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QAUSI CONFORMAL CURVATURE TENSOR ON $N(k)$-CONTACT METRIC MANIFOLDS

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ABSTRACT. The purpose of this paper is to study $N(k)$-contact metric manifolds endowed with a qausi-conformal curvature tensor. Here we consider quasi-conformally flat, Einstein semi-symmetric quasi-conformally flat, quasi-conformally semi-symmetric, and globally ϕ-quasiconformally symmetric $N(k)$-contact metric manifolds.

1. Introduction

In 1968, Yano and Sawaki [**8**] introduced the quasi-conformal curvature tensor given by

$$\begin{aligned}\widetilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY] - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1}$$

where a and b are constants, and R, S, Q, and r are the Riemannian curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and scalar curvature of the manifold, respectively. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1) takes the form

$$\begin{aligned}\widetilde{C}(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad r - g(X,Z)QY] - \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ &= C(X,Y)Z,\end{aligned} \tag{2}$$

where C is the conformal curvature tensor [**9**].
From (1), we obtain

$$\begin{aligned}(\nabla_W\widetilde{C})(X,Y)Z = a(\nabla_W R)(X,Y)Z + b[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y \\ + g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)] \\ - \frac{dr(W)}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3}$$

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In [**11**], De and Matsuyama studied a quasi-conformally flat Riemannian manifold satisfying a certain condition on the Ricci tensor. Again Cihan Ozgar and De [**10**] studied quasi-conformal curvature tensor on Kenmotsu manifold and showed that a Kenmotsu manifold is quasi-conformally flat or quasi-conformally semi-symmetric if and only if it is locally isometric to the hyperbolic space. Recently Ali Akbar and Avijit Sarkar [**16**] studied quasi-conformally flat trans-Sasakian manifold. The geometry of quasi-conformal curvature tensor in a Riemannian manifold with different structures was studied by several authors, viz., [**12, 13, 15, 19, 20**].

The present paper is organized as follows: In Section 2, we give the definitions and some preliminary results that will be needed thereafter. In Section 3, we discuss quasi-conformally flat $N(k)$-contact metric manifold and it is shown that the manifold is η-Einstein. Section 4 is devoted to the study of Einstein semi-symmetric quasi-conformally flat $N(k)$-contact metric manifold and obtains that the scalar curvature is constant in that case. In section 5, we consider quasi-conformally semi-symmetric $N(k)$-contact metric manifold and prove that the manifold is η-Einstein provided $a \neq b$. Finally, in the last section, we show that an $N(k)$-contact metric manifold is globally ϕ-quasiconformally symmetric if and only if it is globally ϕ-symmetric provided $k \neq 0$ and r is constant.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be a contact manifold if it carries a global differentiable 1-form η which satisfies the condition $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Also a Contact manifold admits an almost Contact structure (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a characteristic vector field and η is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{4}$$

An almost Contact structure is said to be normal if the induced almost complex structure J on the product manifold $M \times R$ defined by

$$J\Big(X, \lambda \frac{\mathrm{d}}{\mathrm{d}t}\Big) = \Big(\phi X - \lambda\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\Big),$$

is integrable, where X is tangent to M, t is the coordinate of R and λ a smooth function on $M \times R$. The condition of almost contact metric structure being normal is equivalent to vanishing of the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Let g be the compatible Riemannian metric with almost Contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \xi) = \eta(X) \tag{5}$$

for all vector fields $X, Y \in \chi(M)$. A manifold M together with this almost Contact metric structure is said to be almost Contact metric manifold denoted by $M(\phi, \xi, \eta, g)$. An almost Contact metric structure reduces to a contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds

$$\nabla_X \xi = -\phi X - \phi h X. \tag{6}$$

Blair, Koufogiorgos and Papantoniou [**17**] introduced the (k, μ)-nullity distribution of a Contact metric manifold M that is defined by

$$N(k, \mu)\colon p \to N_p(k, \mu)$$
$$N_p(k, \mu) = \{U \in T_pM \mid R(X, Y)U = (kI + \mu h)g(Y, U)X - g(X, U)Y\}$$

for all $X, Y \in TM$, where $(k, \mu) \in R^2$. A Contact metric manifold with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. If $\mu = 0$, the (k, μ)-nullity distribution reduces to k-nullity distribution. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [**5**]

$$N(k)\colon p \to N_p(k) = \{U \in T_pM \mid R(X, Y)U = k[g(Y, U)X - g(X, U)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold [**18**]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [**3**]. In an $N(k)$-contact metric manifold, the following relations hold:

$$h^2 = (k-1)\phi^2, \tag{7}$$
$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{8}$$
$$S(X, Y) = [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y), \tag{9}$$
$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX, Y), \tag{10}$$
$$S(X, \xi) = 2nk\eta(X), \tag{11}$$
$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y). \tag{12}$$

Definition 2.1. A $(2n+1)$-dimensional $N(k)$-contact metric manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for any vector fields X and Y, where a and b are constants. If $b = 0$, then the manifold M is an Einstein manifold.

Definition 2.2. The Einstein Tensor denoted by E is defined by

$$E(X, Y) = S(X, Y) - \frac{r}{2}g(X, Y), \tag{13}$$

where S is a Ricci tensor and r is the scalar curvature.

3. Quasi-conformally flat $N(k)$-contact metric manifold

Definition 3.1. A $(2n+1)$-dimensional $N(k)$-contact metric manifold is said to be quasi-conformally flat if the quasi conformal curvature tensor $\widetilde{C} = 0$.

Now, we consider an $N(k)$-contact metric manifold which is quasi-conformally flat. Then from (1), we get

$$\begin{aligned} R(X,Y)Z = & -\frac{b}{a}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ & + \frac{r}{a(2n+1)}\Big(\frac{a}{2n} + 2b\Big)\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{14}$$

Taking inner product on both sides of above equation with respect to W, we get

$$\begin{aligned} R(X,Y,Z,W) = & -\frac{b}{a}\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ & + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\} \\ & + \frac{r}{a(2n+1)}\Big(\frac{a}{2n} + 2b\Big)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \end{aligned} \tag{15}$$

where $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $S(X,Y) = g(QX,Y)$.
Now using (8) in the above, we get

$$R(\xi,X,Y,\xi) = kg(\phi X,\phi Y). \tag{16}$$

Putting $X = W = \xi$ in (15), we obtain

$$\begin{aligned} R(\xi,Y,Z,\xi) = & -\frac{b}{a}\{S(Y,Z) - S(\xi,Z)\eta(Y) + g(Y,Z)S(\xi,\xi) \\ & - \eta(Z)S(Y,\xi)\} + \frac{r}{a(2n+1)}\left(\frac{a}{2n} + 2b\right)\{g(Y,Z) - \eta(Z)\eta(Y)\}. \end{aligned} \tag{17}$$

By virtue of (5), (11) and (16), equation (17) becomes

$$\begin{aligned} S(Y,Z) = & \Big[\frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) - 2nk - \frac{ka}{b}\Big]g(Y,Z) \\ & + \Big[4nk + \frac{ka}{b} - \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right)\Big]\eta(Y)\eta(Z). \end{aligned} \tag{18}$$

$$S(Y,Z) = Ag(Y,Z) + B\eta(Y)\eta(Z), \tag{19}$$

where

$$A = \Big[\frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) - 2nk - \frac{ka}{b}\Big], \tag{20}$$

$$B = \Big[4nk + \frac{ka}{b} - \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right)\Big]. \tag{21}$$

Hence we state the following:

Theorem 3.2. *A $(2n+1)$ dimensional quasi-conformally flat $N(k)$-contact metric manifold is an η-Einstein manifold.*

4. Einstein semi-symmetric quasi-conformally flat $N(k)$-contact metric manifold

Definition 4.1. A $(2n+1)$-dimensional quasi-conformally flat $N(k)$-contact metric manifold is called Einstein Semi-symmetric if

$$R(X,Y).E(Z,W) = 0 \tag{22}$$

for any vector fields X, Y, Z and W.

Using $g(QX,Y) = S(X,Y)$ in (19), we get

$$QX = AX + B\eta(X)\xi. \tag{23}$$

Substituting (18) and (23) in (14), we have

$$\begin{aligned} R(X,Y)Z = M\{g(Y,Z)X - g(X,Z)Y\} + N\{g(Y,Z)\eta(X)\xi \\ - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{24}$$

where

$$M = \left\{\frac{4nbk}{a} + 2k - \frac{r}{a(2n+1)}\left(\frac{a}{2n} + 2b\right)\right\},$$
$$N = \left\{\frac{r}{a(2n+1)}\left(\frac{a}{2n} + 2b\right) - \frac{4nbk}{a} - k\right\}.$$

Now, we consider the quasi-conformally flat $N(k)$-contact metric manifold which is Einstein Semi-symmetric, i.e.,

$$R \cdot E = 0. \tag{25}$$

The above equation reduces to

$$E(R(X,Y)Z,U) + E(Z,R(X,Y)U) = 0. \tag{26}$$

In view of (13), equation (26) gives

$$\begin{aligned} S(R(X,Y)Z,U) - \frac{r}{2}g(R(X,Y)Z,U) + S(Z,R(X,Y)U) \\ - \frac{r}{2}g(Z,R(X,Y)U) = 0. \end{aligned} \tag{27}$$

Using (19), we get from the above equations

$$\begin{aligned} \left(A - \frac{r}{2}\right) g(R(X,Y)Z,U) + \left(A - \frac{r}{2}\right) g(Z,R(X,Y)U) \\ +B\eta(R(X,Y)Z)\eta(U) + B\eta(R(X,Y)U)\eta(Z) = 0. \end{aligned} \tag{28}$$

Putting $Z = \xi$ in (28), we get

$$\begin{aligned} \left(A - \frac{r}{2}\right) g(R(X,Y)\xi,U) + \left(A - \frac{r}{2}\right) g(\xi,R(X,Y)U) \\ +B\eta(R(X,Y)\xi)\eta(U) + B\eta(R(X,Y)U) = 0. \end{aligned} \tag{29}$$

Using (24) in (29), we have

$$B\{g(X,U)\eta(Y) - g(Y,U)\eta(X)\} = 0. \tag{30}$$

Putting $Y = \xi$ in (30), we get

$$B\{g(X,U) - \eta(U)\eta(X)\} = 0. \tag{31}$$

Again putting $U = QW$ in (31) and using (23), we get

$$B\{S(X,W) - (A+B)\eta(W)\eta(X)\} = 0. \tag{32}$$

This implies that, either $B = 0$, or $S(X,W) - (A+B)\eta(W)\eta(X) = 0$.
Now if $B = 0$, then from (21), we get that r is constant.
Again if $S(X,W) - (A+B)\eta(W)\eta(X) = 0$, then we have

$$S(X,W) = (A+B)\eta(W)\eta(X). \tag{33}$$

Putting $X = W = e_i$ in (33), where $\{e_i\}$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$r = 2nk.$$

Thus we have the following:

Theorem 4.2. *If a $(2n+1)$-dimensional quasi-conformally flat $N(k)$-contact metric manifold is Einstein Semi-symmetric, then the scalar curvature is constant.*

5. Quasi-conformally semi-symmetric $N(k)$-contact metric manifold

Let us consider a Quasi-conformally semi-symmetric $N(k)$-contact metric manifold. Then the condition

$$R(X,Y).\widetilde{C} = 0 \tag{34}$$

holds for every vector fields X, Y. From the above equation, we have

$$\begin{aligned} 0 &= (R(X,Y)\cdot\widetilde{C})(U,V)W = R(X,Y)\widetilde{C}(U,V)W \\ &\quad - \widetilde{C}(R(X,Y)U,V)W - \widetilde{C}(U,R(X,Y)V)W - \widetilde{C}(U,V)R(X,Y)W. \end{aligned} \tag{35}$$

For $X = \xi$, the above equation gives

$$\begin{aligned} 0 &= R(\xi,Y)\widetilde{C}(U,V)W - \widetilde{C}(R(\xi,Y)U,V)W \\ &\quad - \widetilde{C}(U,R(\xi,Y)V)W - \widetilde{C}(U,V)R(\xi,Y)W. \end{aligned} \tag{36}$$

In view of (8), the equation (36) can be written as

$$\begin{aligned} 0 &= k[\widetilde{C}(U,V,W,Y)\xi - \eta(\widetilde{C}(U,V)W)Y + \eta(U)\widetilde{C}(Y,V)W \\ &\quad - g(Y,W)\widetilde{C}(U,V)\xi - g(Y,U)\widetilde{C}(\xi,V)W + \eta(V)\widetilde{C}(U,Y)W \\ &\quad - g(Y,V)\widetilde{C}(U,\xi)W + \eta(W)\widetilde{C}(U,V)Y], \end{aligned} \tag{37}$$

where $\widetilde{C}(U,V,W,Y) = g(\widetilde{C}(U,V)W,Y)$.

Taking inner product of the above equation with ξ, we get

$$
\begin{aligned}
0 = k[&\widetilde{C}(U,V,W,Y) - \eta(\widetilde{C}(U,V)W)\eta(Y) + \eta(U)\eta(\widetilde{C}(Y,V)W) \\
&- g(Y,U)\eta(\widetilde{C}(\xi,V)W) + \eta(V)\eta(\widetilde{C}(U,Y)W) - g(Y,V)\eta(\widetilde{C}(U,\xi)W) \\
&+ \eta(W)\eta(\widetilde{C}(U,V)Y) - g(Y,W)\eta(\widetilde{C}(U,V)\xi)].
\end{aligned}
\tag{38}
$$

Putting $Y = U$ in (38), we obtain

$$
\begin{aligned}
0 = k[&\widetilde{C}(U,V,W,U) - g(U,U)\eta(\widetilde{C}(\xi,V)W) \\
&- g(U,V)\eta(\widetilde{C}(U,\xi)W) + \eta(W)\eta(\widetilde{C}(U,V)U)].
\end{aligned}
\tag{39}
$$

Let $\{e_i\}$, $1 \leq i \leq 2n+1$, be an orthonormal basis of the tangent space at any point. Then by virtue of (1), (8) and (11), the above equation reduces to

$$
\begin{aligned}
S(V,W) = &\Big\{\frac{2n(ak+2nkb) - rb}{a-b}\Big\}g(V,W) \\
&+ \Big\{\frac{rb - (2n+1)2nkb}{a-b}\Big\}\eta(V)\eta(W).
\end{aligned}
$$

Hence we can state the following result:

Theorem 5.1. *If M is a $(2n+1)$-dimensional quasi-conformally semi-symmetric $N(k)$-contact metric manifold, then the manifold is η-Einstein provided $a \neq b$.*

6. Globally ϕ-quasiconformally symmetric $N(k)$-contact metric manifold

Definition 6.1. An $N(k)$-contact metric manifold is said to be globally ϕ-quasiconformally symmetric if

$$
\phi^2(\nabla_W \widetilde{C})(X,Y)Z = 0, \tag{40}
$$

for any $X, Y, Z \in \chi(M)$.

If X, Y and Z are horizontal vector fields, then the manifold is said to be locally ϕ-quasiconformally symmetric.

Here we consider a $(2n+1)$-dimensional $N(k)$-contact metric manifold which is globally ϕ-quasiconformally symmetric. Then using (4) in (40), we obtain

$$
-(\nabla_W \widetilde{C})(X,Y)Z + \eta((\nabla_W \widetilde{C})(X,Y)Z) = 0. \tag{41}
$$

By virtue of (3) in equation (41) and taking inner product with U, we get

$$\begin{aligned}
&-ag((\nabla_W R)(X,Y)Z,U) - b\Big[(\nabla_W S)(Y,Z)g(X,U)\\
&-(\nabla_W S)(X,Z)g(Y,U) + g(Y,Z)g((\nabla_W Q)X,U)\\
&-g(X,Z)g((\nabla_W Q)Y,U)] + a\eta((\nabla_W R)(X,Y)Z)\eta(U)\\
&+\frac{dr(W)}{2n+1}\Big[\frac{a}{2n}+2b\Big][g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\Big]\\
&+b[(\nabla_W S)(Y,Z)\eta(X)\eta(U) - (\nabla_W S)(X,Z)\eta(Y)\eta(U)\\
&+g(Y,Z)\eta((\nabla_W Q)(X))\eta(U) - g(X,Z)\eta((\nabla_W Q)(Y))\eta(U)]\\
&-\frac{dr(W)}{2n+1}\Big[\frac{a}{2n}+2b\Big][g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U)] = 0.
\end{aligned} \tag{42}$$

Putting $X = U = e_i$ in (42), where $\{e_i\}$, $1 \le i \le 2n+1$, is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\begin{aligned}
&-(a+2nb-b)(\nabla_W S)(Y,Z) - \Big[b\sum_{i=1}^{2n+1} g((\nabla_W Q)e_i,e_i)\\
&-\frac{2n}{2n+1}dr(W)\Big(\frac{a}{2n}+2b\Big) - b\sum_{i=1}^{2n+1}\eta((\nabla_W Q)e_i)\eta(e_i)\\
&+\frac{dr(W)}{2n+1}\Big(\frac{a}{2n}+2b\Big)\Big]g(Y,Z) + bg((\nabla_W Q)Y,Z)\\
&+a\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)Z\eta(e_i) - b(\nabla_W S)(\xi,Z)\eta(Y)\\
&-b\eta((\nabla_W Q)Y)\eta(Z) + \frac{dr(W)}{2n+1}\Big(\frac{a}{2n}+2b\Big)\eta(Y)\eta(Z) = 0.
\end{aligned} \tag{43}$$

Again putting $Z = \xi$ in (43) and by virtue of (4), we obtain

$$\begin{aligned}
&-(a+2nb-b)(\nabla_W S)(Y,\xi) - \Big[bdr(W) - \frac{2n}{2n+1}dr(W)\Big(\frac{a}{2n}+2b\Big)\\
&-b\eta((\nabla_W Q)\xi) + \frac{dr(W)}{2n+1}\Big(\frac{a}{2n}+2b\Big)\Big]\eta(Y)\\
&+ag((\nabla_W R)(\xi,Y)\xi,\xi) - b(\nabla_W S)(\xi,\xi)\eta(Y) + \frac{dr(W)}{2n+1}\Big(\frac{a}{2n}+2b\Big)\eta(Y) = 0.
\end{aligned} \tag{44}$$

Using (4), (5), (8), (11), and $h\xi = 0$, we have the following relations:

$$\eta((\nabla_W Q)\xi) = 0, \tag{45}$$

$$g((\nabla_W R)(\xi,Y)\xi,\xi) = 0, \tag{46}$$

$$\nabla_W S)(\xi,\xi) = 0. \tag{47}$$

In view of (45), (46) and (47), equation (44) gives

$$(\nabla_W S)(Y,\xi) = \frac{1}{2n+1} dr(W)\eta(Y). \tag{48}$$

Putting $Y = \xi$ in (48), we get $dr(W) = 0$. This implies r is constant.
And from (48), we have

$$(\nabla_W S)(Y,\xi) = 0$$

i.e.,

$$\nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi) = 0. \tag{49}$$

Substituting (6) and (11) in (49), we get

$$2nk(\nabla_W \eta)(Y) + S(Y,\phi W) + S(Y,\phi hW) = 0. \tag{50}$$

Using (12) in (50), we obtain

$$2nkg(\phi Y,W) + S(Y,\phi W) + 2nkg(\phi Y,hW) + S(Y,\phi hW) = 0. \tag{51}$$

Replacing W by hW in (51) and using (7), we have

$$2nk^2 g(\phi Y,W) + kS(Y,\phi W) = 0. \tag{52}$$

Again replacing W by ϕW in (52) and using (4), (5) and (6), we get

$$k[2nkg(\phi Y,W) - S(Y,W)] = 0. \tag{53}$$

Therefore, (53) gives either $k = 0$ or

$$S(Y,W) = 2nkg(Y,W). \tag{54}$$

Hence we state the following:

Theorem 6.2. *A globally ϕ-quasiconformally symmetric $N(k)$-contact metric manifold is an Einstein manifold provided $k \neq 0$.*

Again from (54), we have $QX = 2nkX$. Then from (1), we obtain

$$\begin{aligned} \widetilde{C}(X,Y)Z = aR(X,Y)Z + \Big[4nbk - \frac{r}{2n+1}\Big(\frac{a}{2n} + 2b\Big)\Big][g(Y,Z)X \\ - g(X,Z)Y], \end{aligned} \tag{55}$$

from which we have

$$(\nabla_W \widetilde{C})(X,Y)Z = a(\nabla_W R)(X,Y)Z. \tag{56}$$

Therefore

$$\phi^2(\nabla_W \widetilde{C})(X,Y)Z = a\phi^2(\nabla_W R)(X,Y)Z. \tag{57}$$

Hence we state the following.

Theorem 6.3. *An $N(k)$-contact metric manifold is globally ϕ-quasiconformally symmetric if and only if it is globally ϕ-symmetric provided $k \neq 0$ and r is constant.*

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