \frac{t_{kl}}{d_{kl}}, \quad x_{22} = \sum_{k=2}^{m}\sum_{l=2}^{n} \frac{t_{kl}}{d_{kl}}, \quad \ldots, \quad x_{2n} = \sum_{k=1}^{m} \frac{t_{kn}}{d_{kn}};\\
&\vdots\\
x_{m1} &= \sum_{l=1}^{n} \frac{t_{kl}}{d_{kl}}, \qquad x_{m2} = \sum_{l=2}^{n} \frac{t_{kl}}{d_{kl}}, \quad \ldots, \quad x_{mn} = \frac{t_{mn}}{d_{mn}};\\
x_{kl} &= 0 \text{ for } k > m \text{ or } l > n \text{ or both.}
\end{aligned}$$

After straightforward calculation, we obtain

$$\|x\|_d = \sum_{k,l=1}^{\infty} d_{kl}|\Delta x_{kl}| = \sum_{k,l=1}^{m,n} |t_{kl}| \leq 1.$$

So, $x \in D(H_\vartheta^d)$. Thus, we see that the inclusion $K \subset D(H_\vartheta^d)$ holds.

Therefore, we conclude the fact that $K = D(H_\vartheta^d)$. This completes the proof. □

Corollary 2.9. *Take the sequence $s_d^{\mathbf{kl}} \in E$ for $k, l \in \mathbb{N}$. Then, we have*

$$\left(As_d^{\mathbf{kl}}\right)_{mn} = \sum_{i,j=1}^{\infty} a_{mnij} \frac{1}{d_{kl}} \sum_{i,j=1}^{k,l} \mathbf{e}^{\mathbf{ij}} = \frac{1}{d_{kl}} \sum_{i,j=1}^{k,l} a_{mnij}$$

for all $m, n \in \mathbb{N}$. Thus, we obtain that

$$A[E] = \left\{ \frac{1}{d_{kl}} \sum_{i,j=1}^{k,l} a_{mnij} : m, n \in \mathbb{N} \right\}$$

for each $k, l \in \mathbb{N}$.

Referring Wilansky [**15**, Theorem 8.3.4.], we can give the following lemma without proof which is a corrected version of Lemma 2.1 given in [**14**].

Lemma 2.10. *Let V be a BDK-space and E is determining set for V. Let W be an FDK-space and $A = (a_{mnkl})$ be a four dimensional infinite matrix for $m, n, k, l \in \mathbb{N}$. Suppose that either V has $AK(\vartheta)$ or $(a_{mnkl})_{k,l \in \mathbb{N}} \in \Phi$ for all $m, n \in \mathbb{N}$. Then $A \in (V : W)$ if and only if*

(i) $(a_{mnkl})_{m,n \in \mathbb{N}} \in W$ *for all* $k, l \in \mathbb{N}$,

(ii) $A[E]$ *is a bounded set of* W.

Theorem 2.11. $A = (a_{mnkl}) \in (H_\vartheta^d : \mathcal{C}_\vartheta)$ *if and only if the following conditions hold:*

$$\vartheta\text{–}\lim_{m,n\to\infty} a_{mnkl} \text{ exists for all } k, l \in \mathbb{N}, \tag{19}$$

$$\sup_{m,n,k,l \in \mathbb{N}} \frac{1}{d_{kl}} \left| \sum_{i,j=1}^{k,l} a_{mnij} \right| < \infty. \tag{20}$$

Proof. By Theorems 2.2 and 2.3, we have that the space H_ϑ^d is a BDK-space with $AK(\vartheta)$. Therefore, we conclude by Lemma 2.10 and Corollary 2.9 that $A \in (H_\vartheta^d : \mathcal{C}_\vartheta)$ if and only if $(a_{mnkl})_{m,n \in \mathbb{N}}$ in $\mathcal{C}_\vartheta$ for all $k, l \in \mathbb{N}$, and $A[E]$ is a bounded subset of $\mathcal{C}_\vartheta$, that is, $A \in (H_\vartheta^d : \mathcal{C}_\vartheta)$ if and only if the conditions (19) and (20) hold. This establishes the result. □

Let us keep in mind that the spaces $\mathcal{M}_u$ and $\mathcal{L}_u$ are BDK-spaces, in [**19**, p. 37] and [**17**, Theorem 2.1], respectively. So, combining Corollary 2.9 and Lemma 2.10 and omitting the proofs, we formulate the following results.

Theorem 2.12. $A = (a_{mnkl}) \in (H_\vartheta^d : \mathcal{M}_u)$ *if and only if*

$$\sup_{m,n,k,l \in \mathbb{N}} |a_{mnkl}| < \infty. \tag{21}$$

Theorem 2.13. $A = (a_{mnkl}) \in (H_\vartheta^d : \mathcal{L}_u)$ *if and only if*

$$\sum_{m,n=1}^{\infty} |a_{mnkl}| < \infty \text{ for all } k, l \in \mathbb{N}, \tag{22}$$

$$\sup_{k,l \in \mathbb{N}} \frac{1}{d_{kl}} \sum_{m,n=1}^{\infty} \left| \sum_{i,j=1}^{k,l} a_{mnij} \right| < \infty.$$

Theorem 2.14. $A = (a_{mnkl}) \in (H_\vartheta^d : H_\vartheta^d)$ *if and only if the condition* (22) *holds and*

$$\vartheta\text{–}\lim_{m,n\to\infty} a_{mnkl} = 0 \quad \text{for all } k, l \in \mathbb{N},$$

$$\sum_{m,n=1}^{\infty} d_{mn} |\Delta_{11}^{mn} a_{mnkl}| < \infty \quad \text{for all } k, l \in \mathbb{N},$$

$$\sup_{k,l \in \mathbb{N}} \frac{1}{d_{kl}} \sum_{m,n=1}^{\infty} d_{mn} \left| \sum_{i,j=1}^{k,l} \Delta_{11}^{mn} a_{mnij} \right| < \infty,$$

where $\Delta_{11}^{mn} a_{mnkl} = a_{mnkl} - a_{m,n+1,kl} - a_{m+1,nkl} + a_{m+1,n+1,kl}$ *for* $m, n, k, l \in \mathbb{N}$.

Theorem 2.15. *The α-dual of the space H_ϑ^d is the space $\mathcal{M}_u$.*

Proof. Since $H_\vartheta^d \subset \mathcal{L}_u$ and $\mathcal{L}_u^\alpha = \mathcal{M}_u$, we have that $\mathcal{M}_u \subset \{H_\vartheta^d\}^\alpha$.

Conversely, suppose that $a = (a_{kl}) \in \{H_\vartheta^d\}^\alpha \setminus \mathcal{M}_u$. Then, $\sum_{k,l} |a_{kl}y_{kl}| < \infty$ for all $y = (y_{kl}) \in H_\vartheta^d$ and $\sup_{k,l\in\mathbb{N}} |a_{kl}| = \infty$. Choose the sequence a as $a_{kl} > l^3$ for all $k,l \in \mathbb{N}$ and take $y \in H_\vartheta^d$ as in (15). Then, we easily see that $\sum_{k,l} |a_{kl}y_{kl}| = \sum_l |a_{1l}y_{1l}| > \sum_l l = \infty$, that is, $a \notin \{H_\vartheta^d\}^\alpha$, a contradiction.

Therefore, the proof is completed. □

Lemma 2.16 ([**16**, Theorem 4.3(i)])**.** *$A \in (\mathcal{L}_u : \mathcal{M}_u)$ if and only if*

$$\sup_{m,n,k,l\in\mathbb{N}} |a_{mnkl}| < \infty. \tag{23}$$

Lemma 2.17 ([**16**, Theorem 4.1(i)])**.** *$A \in (\mathcal{L}_u : \mathcal{C}_{bp})$ if and only if the condition in (23) holds and*

$$bp\text{-}\lim_{m,n\to\infty} a_{mnkl} \quad \text{exists for all} \quad k,l \in \mathbb{N}.$$

Theorem 2.18. *Define the sets d_1 and d_2 as follows:*

$$d_1 = \left\{a = (a_{kl}) \in \Omega : \sup_{k,l\in\mathbb{N}} \frac{1}{d_{kl}} \left| \sum_{i,j=1}^{k,l} a_{ij} \right| < \infty \right\},$$

$$d_2 = \left\{a = (a_{kl}) \in \Omega : bp - \lim_{k,l\to\infty} \frac{1}{d_{kl}} \sum_{i,j=1}^{k,l} a_{ij} \text{ exists} \right\}.$$

Then $\{H_r^d\}^{\beta(bp)} = d_1 \cap d_2$.

Proof. Let $a = (a_{kl}) \in \Omega$ be an arbitrary sequence and take $x = (x_{kl}) \in H_r^d$. Also, put $d_{kl}\Delta x_{kl} = \eta_{kl}$ for all $k,l \in \mathbb{N}$ and define the four-dimensional matrix $B = (b_{mnkl})$ as follows:

$$b_{mnkl} := \begin{cases} \dfrac{1}{d_{kl}} \sum_{i,j=1}^{k,l} a_{ij}, & 1 \le k \le m \text{ and } 1 \le l \le n, \\ 0, & \text{otherwise}, \end{cases}$$

for all $m,n,k,l \in \mathbb{N}$. Since $x \in H_r^d$, $\eta \in \mathcal{L}_u$. With some straightforward calculation, we have

$$\begin{aligned} \sum_{k,l=1}^{m,n} a_{kl}x_{kl} &= \sum_{k,l=1}^{m,n} \Delta x_{kl} \sum_{i,j=1}^{k,l} a_{kl} + \sum_{k,l=1}^{m,n} a_{kl}x_{m+1,l} + \sum_{k,l=1}^{m,n} a_{kl}x_{k,n+1} - \sum_{k,l=1}^{m,n} a_{kl}x_{m+1,n+1} \\ &= \sum_{k,l=1}^{m,n} d_{kl}\Delta x_{kl} \frac{1}{d_{kl}} \sum_{i,j=1}^{k,l} a_{ij} + \sum_{k,l=1}^{m,n} a_{kl}x_{m+1,l} + \sum_{k,l=1}^{m,n} a_{kl}x_{k,n+1} - \sum_{k,l=1}^{m,n} a_{kl}x_{m+1,n+1} \\ &= \sum_{k,l=1}^{m,n} \eta_{kl} \frac{1}{d_{kl}} \sum_{i,j=1}^{k,l} a_{ij} + \sum_{k,l=1}^{m,n} a_{kl}x_{m+1,l} + \sum_{k,l=1}^{m,n} a_{kl}x_{k,n+1} - \sum_{k,l=1}^{m,n} a_{kl}x_{m+1,n+1} \\ &= (B\eta)_{mn} + \sum_{k,l=1}^{m,n} a_{kl}x_{m+1,l} + \sum_{k,l=1}^{m,n} a_{kl}x_{k,n+1} - \sum_{k,l=1}^{m,n} a_{kl}x_{m+1,n+1}. \end{aligned} \tag{24}$$

If the limit $bp-\lim_{m,n\to\infty}\sum_{k,l=1}^{m,n} a_{kl}x_{kl}$ exists, then the limits

$$bp-\lim_{m,n\to\infty}\sum_{k,l=1}^{m,n} a_{kl}x_{m+1,l},$$
$$bp-\lim_{m,n\to\infty}\sum_{k,l=1}^{m,n} a_{kl}x_{k,n+1},$$
$$\lim_{m,n\to\infty}\sum_{k,l=1}^{m,n} a_{kl}x_{m+1,n+1}$$

also exist. Therefore, we observe by the relation (24) that $a \in \{H_r^d\}^{\beta(bp)}$ if and only if

$$bp-\lim_{m,n\to\infty}\sum_{k,l=1}^{m,n} \eta_{kl}\frac{1}{kl}\sum_{i,j=1}^{k,l} a_{ij} = bp-\lim_{m,n\to\infty}(B\eta)_{mn}$$

exists, that is, $a \in \{H_r^d\}^{\beta(bp)}$ if and only if $B \in (\mathcal{L}_u : \mathcal{C}_{bp})$. Hence, $a \in \{H_r^d\}^{\beta(bp)}$ if and only if the conditions

$$\sup_{k,l\in\mathbb{N}}\left|\frac{1}{d_{kl}}\sum_{i,j=1}^{k,l} a_{ij}\right| < \infty,$$
$$bp-\lim_{k,l\to\infty}\frac{1}{d_{kl}}\sum_{i,j=1}^{k,l} a_{ij}$$

hold from Lemma 2.17. Therefore, $\{H_r^d\}^{\beta(bp)} = d_1 \cap d_2$. □

Theorem 2.19. *The γ-dual of the space H_r^d is the set d_1.*

Proof. This is easily obtained by proceeding as in the proof of Theorem 2.18 above, by using Lemma 2.16 instead of Lemma 2.17. So, we omit details. □

3. Conclusion

In [**18**], Yeşilkayagil Savaşcı and Başar recently introduced the double Hahn sequence space H_ϑ as an extension of Hahn sequence space h defined by Hahn [**8**], where $\vartheta \in \{bp, r\}$. They gave some topological properties of the space H_ϑ and characterized the classes $(H_\vartheta : W)$ of four dimensional matrix transformations, where $W \in \{\mathcal{C}_\vartheta, \mathcal{M}_u, \mathcal{L}_u, H_\vartheta\}$. Finally, they determined the α- and dual of the space H_ϑ and $\beta(bp)$-dual of the space H_r.

In this present paper, as a continuation of Yeşilkayagil Savaşcı and Başar [**18**], we have studied the generalised double Hahn sequence space H_ϑ^d as an extension of generalised Hahn sequence space h_d defined by Goes [**7**], where $\vartheta \in \{bp, r\}$. We have emphasized some topological properties of the space H_ϑ^d and characterizations of the classes $(H_\vartheta^d : W)$ of four dimensional infinite matrices, where $W \in \{\mathcal{C}_\vartheta, \mathcal{M}_u, \mathcal{L}_u, H_\vartheta^d\}$. Also, we have found the α-dual of the space H_ϑ^d and $\beta(bp)$- and γ-duals of the space H_r^d.

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