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# $(\alpha,\beta)$-TYPE ALMOST $\eta$-RICCI–YAMABE SOLITONS IN PERFECT FLUID SPACETIME

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Abstract. In this paper, we consider perfect fluid spacetime admitting $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe solitons by means of some curvature tensors. Ricci pseudosymmetry concepts of perfect fluid spacetime admitting $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton are introduced according to the choice of some special curvature tensors such as Riemann, concircular and projective curvature tensor. After then, according to choosing of the curvature tensors, necessary conditions are given for perfect fluid spacetime admitting $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton to be Ricci semisymmetric. Then, some important characterizations are given for Ricci, Yamabe, Einstein and $\eta$-Einstein solitons on perfect fluid spacetime.

## 1. Introduction

Geometric flows are crucial in scrutinizing the geometric configurations within Riemannian geometry. In 1982, Hamilton presented the idea of Ricci flow in [**7, 8**], which is defined as follows:

$$\frac{\partial}{\partial t}g(t) = -2S(t), \qquad t \geq 0, \quad g(0) = g, \tag{1}$$

where $g$ is the Riemannian metric, $S$ denotes the $(0,2)$-symmetric Ricci tensor. Solitons are waves that propagate with minimal energy loss, maintaining their shape and speed even after colliding with another wave of similar nature. Solitons play a crucial role in the analytical treatment of initial-value problems related to nonlinear partial differential equations that describe the propagation of waves. Indeed, solitons have been instrumental in explaining the recurrence phenomena observed in the Fermi–Pasta–Ulam system.

A Ricci soliton arises as the endpoint of the evolution of a soliton under Ricci flow when it moves solely through a one-parameter group of diffeomorphism and scaling. A Riemannian manifold $(M^n, g)$ is said to be a Ricci soliton if there exist a vector field $V$ and a constant $\lambda$ such that

$$\mathcal{L}_V g + 2S = 2\lambda g, \tag{2}$$

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where $\mathcal{L}_V$ denotes the Lie derivative of $V$. Different type of Ricci solitons for different structures were studied in [**11, 12, 13**]. To address the Yamabe problem of discovering a metric on a specified compact Riemannian manifold $(M^n, g)$ that conforms to $g$ while possessing a constant scalar curvature $r$, Hamilton introduced the concept of Yamabe flow, which is defined as follows [**8**],

$$\frac{\partial}{\partial t} g(t) = -rg(t), \qquad t \geq 0, \quad g(0) = g. \tag{3}$$

Similar to Ricci soliton, a Yamabe soliton is a self-similar solution to the Yamabe flow and is defined as follows:

$$\frac{1}{2}\mathcal{L}_V g = (\lambda - r) g. \tag{4}$$

In two-dimensional analysis, the Ricci soliton and Yamabe soliton coincide, but in higher dimensions, the Yamabe soliton maintains the metric's conformal class while the Ricci soliton doesn't consistently do so. Over the past two decades, geometric flow theories like Ricci flow and Yamabe flow, along with their solitons, have captivated numerous geometers.

Recently, Guler and Crasmareanu introduced a novel geometric flow, termed the Ricci–Yamabe map, which is a scalar amalgamation of Ricci flow and Yamabe flow [**6**]. The Ricci–Yamabe flow of type $(\alpha, \beta)$ is defined as follows.

**Definition 1.** The map $RY^{(\alpha,\beta,g)} \colon I \to T_2^s(M)$ given by

$$RY^{(\alpha,\beta,g)} = \frac{\partial g}{\partial t}(t) + 2\alpha S(t) + \beta r(t) g(t),$$

is called the $(\alpha, \beta)$-Ricci–Yamabe map of the Riemannian manifold $(M, g)$ [**6**]. If

$$RY^{(\alpha,\beta,g)} \equiv 0,$$

then $g$ is called an $(\alpha, \beta)$-Ricci–Yamabe flow.

The Ricci–Yamabe flow's nature can vary, encompassing Riemannian, semi-Riemannian, or singular Riemannian flows contingent upon the signs of the two scalars $\alpha$ and $\beta$. This versatility in offering multiple choices proves beneficial when exploring geometry or when addressing the physical models inherent in relativistic theories. The concept of an $(\alpha, \beta)$-Ricci–Yamabe soliton, or simply the Ricci–Yamabe soliton, is defined in the following manner.

**Definition 2.** A Riemannian or pseudo-Riemannian manifold $(M^n, g)$ is said to be a Ricci–Yamabe soliton $(g, V, \lambda, \alpha, \beta)$ if

$$\mathcal{L}_V g + 2\alpha S = (2\lambda - \beta r) g. \tag{5}$$

If $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$, then the Ricci–Yamabe soliton is expanding, shrinking or steady, respectively [**5**].

A gradient Ricci–Yamabe soliton is characterized by the existence of a smooth function $f \colon M \to \mathbf{R}$ such that $V = Df$, where $D$ represents the gradient operator relative to metric $g$ on the manifold. The Ricci–Yamabe soliton indeed serves as a broader, generalized form encompassing both Ricci and Yamabe solitons within its scope. The Ricci–Yamabe soliton of $(1, -1)$-type is commonly known as an Einstein soliton [**3, 17**]. Therefore, it is worthwhile to study Ricci–Yamabe soliton as it generalizes a large group of solitons.

Cho and Kimura in [**4**], expanded on the concept of Ricci solitons by introducing the $\eta$-Ricci soliton, which emerges from modifying equation (2) through the addition of a multiple of a specific $(0, 2)$-tensor field $\eta \otimes \eta$ [**2**]. Sıddıqi and Akyol achieved a broader extension by generalizing the concept further in [**15**], called such soliton as $\eta$-Ricci–Yamabe soliton of type $(\alpha, \beta)$ which is defined as:

$$\mathcal{L}_V g + 2\alpha S + (2\lambda - \beta r)\, g + 2\mu\eta \otimes \eta = 0. \tag{6}$$

It is worth remarking that $\eta$-Ricci soliton [**15**] and $\eta$-Yamabe soliton of type $(1, 0)$ and $(0, 2)$, respectively [**1**]. If $\mu = 0$ in (6), then it reduces to Ricci–Yamabe soliton.

In recent years, extensive research has focused on $\eta$-Ricci and $\eta$-Yamabe solitons within the realm of Riemannian geometry. Notably, there has been a significant initiation of geometric flow studies applied to cosmological models, specifically perfect fluid spacetimes. In [**1**] and [**2**], Blaga developed the investigation of $\eta$-Ricci and $\eta$-Einstein solitons within perfect fluid spacetime, deriving the Poisson equation from the solitons equation under conditions where the potential vector field $\xi$ follows a gradient type.

Kimura and Venkatesha in [**17**] and [**18**], conducted an analysis on Ricci solitons in perfect fluid spacetime employing a torse-forming vector field. Additionally, research on Conformal Ricci solitons in perfect fluid spacetime is documented in [**16**].

Moreover, Praveena et al. examined solitons in Kahlerian spacetime manifolds. Given that Ricci–Yamabe solitons represent a scalar amalgamation of Ricci and Yamabe solitons, exploring it within the framework of perfect fluid spacetime seems promising, offering a platform to generalize and expand upon the existing findings in this context.

## 2. Preliminaries

Absolutely, in Einstein's field equations, the energy-momentum tensor serves as a fundamental component. It characterizes the curvature of spacetime within the framework of general relativity. The spacetime in general relativity is conceptualized as a connected four-dimensional semi-Riemannian manifold, denoted as $\left(M^4, g\right)$, where $M^4$ represents the manifold and $g$ denotes the Lorentzian metric. This metric $g$ conforms to the signature $(-, +, +, +)$ which reflects the spacetime's Lorentzian or pseudo-Riemannian nature, indicating one negative and three positive eigenvalues associated with its curvature. If the Ricci tensor follows a specific structure as

$$S = ag + b\eta \otimes \eta, \tag{7}$$

a spacetime is classified as a perfect fluid spacetime, where $a, b$ are scalars and $\eta$ is non-zero 1-form.

The general form of energy-momentum tensor $T$ for a perfect fluid is

$$T(X,Y) = \rho g(X,Y) + (\sigma+\rho)\eta(X)\eta(Y) \tag{8}$$

for any $X, Y \in \chi(M)$, where $\sigma$ is the energy density, $\rho$ is the isotropic pressure, $g$ is the metric tensor of Minkowski spacetime, $\eta(X) = -g(X,\xi)$ is 1-form, equivalent to unit vector $\xi$ and $g(\xi,\xi) = -1$ [**9**]. If $\rho = \rho(\sigma)$, then perfect fluid spacetime is called isentropic and if $\sigma = 3\rho$, then it is a radiation fluid [**10**].

The Einstein's field equation governing the perfect fluid motion is defined as:

$$S(X,Y) + \left(\omega - \frac{r}{2}\right) g(X,Y) = kT(X,Y) \tag{9}$$

for any $X, Y \in \chi(M)$, where $\omega$ is the cosmological constant, $k$ is the gravitational constant.

Combining (8) and (9), we obtain

$$S(X,Y) = -\left(\omega - \frac{r}{2} + k\rho\right) g(X,Y) + k(\sigma+\rho)\eta(X)\eta(Y). \tag{10}$$

Taking trace of (10), the scalar curvature becomes $r = 4\omega + k(\sigma - 3\rho)$, using in (10), we infer

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{11}$$

where

$$a = \omega + \frac{k(\sigma-\rho)}{2} \quad \text{and} \quad b = k(\sigma+\rho).$$

**Definition 3.** A vector field $\xi$ is called torse-forming if it satisfies

$$\nabla_X \xi = X + \eta(X)\xi \tag{12}$$

for any $X \in \chi(M)$ [**9**].

**Lemma 1.** *In perfect fluid spacetime with torse-forming vector field $\xi$, the following relations hold:*

$$\eta(\nabla_\xi \xi) = 0, \nabla_\xi \xi = 0, \tag{13}$$

$$(\nabla_X \eta) Y = g(X,Y) + \eta(X)\eta(Y), \tag{14}$$

$$(\mathcal{L}_\xi g)(X,Y) = 2[g(X,Y) + \eta(X)\eta(Y)], \tag{15}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{16}$$

$$\eta(R(X,Y)Z) = g(\eta(X)Y - \eta(Y)X, Z), \tag{17}$$

$$S(X,\xi) = \frac{-1}{2}[2\omega + k(3\sigma+\rho)]\eta(X). \tag{18}$$

## 3. $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe solitons in perfect fluid spacetime with torse-forming vector field

Let $(g, \xi, \lambda, \mu, \alpha, \beta)$ be $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton in perfect fluid spacetime. Thus, in a perfect fluid spacetime, from (6) and (15), we have

$$2\alpha S(X,Y) + (2\lambda - \beta r + 2) g(X,Y) + 2(\mu+1)\eta(X)\eta(Y) = 0. \tag{19}$$

For $Y=\xi$ in (19), this implies that

$$2\alpha S(\xi,X)=(2\lambda-\beta r+2\mu+4)\,\eta(X)\,. \tag{20}$$

Taking into account (18) and (20), we conclude that

$$\alpha\left[2\omega+k(3\sigma+\rho)\right]=\beta r-2(\lambda+\mu+2)\,. \tag{21}$$

**Definition 4.** Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field. If $R\cdot S$ and $Q(g,S)$ are linearly dependent, then the $M^n$ is said to be *Ricci pseudosymmetric.*

In this case, there exists a function $\mathcal{H}_1$ on $M^n$ such that

$$R\cdot S=\mathcal{H}_1 Q(g,S)\,.$$

In particular, if $\mathcal{H}_1=0$, the $M$ is said to be *Ricci semisymmetric.*

**Theorem 1.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. If $M^n$ is Ricci pseudosymmetric, then we get*

$$\mathcal{H}_1=\frac{2\omega+k(3\sigma+\rho)+\beta r-2(\lambda+1)}{2\omega+k(3\sigma+\rho)-\beta r+2(\lambda+1)},$$

*provided* $2\omega+k(3\sigma+\rho)\neq\beta r-2(\lambda+1)\,.$

*Proof.* Let us assume that perfect fluid spacetime with torse-forming vector field $M^n$ is Ricci pseudosymmetric and $(g,\xi,\lambda,\mu,\alpha,\beta)$ is $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. That means,

$$(R(X,Y)\cdot S)(U,V)=\mathcal{H}_1 Q(g,S)(U,V;X,Y)$$

for all $X,Y,U,V\in\Gamma(TM^n)$. From the last equation, we can easily write

$$\begin{aligned}&S(R(X,Y)U,V)+S(U,R(X,Y)V)\\&=\mathcal{H}_1\left\{S((X\wedge_g Y)U,V)+S(U,(X\wedge_g Y)V)\right\}.\end{aligned} \tag{22}$$

Setting $V=\xi$ in (22) and using (16), (18), we get

$$\begin{aligned}&S(U,\eta(Y)X-\eta(X)Y)+\mathcal{F}\eta(R(X,Y)U)\\&=\mathcal{H}_1\left\{\mathcal{F}g(\eta(X)Y-\eta(Y)X,U)+S(U,\eta(X)Y-\eta(Y)X)\right\},\end{aligned} \tag{23}$$

where $\mathcal{F}=\frac{-1}{2}\left[2\omega+k(3\sigma+\rho)\right].$ By using (17) in (23), we get

$$\begin{aligned}&\mathcal{F}g(\eta(X)Y-\eta(Y)X,U)+S(\eta(Y)X-\eta(X)Y,U)\\&=\mathcal{H}_1\left\{\mathcal{F}g(\eta(X)Y-\eta(Y)X,U)+S(U,\eta(X)Y-\eta(Y)X)\right\}.\end{aligned} \tag{24}$$

If we use (19) in (24), we can write

$$\begin{aligned}&\left\{\frac{1}{2}\left[2\omega+k(3\sigma+\rho)\right]+\frac{1}{2\alpha}\left[\beta r-2(\lambda+1)\right]\right.\\&\left.-\mathcal{H}_1\left[\frac{1}{2}\left[2\omega+k(3\sigma+\rho)\right]+\frac{1}{2\alpha}\left[\beta r-2(\lambda+1)\right]\right]\right\}\\&\times g(\eta(Y)X-\eta(X)Y,U)=0.\end{aligned} \tag{25}$$

This completes the proof of theorem. □

**Corollary 1.** *Let $M^n$ be an n-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, \alpha, \beta)$ be $(\alpha, \beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. If $M^n$ is Ricci semisymmetric, then we observe the following situations:*

i) *$M^n$ is expanding if $\alpha [2\omega + k (3\sigma + \rho)] + \beta r > 2$,*

ii) *$M^n$ is steady if $\alpha [2\omega + k (3\sigma + \rho)] + \beta r = 2$,*

iii) *$M^n$ is shrinking if $\alpha [2\omega + k (3\sigma + \rho)] + \beta r < 2$.*

**Theorem 2.** *Let $M^n$ be an n-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 1, 0)$ be almost Ricci soliton on $M^n$. If $M^n$ is Ricci pseudosymmetric, then we get*

$$\mathcal{H}_1 = \frac{2\omega + k (3\sigma + \rho) - 2 (\lambda + 1)}{2\omega + k (3\sigma + \rho) + 2 (\lambda + 1)},$$

*provided $2\omega + k (3\sigma + \rho) \neq -2 (\lambda + 1)$.*

**Corollary 2.** *Let $M^n$ be an n-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 1, 0)$ be almost Ricci soliton on $M^n$. If $M^n$ is Ricci semisymmetric, then we observe the following situations:*

i) *$M^n$ is expanding if $2\omega + k (3\sigma + \rho) > 2$,*

ii) *$M^n$ is steady if $2\omega + k (3\sigma + \rho) = 2$,*

iii) *$M^n$ is shrinking if $2\omega + k (3\sigma + \rho) < 2$.*

**Theorem 3.** *Let $M^n$ be an n-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 0, 1)$ be almost Yamabe soliton on $M^n$. If $M^n$ is Ricci pseudosymmetric, then we get $\mathcal{H}_1 = -1$.*

**Corollary 3.** *Let $M^n$ be an n-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 0, 1)$ be almost Yamabe soliton on $M^n$. If $M^n$ is Ricci semisymmetric, then we observe the following situations:*

i) *$M^n$ is expanding if $r > 2$,*

ii) *$M^n$ is steady if $r = 2$,*

iii) *$M^n$ is shrinking if $r < 2$.*

**Theorem 4.** *Let $M^n$ be an n-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 1, 1)$ be almost Einstein soliton on $M^n$. If $M^n$ is Ricci pseudosymmetric, then we get*

$$\mathcal{H}_1 = \frac{2\omega + k (3\sigma + \rho) + r - 2 (\lambda + 1)}{2\omega + k (3\sigma + \rho) - r + 2 (\lambda + 1)},$$

*provided $2\omega + k (3\sigma + \rho) \neq r - 2 (\lambda + 1)$.*

**Corollary 4.** *Let $M^n$ be an n-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 1, 1)$ be almost Einstein soliton on $M^n$. If $M^n$ is Ricci semisymmetric, then we observe the following situations:*

i) *$M^n$ is expanding if $2\omega + k (3\sigma + \rho) + r > 2$,*

ii) *$M^n$ is steady if $2\omega + k (3\sigma + \rho) + r = 2$,*

iii) $M^n$ *is shrinking if* $2\omega + k(3\sigma+\rho) + r < 2$.

**Theorem 5.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,1)$ be almost $\eta$-Einstein soliton on $M^n$. If $M^n$ is Ricci pseudosymmetric, then we get*

$$\mathcal{H}_1 = \frac{2\omega + k(3\sigma+\rho) - 2r^2 - 2(\lambda+1)}{2\omega + k(3\sigma+\rho) + 2r^2 + 2(\lambda+1)},$$

*provided* $2\omega + k(3\sigma+\rho) \neq -2r^2 - 2(\lambda+1)$.

**Corollary 5.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,1)$ be almost $\eta$-Einstein soliton on $M^n$. If $M^n$ is Ricci semisymmetric, then we observe the following situations:*

i) $M^n$ *is expanding if* $2\omega + k(3\sigma+\rho) - 2r^2 > 2$,
ii) $M^n$ *is steady if* $2\omega + k(3\sigma+\rho) - 2r^2 = 2$,
iii) $M^n$ *is shrinking if* $2\omega + k(3\sigma+\rho) - 2r^2 < 2$.

For an $n$-dimensional semi-Riemann manifold $M$, the concircular curvature tensor is defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{26}$$

For an $n$-dimensional perfect fluid spacetime with torse-forming vector field, if we choose $Z=\xi$ in (26), we can write

$$C(X,Y)\xi = \frac{n(n-1)+r}{n(n-1)}\left[\eta(Y)X - \eta(X)Y\right], \tag{27}$$

and similarly if we take the inner product of both sides of (26) by $\xi$, we get

$$\eta(C(X,Y)Z) = \frac{n(n-1)+r}{n(n-1)} g(\eta(X)Y - \eta(Y)X, Z). \tag{28}$$

**Theorem 6.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. If $M^n$ is concircular Ricci pseudosymmetric, then*

$$\mathcal{H}_2 = \frac{[n(n-1)+r]\left[-\alpha(2\omega + k(3\sigma+\rho)) + (2\lambda - \beta r + 2)\right]}{n(n-1)\left[-\alpha(2\omega + k(3\sigma+\rho)) - (2\lambda - \beta r + 2)\right]},$$

*provided* $-(2\lambda - \beta r + 2) \neq -\alpha(2\omega + k(3\sigma+\rho))$.

*Proof.* Let us assume that perfect fluid spacetime with torse-forming vector field $M^n$ is concircular Ricci pseudosymmetric and $(g,\xi,\lambda,\mu,\alpha,\beta)$ is $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. That means,

$$(C(X,Y)\cdot S)(U,V) = \mathcal{H}_2 Q(g,S)(U,V;X,Y)$$

for all $X,Y,U,V \in \Gamma(TM^n)$. From the last equation, we can easily write

$$\begin{aligned} &S(C(X,Y)U,V) + S(U,C(X,Y)V) \\ &= \mathcal{H}_2\left\{S((X\wedge_g Y)U,V) + S(U,(X\wedge_g Y)V)\right\}. \end{aligned} \tag{29}$$

If we choose $V=\xi$ in (29), and use (18), (27), then we get

$$\begin{aligned} &\mathcal{F}_1\eta\left(C\left(X,Y\right)U\right)+\mathcal{F}_2S\left(U,\eta\left(Y\right)X-\eta\left(X\right)Y\right)\\ &=\mathcal{H}_2\left\{\mathcal{F}_1g\left(\eta\left(X\right)Y-\eta\left(Y\right)X,U\right)+S\left(U,\eta\left(X\right)Y-\eta\left(Y\right)X,U\right)\right\}, \end{aligned} \tag{30}$$

where $\mathcal{F}_1=\frac{-1}{2}\left[2\omega+k\left(3\sigma+\rho\right)\right]$ and $\mathcal{F}_2=\frac{n(n-1)+r}{n(n-1)}$. Substituting (28) into (30), we have

$$\begin{aligned} &\mathcal{F}_1\mathcal{F}_2g\left(\eta\left(X\right)Y-\eta\left(Y\right)X,U\right)+\mathcal{F}_2S\left(\eta\left(Y\right)X-\eta\left(X\right)Y,U\right)\\ &=\mathcal{H}_2\left\{\mathcal{F}_1g\left(\eta\left(X\right)Y-\eta\left(Y\right)X,U\right)+S\left(\eta\left(X\right)Y-\eta\left(Y\right)X,U\right)\right\}. \end{aligned} \tag{31}$$

If we use (19) in (31), we can write

$$\left[\mathcal{F}_1\mathcal{F}_2+\frac{\mathcal{F}_2\left(2\lambda-\beta r+2\right)}{2\alpha}-\mathcal{H}_2\left(\mathcal{F}_1-\frac{2\lambda-\beta r+2}{2\alpha}\right)\right]g\left(\eta\left(X\right)Y-\eta\left(Y\right)X,U\right)=0.$$

This completes the proof of theorem. □

**Corollary 6.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. If $M^n$ is concircular Ricci semisymmetric, then $M^n$ has either constant scalar curvature $r=-n\left(n-1\right)$ or we observe the following situations:*

i) *$M^n$ is expanding if $\alpha\left[2\omega+k\left(3\sigma+\rho\right)\right]+\beta r>2$,*
ii) *$M^n$ is steady if $\alpha\left[2\omega+k\left(3\sigma+\rho\right)\right]+\beta r=2$,*
iii) *$M^n$ is shrinking if $\alpha\left[2\omega+k\left(3\sigma+\rho\right)\right]+\beta r<2$.*

**Theorem 7.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,0)$ be almost Ricci soliton on $M^n$. If $M^n$ is concircular Ricci pseudosymmetric, then $M^n$ has either constant scalar curvature $r=-n\left(n-1\right)$ or*

$$\mathcal{H}_2=-\frac{\left[n\left(n-1\right)+r\right]\left[2\left(\lambda+1\right)-\left(2\omega+k\left(3\sigma+\rho\right)\right)\right]}{n\left(n-1\right)\left[2\left(\lambda+1\right)+\left(2\omega+k\left(3\sigma+\rho\right)\right)\right]},$$

*provided $2\omega+k\left(3\sigma+\rho\right)\neq-2\left(\lambda+1\right)$.*

**Corollary 7.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,0)$ be almost Ricci soliton on $M^n$. If $M^n$ is concircular Ricci semisymmetric, then $M^n$ has either constant scalar curvature $r=-n\left(n-1\right)$ or we observe the following situations:*

i) *$M^n$ is expanding if $2\omega+k\left(3\sigma+\rho\right)>2$,*
ii) *$M^n$ is steady if $2\omega+k\left(3\sigma+\rho\right)=2$,*
iii) *$M^n$ is shrinking if $2\omega+k\left(3\sigma+\rho\right)<2$.*

**Theorem 8.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,0,1)$ be almost Yamabe soliton on $M^n$. If $M^n$ is concircular Ricci pseudosymmetric, then $M^n$ has either constant scalar curvature $r=-n\left(n-1\right)$ or*

$$\mathcal{H}_2=\frac{\left[n\left(n-1\right)+r\right]\left(2\lambda-r+2\right)}{n\left(n-1\right)\left(r-2\lambda-2\right)},$$

*provided $r\neq2\left(\lambda+1\right)$.*

**Corollary 8.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,0,1)$ be almost Yamabe soliton on $M^n$. If $M^n$ is concircular Ricci semisymmetric, then $M^n$ has either constant scalar curvature $r=-n(n-1)$ or we observe the following situations:*

i) *$M^n$ is expanding if $r>2$,*
ii) *$M^n$ is steady if $r=2$,*
iii) *$M^n$ is shrinking if $r<2$.*

**Theorem 9.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,1)$ be almost Einstein soliton on $M^n$. If $M^n$ is concircular Ricci pseudosymmetric, then $M^n$ has either constant scalar curvature $r=-n(n-1)$ or*

$$\mathcal{H}_2=-\frac{[n(n-1)+r][(2\lambda-r+2)-(2\omega+k(3\sigma+\rho))]}{n(n-1)[(2\lambda-r+2)+(2\omega+k(3\sigma+\rho))]},$$

*provided $2\omega+k(3\sigma+\rho)\neq r-2(\lambda+1)$.*

**Corollary 9.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,1)$ be almost Einstein soliton on $M^n$. If $M^n$ is concircular Ricci semisymmetric, then $M^n$ has either constant scalar curvature $r=-n(n-1)$ or we observe the following situations:*

i) *$M^n$ is expanding if $2\omega+k(3\sigma+\rho)+r>2$,*
ii) *$M^n$ is steady if $2\omega+k(3\sigma+\rho)+r=2$,*
iii) *$M^n$ is shrinking if $2\omega+k(3\sigma+\rho)+r<2$.*

**Theorem 10.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,1)$ be almost $\eta$-Einstein soliton on $M^n$. If $M^n$ is concircular Ricci pseudosymmetric, then we get*

$$\mathcal{H}_2=-\frac{[n(n-1)+r]\left[2\left(\lambda+r^2+1\right)-(2\omega+k(3\sigma+\rho))+\right]}{n(n-1)[(2\omega+k(3\sigma+\rho))+2(\lambda+r^2+1)]},$$

*provided $2\omega+k(3\sigma+\rho)\neq -2r^2-2(\lambda+1)$.*

**Corollary 10.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,1)$ be almost $\eta$-Einstein soliton on $M^n$. If $M^n$ is concircular Ricci semisymmetric, then $M^n$ has either constant scalar curvature $r=-n(n-1)$ or we observe the following situations:*

i) *$M^n$ is expanding if $2\omega+k(3\sigma+\rho)-2r^2>2$,*
ii) *$M^n$ is steady if $2\omega+k(3\sigma+\rho)-2r^2=2$,*
iii) *$M^n$ is shrinking if $2\omega+k(3\sigma+\rho)-2r^2<2$.*

For an $n$-dimensional semi-Riemann manifold $M$, the projective curvature tensor is defined as

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{n-1}[S(Y,Z)X-S(X,Z)Y]. \tag{32}$$

For an $n$-dimensional perfect fluid spacetime with torse-forming vector field, if we choose $Z=\xi$ in (32), we can write

$$P(X,Y)\xi=\left[\frac{2\omega+k(3\sigma+\rho)+2(n-1)}{2(n-1)}\right][\eta(Y)X-\eta(X)Y], \tag{33}$$

and similarly, if we take the inner product of both sides of (32) by $\xi$, we get

$$\eta(P(X,Y)Z)=\left[\frac{2\omega+k(3\sigma+\rho)+2(n-1)}{2(n-1)}\right]g(\eta(X)Y-\eta(Y)X). \tag{34}$$

**Theorem 11.** *Let $M^n$ be a perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. If $M^n$ is projective Ricci pseudosymmetric, then we have*

$$\mathcal{H}_3=\frac{\alpha(n-1)[2\omega+k(3\sigma+\rho)]^2-[2\omega+k(3\sigma+\rho)+2(n-1)](2\lambda-\beta r+2)}{2(n-1)[2\lambda-\beta r+2-\alpha(2\omega+k(3\sigma+\rho))]},$$

*provided* $2\lambda-\beta r+2\neq\alpha[2\omega+k(3\sigma+\rho)]$.

*Proof.* Let's assume that perfect fluid spacetime with torse-forming vector field $M^n$ is projective Ricci pseudosymmetric and $(g,\xi,\lambda,\mu,\alpha,\beta)$ is $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. That means,

$$(P(X,Y)\cdot S)(U,V)=\mathcal{H}_3Q(g,S)(U,V;X,Y)$$

for all $X,Y,U,V\in\Gamma(TM^n)$. From the last equation, we can easily write

$$\begin{aligned}&S(P(X,Y)U,V)+S(U,P(X,Y)V)\\&=\mathcal{H}_3\{S((X\wedge_g Y)U,V)+S(U,(X\wedge_g Y)V)\}.\end{aligned} \tag{35}$$

If we choose $V=\xi$ in (35), and use (18), (33), we get

$$\begin{aligned}&\mathcal{F}_1\eta(P(X,Y)U)+\mathcal{F}_2S(\eta(Y)X-\eta(X)Y)\\&=\mathcal{H}_3\{\mathcal{F}_1g(\eta(X)Y-\eta(Y)X,U)+S(\eta(X)Y-\eta(Y)X,U)\},\end{aligned} \tag{36}$$

where $\mathcal{F}_1=\frac{-1}{2}[2\omega+k(3\sigma+\rho)]$ and $\mathcal{F}_2=\frac{2\omega+k(3\sigma+\rho)+2(n-1)}{2(n-1)}$. If we use (34) in (36), we get

$$\begin{aligned}&\mathcal{F}_1^2g(\eta(X)Y-\eta(Y)X,U)+\mathcal{F}_2S(\eta(Y)X-\eta(X)Y)\\&=\mathcal{H}_3\{\mathcal{F}_1g(\eta(X)Y-\eta(Y)X,U)+S(\eta(X)Y-\eta(Y)X,U)\}.\end{aligned} \tag{37}$$

If we use (19) in (37), we can write

$$\left[\mathcal{F}_1^2-\frac{\mathcal{F}_2(2\lambda-\beta r+2)}{2\alpha}-\mathcal{H}_3\left(\mathcal{F}_1+\frac{2\lambda-\beta r+2}{2\alpha}\right)\right]g(\eta(X)Y-\eta(Y)X,U)=0.$$

This completes the proof of theorem. $\square$

**Corollary 11.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,\alpha,\beta)$ be $(\alpha,\beta)$-type almost $\eta$-Ricci–Yamabe soliton on $M^n$. If $M^n$ is projective Ricci semisymmetric, then we observe the following situations:*

i) *$M^n$ is expanding if* $\frac{\alpha(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)}+\beta r>2$,

ii) $M^n$ *is steady if* $\frac{\alpha(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)} + \beta r = 2$,

iii) $M^n$ *is shrinking if* $\frac{\alpha(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)} + \beta r < 2$.

**Theorem 12.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 1, 0)$ be almost Ricci soliton on $M^n$. If $M^n$ is projective Ricci pseudosymmetric, then we get*

$$\mathcal{H}_3 = \frac{(n-1)\left[2\omega + k\left(3\sigma+\rho\right)\right]^2 - \left[2\omega + k\left(3\sigma+\rho\right) + 2\left(n-1\right)\right]\left(2\lambda+2\right)}{2\left(n-1\right)\left[2\lambda + 2 - \left(2\omega + k\left(3\sigma+\rho\right)\right)\right]},$$

*provided* $2\omega + k\left(3\sigma+\rho\right) \neq 2\left(\lambda+1\right)$.

**Corollary 12.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 1, 0)$ be almost Ricci soliton on $M^n$. If $M^n$ is projective Ricci semisymmetric, then we observe the following situations:*

i) $M^n$ *is expanding if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)} > 2$,

ii) $M^n$ *is steady if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)} = 2$,

iii) $M^n$ *is shrinking if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)} < 2$.

**Theorem 13.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 0, 1)$ be almost Yamabe soliton on $M^n$. If $M^n$ is projective Ricci pseudosymmetric, then we get*

$$\mathcal{H}_3 = \frac{\left[2\omega + k\left(3\sigma+\rho\right) + 2\left(n-1\right)\right]\left(-2\lambda + r - 2\right)}{2\left(n-1\right)\left(2\lambda - r + 2\right)},$$

*provided* $2\lambda \neq r - 2$.

**Corollary 13.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 0, 1)$ be almost Yamabe soliton on $M^n$. If $M^n$ is projective Ricci semisymmetric, then we observe the following situations:*

i) $M^n$ *is expanding if* $r > 2$,

ii) $M^n$ *is steady if* $r = 2$,

iii) $M^n$ *is shrinking if* $r < 2$.

**Theorem 14.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 1, 1)$ be almost Einstein soliton on $M^n$. If $M^n$ is projective Ricci pseudosymmetric, then we get*

$$\mathcal{H}_3 = \frac{(n-1)\left[2\omega + k\left(3\sigma+\rho\right)\right]^2 - \left[2\omega + k\left(3\sigma+\rho\right) + 2\left(n-1\right)\right]\left(2\lambda - r + 2\right)}{2\left(n-1\right)\left[2\lambda - r + 2 - \left(2\omega + k\left(3\sigma+\rho\right)\right)\right]},$$

*provided* $2\omega + k\left(3\sigma+\rho\right) \neq r - 2\left(\lambda+1\right)$.

**Corollary 14.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g, \xi, \lambda, \mu, 1, 1)$ be almost Einstein soliton on $M^n$. If $M^n$ is projective Ricci semisymmetric, then we observe the following situations:*

i) $M^n$ *is expanding if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)} + r > 2$,

ii) $M^n$ *is steady if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)}+r=2$,

iii) $M^n$ *is shrinking if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)}+r<2$.

**Theorem 15.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,1)$ be almost $\eta$-Einstein soliton on $M^n$. If $M^n$ is projective Ricci pseudosymmetric, then we get*

$$\mathcal{H}_3=\frac{(n-1)\left[2\omega+k\left(3\sigma+\rho\right)\right]^2-\left[2\omega+k\left(3\sigma+\rho\right)+2\left(n-1\right)\right]\left(2\lambda+2r^2+2\right)}{2\left(n-1\right)\left[2\lambda+2r^2+2-\left(2\omega+k\left(3\sigma+\rho\right)\right)\right]},$$

*provided* $2\omega+k\left(3\sigma+\rho\right)\neq-2\left(r^2+\lambda+1\right)$.

**Corollary 15.** *Let $M^n$ be an $n$-dimensional perfect fluid spacetime with torse-forming vector field and $(g,\xi,\lambda,\mu,1,1)$ be almost $\eta$-Einstein soliton on $M^n$. If $M^n$ is a projective Ricci semisymmetric, then we observe the following situations:*

i) $M^n$ *is expanding if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)}-2r^2>2$,

ii) $M^n$ *is steady if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)}-2r^2=2$,

iii) $M^n$ *is shrinking if* $\frac{(n-1)}{4\omega+2k(3\sigma+\rho)+2(n-1)}-2r^2<2$.

**Example 1.** Let $M=\left\{(x,y,z,t)\in\mathbf{R}^4\,\middle|\,t\neq0\right\}$, where $(x,y,z,t)$ are the standard coordinates of $\mathbf{R}^4$. Consider a Lorentzian metric $g$ on $M$ is given by

$$\mathrm{d}s^2=\mathrm{e}^{2t}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)-\mathrm{d}t^2. \tag{38}$$

The non-vanishing components of the Christoffel symbol, the curvature tensor and Ricci tensor are

$$\begin{aligned}
&\Gamma^4_{11}=\Gamma^4_{22}=\Gamma^4_{33}=\mathrm{e}^{2t},\\
&\Gamma^1_{14}=\Gamma^2_{24}=\Gamma^3_{34}=1,\\
&R_{1411}=R_{2442}=R_{3443}=\mathrm{e}^{2t},\\
&R_{1221}=R_{1331}=R_{2332}=-\mathrm{e}^{4t},\\
&S_{11}=S_{22}=S_{33}=-3\mathrm{e}^{2t},\quad S_{44}=3.
\end{aligned}$$

Therefore, the scalar curvature of the manifold is $r=-12$. Thus, $\left(M^4,g\right)$ is a perfect fluid spacetime whose isotropic pressure and energy density are

$$\rho=\frac{1}{k}\left(\lambda+3\right)\quad\text{and}\quad\sigma=-\frac{1}{k}\left(\lambda+3\right),$$

respectively.

Let $\eta$ be the 1-form defined by $\eta\left(Z\right)=-g\left(Z,t\right)$ for any $Z\in\chi\left(M\right)$. Take $\xi=t$. Replacing $V=\xi$ in (6) and using

$$\left(\mathcal{L}_\xi g\right)\left(X,Y\right)=2\left[g\left(X,Y\right)+\eta\left(X\right)\eta\left(Y\right)\right],$$

we see that the soliton equation becomes

$$2\left[g_{ii}+\eta_i\otimes\eta_i\right]+2\alpha S_{ii}+\left(2\lambda-\beta r\right)g_{ii}+2\mu\eta_i\otimes\eta_i=0 \tag{39}$$

for all $i \in \{1, 2, 3, 4\}$ . Thus the data $(\xi, g, \lambda, \mu, \alpha, \beta)$ is $\eta$-Ricci–Yamabe soliton on $(M^4, g)$, where

$$\lambda = 3\alpha - 4\beta - 1 \quad \text{and} \quad \mu = -1,$$

which is expanding if $3\alpha - 4\beta > 1$, shrinking if $3\alpha - 4\beta < 1$, and steady if $3\alpha - 4\beta = 1$ [**16**].

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