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ON THE EXISTENCE AND UNIQUENESS OF A SOLUTION TO THE BOUNDARY VALUE PROBLEM FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS

N. A. GASILOV

Abstract. In this study, we investigate the Boundary Value Problem (BVP) for second order non-homogeneous linear differential equation with Dirichlet conditions. We derive a novel sufficient condition for the existence and uniqueness of a solution. The condition is formulated in terms of input parameters (coefficient functions and the length l of the interval, where the BVP is considered), not in secondary terms as Lipschitz coefficients. We compare the obtained sufficient condition with those for non-linear BVPs and demonstrate that it covers a significantly wider class of BVPs.

1. Introduction

1.1. Formulation of the problem

Consider the following BVP for a second order non-homogeneous linear differential equation (LDE) with Dirichlet boundary conditions:

$$
\left\{
\begin{array}{l}
y'' + p(x)\, y' + q(x)\, y = r(x), \\
y(0) = A, \\
y(l) = B,
\end{array}
\right.
\tag{1}
$$

where $l > 0$, A, B are given real numbers. Here and throughout the article, we assume that $p(x)$, $q(x)$, $r\,(x)$ are real functions continuous on $[0, l]$. Below we will often refer to l, A, B, $p(x)$, $q\,(x)$ and $r(x)$ as input parameters.

We will investigate the existence and uniqueness issue for BVP (1). The issue is as follows. It is well known that for an Initial Value Problem (IVP), when the given functions p, q and r are continuous, the solution exists and is unique. For a BVP, however, the situation is uncertain. Depending on the input parameters, three cases are possible:

1. There exists precisely one solution.
2. There exist infinitely many solutions.
3. There exists no solution.

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Currently, there is no decision mechanism that determines, based on the input parameters, which case will occur for a problem under consideration. In this study, we will attempt to establish a sufficient condition of existence and uniqueness for BVP (1). Namely, we will derive under what conditions on the right-hand side and coefficient functions, boundary values and l we can state that the solution exists and is unique. Generally, the issue is treated as a special case of non-linear BVPs.

1.2. Existence and uniqueness results for non-linear BVPs

In this subsection, we give a brief summary of results for non-linear BVPs.

Let us consider the two-point boundary value problem

$$\begin{cases} y''(x) + f\left(x, y(x), y'(x)\right) = 0, \\ y\left(a\right) = A, \\ y\left(b\right) = B. \end{cases} \tag{2}$$

First we provide a widely used result on the issue [**5**].

Theorem 1.1. *Let the function $f(x,y,z)$ in (2) be continuous on the domain $D = \left\{(x,y,z) \;\middle|\; a \le x \le b,\ y^2+z^2 < \infty\right\}$ and satisfy there a uniform Lipschitz condition with respect to y and z:*

$$\left|f\left(x,y_1,z_1\right) - f\left(x,y_2,z_2\right)\right| \le K\left|y_1-y_2\right| + L\left|z_1-z_2\right|. \tag{3}$$

Also, let the function $f(x,y,z)$ have continuous partial derivatives on D which satisfy, for some positive constant M,

$$\frac{\partial f}{\partial y} < 0, \qquad \left|\frac{\partial f}{\partial z}\right| \le M.$$

Then the BVP (2) has a unique solution.

A very important special case of Theorem 1.1 occurs for second-order linear BVPs.

Corollary 1.2. *Let the functions $p(x)$, $q\left(x\right)$ and $r(x)$ be continuous on $[a,b]$ with $q(x) < 0$, $a \le x \le b$. Then the BVP*

$$\begin{cases} y'' + p(x)\,y' + q(x)\,y = r\left(x\right), \quad a < x < b, \\ y\left(a\right) = A, \\ y\left(b\right) = B \end{cases} \tag{4}$$

has a unique solution for each A, B.

The above theorem covers a narrow class of BVPs. Different approaches are applied to get better results. One of them deals with the sequence of functions defined by Picard's iteration. Picard [**7, 8**] showed that for the class of functions $f(x,y,z)$ which are continuous and satisfy the uniform Lipschitz condition (3), his iteration procedure converges (to the solution) whenever the length $l = b-a$ of the interval $[a,b]$ is small enough. (We notice that the expression "whenever the length of the interval $[a, b]$ is small enough" can be commented also as "whenever the function $f(x, y, z)$ changes slowly enough".) In this way, he established that

$$\frac{1}{8}Kl^2 + \frac{1}{2}Ll < 1 \tag{5}$$

suffices for the existence and uniqueness of a solution to (2). After that, the question of a maximum bound for l became a subject of continuing interest. Lettenmeyer [**6**] improved the above estimate to

$$\frac{1}{\pi^2}Kl^2 + \frac{4}{\pi^2}Ll < 1. \tag{6}$$

Further improvement over (6) was done by Coles and Sherman [**2**]:

$$\begin{aligned} \frac{1}{12}Kl^2 + \frac{1}{3}Ll < 1 \quad &\text{for } K \le L^2, \\ \frac{3-\sqrt{3}}{12}Kl^2 + \frac{\sqrt{3}}{6}Ll < 1 \quad &\text{for } K > L^2. \end{aligned} \tag{7}$$

The approach based on Picard's iteration convergence has some restrictions, which were expressed by Bailey, Shampine and Waltman [**1**] as follows: "Although this convergence question has been investigated by a number of people over many years (see [**2, 6, 7, 8, 10**], for example), the maximum interval for which Picard's iteration procedure converges is still not known. And even if it were, that fact alone would not necessarily tell us anything about the maximum interval for which the boundary value problem has a unique solution, other than that it provides a lower bound. Thus Picard's method, though extremely useful for a wide class of problems, does have the one serious limitation of being applicable to only those problems for which the iteration procedure happens to converge".

For linear BVP (1), besides the above results, some independent results were also obtained. By using a Riccati equation, de la Valleé Poussin [**2, 10**] obtained the following estimate:

$$l < 2\int_0^\infty \frac{\mathrm{d}x}{x^2 + Lx + K}, \tag{8}$$

where $K \ge |q(x)|$ and $L \ge |p(x)|$. This estimate for the uniqueness interval is better than (5), (6) and (7).

In particular, for constant coefficient equations, the following estimates can be obtained from (8) by taking $K = |q|$, $L = |p|$, and using the notation $\Delta = L^2 - 4K$:

$$\begin{aligned} &l < \frac{2}{\sqrt{\Delta}}\ln\left(1 + \frac{2\sqrt{\Delta}}{L - \sqrt{\Delta}}\right) && \text{if } \Delta > 0; \\ &l < \frac{4}{L} && \text{if } \Delta = 0; \\ &l < \frac{2}{\sqrt{|\Delta|}}\left(\pi - 2\arctan\frac{L}{\sqrt{|\Delta|}}\right) && \text{if } \Delta < 0. \end{aligned} \tag{9}$$

1.3. Preliminaries on linear BVPs

In this subsection, we give some basic information on the existence and uniqueness issue for the considered BVP (1).

The IVP

$$\begin{cases} y'' + p(x)\, y' + q(x)\, y = r(x), \\ y(0) = 0, \\ y'(0) = 0 \end{cases} \tag{10}$$

has a unique solution. Denote it as $y_p(x)$. Then, the solution of (1) (if any) can be represented as

$$y = y_p(x) + y_c(x),$$

where $y_c(x)$ is the solution of the BVP

$$\begin{cases} y'' + p(x)\, y' + q(x)\, y = 0, \\ y(0) = \alpha, \\ y(l) = \beta \end{cases} \tag{11}$$

with $\alpha = A$ and $\beta = B - y_p(l)$.

Thus, the following conclusion implies.

Lemma 1.3. *The existence and uniqueness issue of BVP* (1) *for a non-homogeneous linear differential equation is the same as that of BVP* (11) *for the associated homogeneous equation (provided that all the involved functions are continuous).*

We can also interpret this fact in such a way that the existence and uniqueness of the solution of BVP (1) is determined by the coefficient functions, the boundary values and l, not by the right-hand side function $r(x)$. Based on this circumstance, we will further focus on the question of the existence and uniqueness of the solution of BVP (11) for a homogeneous differential equation.

Solutions to the IVPs

$$\begin{cases} y'' + p(x)\, y' + q(x)\, y = 0, \\ y(0) = 1, \\ y'(0) = 0 \end{cases} \tag{12}$$

and

$$\begin{cases} y'' + p(x)\, y' + q(x)\, y = 0, \\ y(0) = 0, \\ y'(0) = 1 \end{cases} \tag{13}$$

exist and are unique. Denote the first as $y_1(x)$ and the second as $y_2(x)$. Then, the solution to BVP (11) (if any) can be represented as

$$y = c_1 y_1(x) + c_2 y_2(x)\,.$$

Coefficients c_1 and c_2 are solutions to the linear system

$$\begin{cases} c_1 y_1(0) + c_2 y_2(0) = \alpha, \\ c_1 y_1(l) + c_2 y_2(l) = \beta, \end{cases} \iff \begin{cases} c_1 = \alpha, \\ c_2 y_2(l) = \beta - \alpha y_1(l). \end{cases} \tag{14}$$

Consequently, the conditions for the existence and uniqueness of the solution to BVP (11) are the same as for the linear system (14). Hence we get the following result [**4**].

Lemma 1.4.

(i) *BVP* (11) *has a unique solution for all* α *and* β *if and only if* $y_2(l) \neq 0$ *for the solution of IVP* (13).

(ii) *BVP* (11) *has an infinite number of solutions if and only if* $y_2\,(l) = 0$ *and* $\beta = \alpha y_1(l)$. *In this case, the general solution is* $y = \alpha y_1(x) + c y_2\,(x)$, $c \in R$.

(iii) *BVP* (11) *has no solution if and only if* $y_2(l) = 0$ *but* $\beta \neq \alpha y_1(l)$.

We emphasize that it is decisive for existence and uniqueness whether $y_2(l)$ equals 0 or not.

1.4. Existence and uniqueness of a solution of BVP for linear differential equation with constant coefficients

Consider the BVP

$$\begin{cases} y'' + p\,y' + q\,y = 0, \\ y(0) = \alpha, \\ y(l) = \beta, \end{cases} \tag{15}$$

where p and q are constants.

The solutions of the differential equation are of the form $y = \mathrm{e}^{rx}$. The characteristic equation is $r^2 + pr + q = 0$. Its discriminant is $\Delta = p^2 - 4q$, and roots are $r_{1,2} = -\frac{p}{2} \pm \frac{\sqrt{\Delta}}{2}$.

The solutions of IVPs (12) and (13) for the considered case are as follows:

If $\Delta > 0$, then $y_1 = \mathrm{e}^{-\frac{p}{2}x}\left[\cosh\left(\frac{\sqrt{\Delta}}{2}x\right) + \frac{p}{\sqrt{\Delta}}\sinh\left(\frac{\sqrt{\Delta}}{2}x\right)\right]$,

$y_2 = \frac{2}{\sqrt{\Delta}}\mathrm{e}^{-\frac{p}{2}x}\sinh\left(\frac{\sqrt{\Delta}}{2}x\right)$.

If $\Delta = 0$, then $y_1 = \left(1 + \frac{p}{2}x\right)\mathrm{e}^{-\frac{p}{2}x}$, $\quad y_2 = x\mathrm{e}^{-\frac{p}{2}x}$.

If $\Delta < 0$, then $y_1 = \mathrm{e}^{-\frac{p}{2}x}\left[\cos\left(\frac{\sqrt{|\Delta|}}{2}x\right) + \frac{p}{\sqrt{|\Delta|}}\sin\left(\frac{\sqrt{|\Delta|}}{2}x\right)\right]$,

$y_2 = \frac{2}{\sqrt{|\Delta|}}\mathrm{e}^{-\frac{p}{2}x}\sin\left(\frac{\sqrt{|\Delta|}}{2}x\right)$.

It can be seen that if $\Delta \geq 0$, then $y_2(l) > 0$, and consequently $y_2(l) \neq 0$. Thus, from the formulas given above and Lemma 1.4, the following conclusion can be drawn [**4**].

Corollary 1.5. *In the case* $\Delta \geq 0$, *the solution of BVP* (15) *exists, and it is unique regardless of the value of* l *(as well as the values of* α *and* β*).*

In the case $\Delta < 0$*:*

(a) *If* $\frac{\sqrt{|\Delta|}\ l}{2\pi}$ *is not an integer number, then the solution exists and is unique regardless of the values of* α *and* β*.*

(b) *If* $\frac{\sqrt{|\Delta|}\ l}{2\pi}$ *is an integer number and* $\beta = \alpha \mathrm{e}^{-\frac{pl}{2}} \cos\left(\frac{\sqrt{|\Delta|}\ l}{2}\right)$*, then an infinite number of solutions exist:*

$$y = \mathrm{e}^{-\frac{p}{2}x}\left[\alpha \cos\left(\frac{\sqrt{|\Delta|}}{2}x\right) + c \sin\left(\frac{\sqrt{|\Delta|}}{2}x\right)\right], \quad c \in R.$$

(c) *If* $\frac{\sqrt{|\Delta|}\ l}{2\pi}$ *is an integer number but* $\beta \neq \alpha \mathrm{e}^{-\frac{pl}{2}} \cos\left(\frac{\sqrt{|\Delta|}\ l}{2}\right)$*, then no solution exists.*

We notice that existence and uniqueness are violated only when $\Delta < 0$ and at points $l_n^* = \frac{2\pi}{\sqrt{|\Delta|}}n,\ \ n \geq 1$. They are called resonance points by some researchers. We also notice that, in particular, if $l < l_1^*$, then the solution exists and is unique. Taking this circumstance into account, it can be seen that (5)–(9) are, in fact, estimates for the first resonance point l_1^*.

Remark 1. By Corollary 1.5(a), in particular, if $\Delta < 0$ and $\frac{\sqrt{|\Delta|}\ l}{2\pi} < 1 \iff \frac{|\Delta|}{4} < \frac{\pi^2}{l^2}$, then the solution of BVP (15) exists and is unique. This condition can be reduced to the form $0 < q - \frac{p^2}{4} < \frac{\pi^2}{l^2}$. By adding the case $\Delta \geq 0$, the following conclusion can be reached. If

$$q - \frac{p^2}{4} < \frac{\pi^2}{l^2}\,, \tag{16}$$

then BVP (15) has a unique solution.

1.5. Existence and uniqueness of a solution of BVP for Cauchy-Euler equation

We represent a Cauchy-Euler equation (also referred as an equidimensional equation) in the form

$$y'' + \frac{b}{x+\varepsilon}\,y' + \frac{c}{(x+\varepsilon)^2}\,y = 0\,.$$

Notice that we are using a shift of ε in x to make the equation normal (well-behaved) at $x = 0$.

The solution of the Cauchy-Euler equation is of the form $y = (x+\varepsilon)^r$. The characteristic equation is $r\,(r-1) + br + c = 0 \iff r^2 + (b-1)\,r + c = 0$, its discriminant is $\Delta = (b-1)^2 - 4c$ and the roots are $r_{1,2} = \frac{1-b}{2} \pm \frac{\sqrt{\Delta}}{2}$.

The solutions of IVPs (12) and (13) are as follows:

If $\Delta > 0$, then $y_1 = \left(\frac{x}{\varepsilon}+1\right)^{\frac{1-b}{2}}\left[\cosh\left(\frac{\sqrt{\Delta}}{2}\ln\left(\frac{x}{\varepsilon}+1\right)\right) - \frac{1-b}{\sqrt{\Delta}}\sinh\left(\frac{\sqrt{\Delta}}{2}\ln\left(\frac{x}{\varepsilon}+1\right)\right)\right]$,

$$y_2 = \frac{2\varepsilon}{\sqrt{\Delta}}\left(\frac{x}{\varepsilon}+1\right)^{\frac{1-b}{2}}\sinh\left(\frac{\sqrt{\Delta}}{2}\ln\left(\frac{x}{\varepsilon}+1\right)\right).$$

If $\Delta = 0$, then $y_1 = \left(\frac{x}{\varepsilon}+1\right)^{\frac{1-b}{2}}\left[1 - \frac{1-b}{2}\ln\left(\frac{x}{\varepsilon}+1\right)\right]$, $\quad y_2 = \varepsilon\left(\frac{x}{\varepsilon}+1\right)^{\frac{1-b}{2}}\ln\left(\frac{x}{\varepsilon}+1\right)$.

If $\Delta < 0$, then $y_1 = \left(\frac{x}{\varepsilon}+1\right)^{\frac{1-b}{2}} \left[\cos\left(\frac{\sqrt{|\Delta|}}{2}\ln\left(\frac{x}{\varepsilon}+1\right)\right) - \frac{1-b}{\sqrt{|\Delta|}}\sin\left(\frac{\sqrt{|\Delta|}}{2}\ln\left(\frac{x}{\varepsilon}+1\right)\right)\right]$,

$$y_2 = \frac{2\varepsilon}{\sqrt{|\Delta|}}\left(\frac{x}{\varepsilon}+1\right)^{\frac{1-b}{2}} \sin\left(\frac{\sqrt{|\Delta|}}{2}\ln\left(\frac{x}{\varepsilon}+1\right)\right).$$

From here and from Lemma 1.4, for the BVP

$$\begin{cases} y'' + \frac{b}{x+\varepsilon}\, y' + \frac{c}{(x+\varepsilon)^2}\, y = 0, \\ y(0) = \alpha, \\ y(l) = \beta, \end{cases} \tag{17}$$

we obtain the following result.

Corollary 1.6. *In the case $\Delta \geq 0$, the solution of BVP (17) for a Cauchy-Euler equation exists, and it is unique regardless of the value of l (as well as the values of α and β).*

In the case $\Delta < 0$:

(a) *If $\frac{\sqrt{|\Delta|}\ \ln\left(\frac{l}{\varepsilon}+1\right)}{2\pi}$ is not an integer number, then the solution exists and is unique regardless of the values of α and β.*

(b) *If $\frac{\sqrt{|\Delta|}\ \ln\left(\frac{l}{\varepsilon}+1\right)}{2\pi}$ is an integer number and $\beta = \alpha\left(\frac{l}{\varepsilon}+1\right)^{\frac{1-b}{2}}\cos\left(\frac{\sqrt{|\Delta|}\ \ln\left(\frac{l}{\varepsilon}+1\right)}{2}\right)$, then an infinite number of solutions exist:*

$$y = \left(\frac{x}{\varepsilon}+1\right)^{\frac{1-b}{2}} \left[\alpha\ \cos\left(\frac{\sqrt{|\Delta|}}{2}\ \ln\left(\frac{x}{\varepsilon}+1\right)\right) + c\ \sin\left(\frac{\sqrt{|\Delta|}}{2}\ \ln\left(\frac{x}{\varepsilon}+1\right)\right)\right], \quad c \in R.$$

(c) *If $\frac{\sqrt{|\Delta|}\ \ln\left(\frac{l}{\varepsilon}+1\right)}{2\pi}$ is an integer number but $\beta \neq \alpha\ \left(\frac{l}{\varepsilon}+1\right)^{\frac{1-b}{2}}\cos\left(\frac{\sqrt{|\Delta|}\ \ln\left(\frac{l}{\varepsilon}+1\right)}{2}\right)$, then no solution exists.*

One can see that for Cauchy-Euler equation, the resonance points are $l_n^* = \varepsilon\left(e^{\frac{2\pi}{\sqrt{|\Delta|}}n} - 1\right)$, $n \geq 1$. Similarly to Remark 1, we can obtain the following result.

Remark 2. If

$$c - \frac{(b-1)^2}{4} < \frac{\pi^2}{\ln^2\left(\frac{l}{\varepsilon}+1\right)}, \tag{18}$$

then BVP (17) has a unique solution.

2. New sufficient conditions for existence and uniqueness

In this section, we investigate existence-uniqueness for linear BVPs directly, not in the frame of non-linear BVPs.

For equations with constant coefficients, the condition $p^2 - 4q \geq 0$ is one (indeed, the first) of the sufficient conditions for the existence and uniqueness of the solution. Regarding this, the following question may arise.

Question. If $p^2(x) - 4q(x) \geq 0$ for all $x \in [0, l]$, can it be stated that BVP (11) has a solution and it is unique (regardless of the values α and β)?

The answer to this question is "No". To see this, consider the following Cauchy-Euler equation:

$$y'' + \frac{3}{x+1}\,y' + \frac{2}{(x+1)^2}\,y = 0.$$

For this equation, the above condition holds (for every l), but the existence and uniqueness of the solution occurs not for all values of the parameters. Indeed, the characteristic equation for the considered equation is $r\,(r-1) + 3r + 2 = 0 \iff r^2 + 2r + 2 = 0$. Its discriminant is $\Delta = -4$ and roots are $r_{1,2} = -1 \pm i$. The solution of IVP (13) is $y_2(x) = (x+1)^{-1}\,\sin(\ln(x+1))$. If we take, for example, $l = \mathrm{e}^{\pi} - 1$, then $y_2(l) = 0$. Therefore, not for all boundary conditions, the existence and uniqueness of a solution to BVP (11) occurs.

Lemma 1.4 is formulated in terms of solutions to IVPs (12) and (13). Below we will attempt to get results in terms of input parameters (in terms of l, a, b and the coefficient functions $p(x)$ and $q\,(x)$).

Lemma 2.1. *If $q(x) \le 0$ for all $x \in [0,l]$, then BVP (11) has a solution, and it is unique (regardless of what the values of α and β are).*

Proof. Let $u(x) = \mathrm{e}^{\int_0^x p(x)\mathrm{d}x}$. Then, $u' = p(x)u(x)$. Multiply equation (13) by $u(x)$: $u\,y'' + pu\,y' + qu\,y = 0$, and write it as $(u(x)y'(x))' = -q(x)u(x)y(x)$. Integrating from 0 to x gives

$$u(x)\,y'(x) - u\,(0)\;y'(0) = \int\limits_0^x (-q\,(x))\,u(x)y(x)\mathrm{d}x.$$

From here and $u(0) = 1,\;\; y'(0) = 1$, we have

$$y'(x) = \frac{1 + \int\limits_0^x (-q(x))\,u(x)y(x)\mathrm{d}x}{u(x)}.$$

Since $y(0) = 0$, $u(x) > 0$ and $-q(x) \ge 0$, as long as $y(x) \ge 0$, we get $y'(x) > 0$. Therefore, the solution of IVP (13), y_2, is a strictly increasing function. From here, $y_2(l) \ne 0$. Then, according to Lemma 1.4, the solution of the BVP exists and is unique. □

Lemma 2.1 is almost identical to Corollary 1.2, obtained as a special case of the results for non-linear BVPs. The lemma's condition is more efficient, since it also includes the equality case.

Let us reduce equation (11) to a simpler form (canonical form) to obtain new results in the future.

Denote $v = \mathrm{e}^{\frac{1}{2}\int_0^x p(x)\mathrm{d}x}$. Then, $v' = \frac{1}{2}p(x)v(x)$. Put $z = v\,(x)\,y(x)$. The derivatives are $z' = v\,y' + \frac{1}{2}pv\,y$ and $z'' = v\,y'' + pv\,y' + \frac{1}{4}\left(p^2 + 2p'\right)v\,y$.

Multiply equation (11) by $v(x)$:

$$v\,y'' + pv\,y' + qv\,y = 0$$

and move on to the variable z:

$$z'' + \frac{1}{4}\left(4q - p^2 - 2p'\right) z = 0. \tag{19}$$

After the variable change, BVP (11) is transformed into the following form:

$$\begin{cases} z'' + g(x)\, z = 0, \\ z(0) = \alpha, \\ z(l) = \tilde{\beta}, \end{cases} \tag{20}$$

where $g(x) = \frac{1}{4}\left(4q(x) - p^2(x) - 2p'(x)\right)$ and $\tilde{\beta} = \beta\, v(l)$.

From here we can obtain the following result.

Lemma 2.2. *If* $4q(x) - p^2(x) - 2p'(x) \le 0$ *for all* $x \in [0, l]$, *then BVP* (11) *has a solution and it is unique (no matter what the values of* α *and* β *are).*

Proof. In equation (20) obtained for $z(x)$, the coefficient functions are $\tilde{p}(x) = 0$ and $\tilde{q}\,(x) = \frac{1}{4}\left(4q - p^2 - 2p'\right)$. According to Lemma 2.1, if $\tilde{q}(x) \le 0$, i.e., if $4q - p^2 - 2p' \le 0$, then the solution $z(x)$ exists and is unique. From here and from $v(x) > 0$, the solution $y(x) = \frac{z(x)}{v(x)}$ also exists and is unique. □

Using the variable z, we reduced BVP (11) to BVP (20). Let us also express IVP (13) in terms of z. Since $v(0) = 1$, we have $z\,(0) = 0$ and $z'(0) = 1$. Thus, IVP (13) becomes as follows:

$$\begin{cases} z'' + g(x)\, z = 0, \\ z(0) = 0, \\ z'(0) = 1. \end{cases} \tag{21}$$

Let us reformulate condition (16) in Remark 1 in terms of the function $g(x)$.

For BVP (15), if

$$g(x) \equiv q - \frac{p^2}{4} < \frac{\pi^2}{l^2}\,, \tag{22}$$

then the solution exists and is unique.

Similarly, condition (18) in Remark 2 is formulated as follows. For BVP (17) with the Cauchy-Euler equation, if

$$g(x) \equiv \left(c - \frac{b^2 - 2b}{4}\right) \frac{1}{(x+\varepsilon)^2} < \left[\frac{\pi^2}{\ln^2\left(\frac{l}{\varepsilon} + 1\right)} + \frac{1}{4}\right] \frac{1}{(x+\varepsilon)^2}\,, \tag{23}$$

then the solution exists and is unique.

Below we will show that conditions (22) and (23) are applicable to any linear equation, and not only to equations of the indicated types.

Let us examine Lemma 2.2's effectiveness.

For equations with constant coefficients, the sufficient condition in Lemma 2.2 takes the form $p^2 - 4q \ge 0$. This is the same as the first condition ($\Delta \ge 0$) at the beginning of the independently obtained Corollary 1.5.

The first sufficient condition in Corollary 1.6 for the Cauchy-Euler equation $y'' + \frac{b}{x+\varepsilon}\, y' + \frac{c}{(x+\varepsilon)^2}\, y = 0$ is $(b-1)^2 - 4c \ge 0 \iff 4c - (b-1)^2 \le 0$. Since

$p(x) = \frac{b}{x+\varepsilon}$ and $q(x) = \frac{c}{(x+\varepsilon)^2}$, the condition $4q - p^2 - 2p' \le 0$ in Lemma 2.2 takes the form $4\frac{c}{(x+\varepsilon)^2} - \frac{b^2}{(x+\varepsilon)^2} + 2\frac{b}{(x+\varepsilon)^2} \le 0 \iff 4c - (b-1)^2 \le 1$. Thus, the condition in Lemma 2.2 is rough than that in Corollary 1.6.

The condition in Corollary 1.6 can be represented as $c \le \frac{1}{4}(b-1)^2$, and the condition obtained from Lemma 2.2 can be expressed as $c \le \frac{1}{4}(b-1)^2 - \frac{1}{4}$. Therefore, the region $\frac{1}{4}(b-1)^2 - \frac{1}{4} \le c \le \frac{1}{4}(b-1)^2$, where there is only one solution, falls outside the scope of Lemma 2.2. Figure 1 shows the situation graphically.

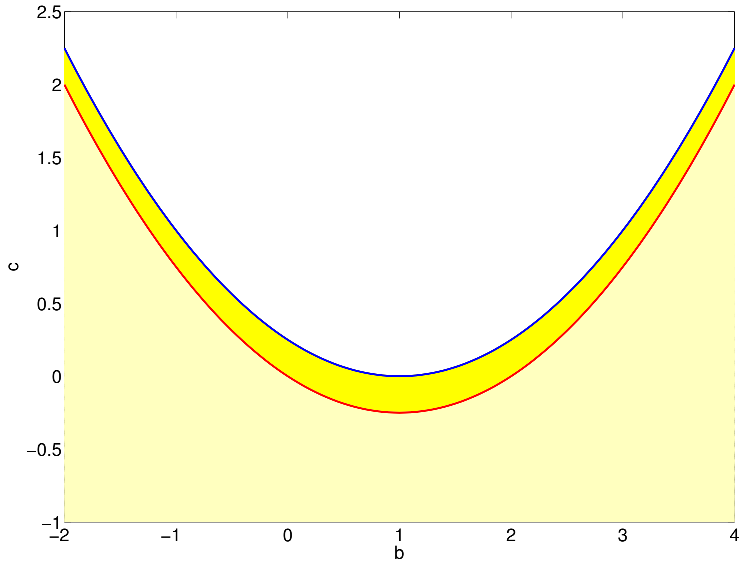

Figure 1. In the marked region, the solution of BVP (17) for the Cauchy-Euler equation is unique. In the more contrasting region between the two curves, the solution also exists and is unique, but Lemma 2.2 does not cover this region.

Let us explain the above circumstance through an example. For the equation $y'' + \frac{2}{x+1}y' + \frac{1}{9}\frac{1}{(x+1)^2}y = 0$ (i.e., in the case of $b = 2$ and $c = \frac{1}{9}$), since the discriminant of the characteristic equation, $\Delta = (b-1)^2 - 4c = \frac{5}{9}$, is positive, the BVP (17) has a solution and it is unique. However, the condition obtained from Lemma 2.2 ($c \le \frac{1}{4}(b-1)^2 - \frac{1}{4} \iff \frac{1}{9} \le 0$) does not hold.

Let us investigate how Lemma 2.2 can be improved. To do this, we will try to use the estimate (6) obtained by Lettenmeier [**6**] for the non-linear BVP (2).

For BVP (20), $f(x, z(x), z'(x)) = \frac{1}{4}\left(4q(x) - p^2(x) - 2p'(x)\right) z(x)$. If we take $K = \frac{1}{4}\max_{x\in[0,l]}\left|4q(x) - p^2(x) - 2p'(x)\right|$ and $L = 0$, the Lipschitz condition (3) is satisfied. Then, condition (6) becomes as $\frac{1}{\pi^2}Kl^2 = \frac{l^2}{4\pi^2}\max_{x\in[0,l]}\left|4q(x) - p^2(x) - 2p'(x)\right| < 1$. This implies the following lemma.

Lemma 2.3. *If* $\max_{x\in[0,\ l]}\left|4q(x) - p^2(x) - 2p'(x)\right| < \frac{4\pi^2}{l^2}$, *then BVP* (11) *has a solution and it is unique.*

To obtain the next result, we use the following theorem and corollary given in [**9**].

Consider the equation

$$[k(x)y']' + g(x)\,y = 0 \qquad (a \le x \le b), \tag{24}$$

where the function $k(x)$ is positive and continuously differentiable, and the function $g(x)$ is continuous.

Theorem 2.4 (Sturm Comparison Theorem). *Let $y_n = y_n(x)$ be nonzero solutions of the linear equations*

$$[k_n(x)y_n']' + g_n(x)y_n = 0 \qquad (n = 1,\ 2),$$

and let the inequalities $k_1(x) \ge k_2(x) > 0$ and $g_1(x) \le g_2(x)$ hold. Then the function y_2 has at least one zero lying between any two adjacent zeros, x_1 and x_2, of the function y_1 (it is assumed that the identities $k_1 \equiv k_2$ and $g_1 \equiv g_2$ are not satisfied on any interval simultaneously).

Corollary 2.5. *If $g(x) \le 0$ or there exists a constant c such that*

$$k(x) \ge c > 0, \qquad g(x) < c\left(\frac{\pi}{b-a}\right)^2,$$

then every nontrivial solution to equation (24) *has no more than one zero on the interval $[a, b]$.*

For IVP (21), $k(x) = 1$ and $g(x) = \frac{1}{4}\left(4q(x) - p^2(x) - 2p'(x)\right)$. If to take $c = 1$, then the condition $k(x) = 1 \ge 1 > 0$ holds. Besides, if $g(x) = \frac{1}{4}\left(4q(x) - p^2(x) - 2p'(x)\right) < \frac{\pi^2}{l^2} \iff 4q(x) - p^2(x) - 2p'(x) < \frac{4\pi^2}{l^2}$, then, according to Corollary 2.5, each non-trivial solution $z(x)$ of the differential equation has at most one zero in the interval $[0, l]$. Then, since $z(0) = 0$ at 0, the solution of IVP (21) has no other zeros. From here, according to Lemma 1.4, BVP (20) has a solution and it is unique. Summarizing, we can express the result obtained as follows.

Lemma 2.6. *If* $\max_{x\in[0,l]}\left(4q(x) - p^2(x) - 2p'(x)\right) < \frac{4\pi^2}{l^2}$, *then BVP* (11) *has a solution and it is unique.*

This result (Lemma 2.6) is stronger than Lemma 2.3 obtained on the basis of the study by Lettenmeyer [**6**]. The result is also stronger than Lemma 2.2.

Let us consider a numerical example to evaluate the impact of the result achieved (i.e., of Lemma 2.6). For the differential equation $y'' + \frac{1}{2}\frac{1}{(x+1)^2}\, y = 0$, the solution of IVP (13) is $y_2(x) = 2\sqrt{x+1}\, \sin\left(\frac{1}{2}\ln(x+1)\right)$. If $l < \mathrm{e}^{2\pi} - 1 \approx 534.49$, then $y_2(x) \neq 0$. Therefore, BVP has a solution and it is unique.

Now, let us check the condition of Lemma 2.6:

$$\max_{x\in[0,l]}\left(4q(x) - p^2(x) - 2p'(x)\right) = \max_{x\in[0,l]} \frac{2}{(x+1)^2} < \frac{4\pi^2}{l^2} \Longleftrightarrow 2 < \frac{4\pi^2}{l^2} \Longleftrightarrow l^2 < 2\pi^2.$$

Thus, Lemma 2.6 guarantees the existence-uniqueness of the solution only when $l < \sqrt{2}\pi \approx 4.44$ (that is, for relatively small values of l). As a result, Lemma 2.6 has the potential to be developed further.

Let us reduce the solution of the 2nd order differential equation (21) in canonical form to the solution of the Riccati equation. With $k(x)$ being a function, subtract $(kz)'$ from the equation and add it back to get

$$z'' - (kz)' + (kz)' + gz = 0$$

and group the terms as follows:

$$(z' - kz)' + k\left(z' + \frac{k' + g}{k} z\right) = 0\,. \tag{25}$$

Choose the function $k(x)$ so that $\frac{k'+g}{k} = -k$. This function solves the Riccati equation $k' + k^2 + g = 0$. Now, assume that it is found.

Put $w = z' - kz$. Equation (25) becomes

$$w' + kw = 0.$$

Since $w(0) = 1 - 0 = 1$ for IVP (21), we have

$$w = \mathrm{e}^{-\int_0^x k(x)\,\mathrm{d}x} \qquad \text{(note: } w(x) > 0\text{)}.$$

Using the function w, (21) is transformed into the following first-order linear IVP:

$$z' - kz = w, \qquad z(0) = 0\,.$$

The integrating factor is

$$u = \mathrm{e}^{-\int_0^x k(x)\,\mathrm{d}x} = w(x),$$

the general solution is

$$z = \frac{1}{u(x)}\left(C + \int w(x)\, u(x)\,\mathrm{d}x\right) = \frac{1}{w(x)}\left(C + \int w^2(x)\,\mathrm{d}x\right).$$

Since $z(0) = 0$, we get

$$z = \frac{\int_0^x w^2(x)\,\mathrm{d}x}{w(x)}.$$

From here, we establish that $z_2(x) > 0$ (as long as the function $k(x)$ is defined). Then, based on Lemma 1.4, we arrive at the following conclusion.

Lemma 2.7. *If the Riccati equation $k' + k^2 + g(x) = 0$ has a solution defined on $[0, l]$, then BVP (20) has a solution and it is unique.*

Below, we recall a theorem from [**3**] that we will use in the future.

Theorem 2.8 (Comparison theorem)**.** *Let the functions $f_1(x, y)$, $f_2(x, y)$ be continuous in $Q = \{(x, y) \mid a \le x \le b,\ u \le y \le v\}$, and $f_1(x, y)$ has partial derivative $\frac{\partial f_1}{\partial y}(x, y)$ in Q. Then, if $y_1(x)$, $y_2(x)$ on interval $[a, b]$ are solutions of Cauchy problems*

$$\left\{ \begin{array}{l} y_1'(x) = f_1(x, y_1(x)), \\ y_1(a) = y_{01}, \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{l} y_2'(x) = f_2(x, y_2(x)), \\ y_2(a) = y_{02}, \end{array} \right.$$

where $f_1(x, y) \ge f_2(x, y)$, $(x, y) \in Q$, and $y_{01} \ge y_{02}$, the following inequality holds:

$$y_1(x) \ge y_2(x), \qquad x \in [a, b].$$

Based on this Theorem, we prove the following lemma.

Lemma 2.9. *If the initial value problem $k' = -\left(k^2 + g(x)\right)$, $k(0) = k_0$ for the Riccati equation has solutions defined in $[0, l]$, for continuous functions $g = g_1(x)$ and $g = g_2(x)$ (where $g_1(x) \le g_2(x)$), then it also has a solution for any continuous function $g(x)$ satisfying the condition $g_1(x) \le g(x) \le g_2(x)$.*

Proof. Let a function $g(x)$, satisfying the lemma's condition, be given. According to the classical existence and uniqueness theorem, there exists a solution to the initial value problem in an interval $[0, \tilde{\delta})$. Let us show that this solution can be extended to the interval $[0, l]$. Assume the opposite is true: Let the widest interval over which the solution can be extended be narrower than $[0, l]$. This widest interval can be (1) closed or (2) open.

(1) Let us see if this case is possible. Let the interval in question be $[0, m]$ $(m < l)$. Taking m as the starting point, the solution of the initial value problem under investigation can be extended to an interval $[m, m + \delta)$. As a result, a solution defined in the interval $[0, m + \delta)$ is obtained. This contradicts the assumption that the widest interval to which the solution can be extended is $[0, m]$. Hence, case (1) cannot occur.

Consider case (2). Let $[0, m)$ $(m \le l)$ be the widest interval over which the solution can be extended. Let us denote the solutions corresponding to the functions $g = g_1(x)$ and $g = g_2(x)$ as $k_1(x)$ and $k_2(x)$, respectively. These continuous functions are bounded because they are defined on the closed interval $[0, l]$. By the Comparison theorem (Theorem 2.8) [**3**], the solution $k(x)$ is trapped between $k_1(x)$ and $k_2(x)$ on each interval $[0, \underline{m}]$ $(\underline{m} < m)$, hence, on $[0, m)$: $k_2(x) \le k(x) \le k_1(x)$ for all $x \in [0, m)$. Consider the limit of the solution function $k(x)$ when $x \to m^-$. This limit $(\lim\limits_{x \to m^-} k(x))$ (i) may exist or (ii) may not exist. If the limit exists (let us call it k_m), by taking $k(m) = k_m$, the solution could be extended to the closed interval $[0, m]$ (note that in this case, the solution could be further extended to an interval $[0, m + \delta)$). Thus, in case (i) there is a contradiction that the widest interval over which the solution can be extended is $[0, m)$.

Let us look at situation (ii). In this case, when $x \to m^-$, the solution function $k(x)$ has no limit. On the other hand, $k(x)$ is bounded. According to these two facts, we can choose two increasing sequences x_n and z_n that converge to m, so that the limits of $k(x_n)$ and $k(z_n)$ are different (we denote them as k_x and k_z). Let us denote the absolute value of the difference between the limits by d, i.e., $d = |k_z - k_x|$.

Put $M = \sup\limits_{x\in[0,\ m)} \left|k^2(x) + g(x)\right|$. For each ε, a $\delta > 0$ can be chosen such that $M\delta < \frac{\varepsilon}{4}$. Also, for each ε, one can choose an N such that for all n greater than N, the following inequalities hold: $|x_n - m| < \delta$, $|z_n - m| < \delta$ and $|k(x_n) - k_x| < \frac{\varepsilon}{4}$, $|k(z_n) - k_z| < \frac{\varepsilon}{4}$. Let us take two elements, x_i and z_j, with numbers greater than N and such that $x_i \neq z_j$. Either $x_i < z_j$, or $z_j < x_i$. For clarity, let $x_i < z_j$. Then,

$$\begin{aligned} |k(z_j) - k(x_i)| &= \left|\int\limits_{x_i}^{z_j} k'(x)\,\mathrm{d}x\right| = \left|\int\limits_{x_i}^{z_j} \left(k^2(x) + g(x)\right)\mathrm{d}x\right| \le \int\limits_{x_i}^{z_j} \left|k^2(x) + g(x)\right|\mathrm{d}x \\ &\le \int\limits_{x_i}^{z_j} M\mathrm{d}x = M(z_j - x_i) \le M\delta \\ &< \frac{\varepsilon}{4}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |k(z_j) - k(x_i)| &= |k(z_j) - k_z + k_z - k_x + k_x - k(x_i)| \\ &= |k_z - k_x + (k(z_j) - k_z + k_x - k(x_i))| \\ &\overset{|a+b|\ge|a|-|b|}{\ge} |k_z - k_x| - |k(z_j) - k_z + k_x - k(x_i)| \\ &\ge |k_z - k_x| - (|k(z_j) - k_z| + |k(x_i) - k_x|) \\ &\ge d - \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) \\ &= d - \frac{\varepsilon}{2}, \end{aligned}$$

If we take $\varepsilon < d$, then on the one hand $|k(z_j) - k(x_i)| < \frac{\varepsilon}{4} < \frac{d}{4}$, and on the other hand $|k(z_j) - k(x_i)| \ge d - \frac{\varepsilon}{2} > d - \frac{d}{2} = \frac{d}{2}$. This is a contradiction (the same expression cannot be less than $\frac{d}{4}$ and greater than $\frac{d}{2}$, where d is a positive number). Therefore, case (2)(ii) also cannot occur.

As a result, the assumption that the solution of the initial value problem cannot be extended to the interval $[0, l]$ is false. The lemma is proved. ☐

And now we formulate the key assertion of our work.

Lemma 2.10. *If there exists $\varepsilon > 0$ such that $g(x) < \left[\frac{\pi^2}{\ln^2\left(\frac{l}{\varepsilon}+1\right)} + \frac{1}{4}\right]\frac{1}{(x+\varepsilon)^2}$ for all $x \in [0, l]$, then BVP (20) has a solution and it is unique (regardless of what the values of α and $\tilde{\beta}$ are).*

Proof. We will carry out the proof in 4 steps.

(1) Take an ε that satisfies the lemma's condition. Let us show that

$$\exists\delta > 0\ \forall x \in [0,\ l] : g(x) \leq \left[\frac{\pi^2}{\ln^2\left(\frac{l+\delta}{\varepsilon}+1\right)} + \frac{1}{4}\right]\frac{1}{(x+\varepsilon)^2}. \tag{26}$$

We will prove it by contradiction. Assume that the statement does not hold, i.e.,

$$\forall\delta > 0\ \exists x \in [0, l] : g(x) > \left[\frac{\pi^2}{\ln^2\left(\frac{l+\delta}{\varepsilon}+1\right)} + \frac{1}{4}\right]\frac{1}{(x+\varepsilon)^2}. \tag{27}$$

Let us take a sequence of positive numbers δ_n approaching to 0. For each δ_n, taking an x_n that satisfies the condition (27), we create a sequence x_n. Since $x_n \in [0, l]$, we can choose a convergent subsequence of this sequence: let it be the sequence z_n and its limit be z. Since $z \in [0, l]$, we have $g\left(z\right) \geq \left[\frac{\pi^2}{\ln^2\left(\frac{l}{\varepsilon}+1\right)} + \frac{1}{4}\right]\frac{1}{(z+\varepsilon)^2}$ according to (27). This is a contradiction to the Lemma's condition. Therefore, (26) is satisfied.

(2) Let us take $g = g_2(x) = \left[\frac{\pi^2}{\ln^2\left(\frac{l+\delta}{\varepsilon}+1\right)} + \frac{1}{4}\right]\frac{1}{(x+\varepsilon)^2}$ and consider the Riccati equation $k' + k^2 + g(x) = 0$. It can be seen that

$$k = k_2(x) = \frac{1}{x+\varepsilon}\left\{\frac{1}{2} - \frac{\pi}{\ln\left(\frac{l+\delta}{\varepsilon}+1\right)}\tan\left(\frac{\pi}{\ln\left(\frac{l+\delta}{\varepsilon}+1\right)}\,\frac{2\ln\left(x+\varepsilon\right) - \left(\ln\left(l+\varepsilon\right) + \ln\left(\varepsilon\right)\right)}{2}\right)\right\} \tag{28}$$

is a solution with the initial value

$$k_0 = k(0) = \frac{1}{\varepsilon}\left\{\frac{1}{2} + \frac{\pi}{\ln\left(\frac{l+\delta}{\varepsilon}+1\right)}\tan\left(\frac{\pi}{2}\,\frac{\ln\left(\frac{l}{\varepsilon}+1\right)}{\ln\left(\frac{l+\delta}{\varepsilon}+1\right)}\right)\right\} = B,$$

where $B > 0$. (The solution function k_2 is defined on $[0, l]$. This proposition can be proved as follows. In the expression under the tangent function in (28), the function $s\left(x\right) = \frac{2\ln(x+\varepsilon)-(\ln(l+\varepsilon)+\ln(\varepsilon))}{2}$, which is the second factor, is an increasing function of x, and $s(0) = \frac{2\ln(\varepsilon)-(\ln(l+\varepsilon)+\ln(\varepsilon))}{2} = -\frac{1}{2}\ln\left(\frac{l}{\varepsilon}+1\right)$, $s\left(l\right) = \frac{2\ln(l+\varepsilon)-(\ln(l+\varepsilon)+\ln(\varepsilon))}{2} = \frac{1}{2}\ln\left(\frac{l}{\varepsilon}+1\right)$. Therefore, $-\frac{1}{2}\ln\left(\frac{l}{\varepsilon}+1\right) \leq s(x) \leq \frac{1}{2}\ln\left(\frac{l}{\varepsilon}+1\right)$ on $[0, l]$. From here, for the entire expression under the tangent function, we get: $-\frac{\pi}{2} < \frac{\pi}{\ln\left(\frac{l+\delta}{\varepsilon}+1\right)}s(x) < \frac{\pi}{2}$. Thus, tangent is defined. Consequently, k_2 is defined on the interval $[0, l]$.)

(3) Let us take a function $g(x)$ that satisfies the lemma's condition. Put $\underline{G} = \min\limits_{x\in[0,\ l]} g(x)$ and $A = \min\left\{-1,\ \underline{G}\right\}$. By this definition, $A < 0$. Put $a = \sqrt{-A} > 0$.

For the function $g = g_1(x) = -a^2$, the solution of the Riccati equation $k' + k^2 + g(x) = 0$ corresponding to the initial condition $k(0) = k_0 = B$ is $k = k_1\left(x\right) = a\left(1 - \frac{2(a-B)}{a-B+(a+B)e^{2ax}}\right)$. (One can see that if $a = B$, then $k_1(x) = B$). Since $a > 0$ and $B > 0$, it is easy to see that the solution function k_1 is defined in $[0, l]$.

(4) According to the above, for a function $g(x)$ satisfying the lemma's condition, we have $g_1(x) \leq g(x) \leq g_2(x)$ for all $x \in [0, l]$. Then, according to Lemma 2.9, the

Riccati equation for $g = g(x)$ has a solution in $[0, l]$. From here and Lemma 2.7 it follows that BVP (20) has only one solution. $\square$

3. Main results

Based on Lemmas 1.3 and 2.1, taking into account that we have denoted $g(x) = \frac{1}{4}\left(4q(x) - p^2(x) - 2p'(x)\right)$ and expressing Lemma 2.10 in terms of the functions $p(x)$ and $q(x)$, we get the following result.

Theorem 3.1. *Let the functions $p(x)$, $q(x)$, $r(x)$ and $p'(x)$ be continuous on $[0, l]$. If*

(i) $q(x) \leq 0$ *for all* $x \in [0, l]$,

or

(ii) *there exists* $\varepsilon > 0$ *such that* $4q(x) - p^2(x) - 2p'(x) < \left[\frac{4\pi^2}{\ln^2\left(\frac{l}{\varepsilon}+1\right)} + 1\right] \frac{1}{(x+\varepsilon)^2}$ *for all* $x \in [0, l]$,

then BVP (1) *has a solution and it is unique (regardless of what the values of A and B are).*

Below we derive two particular sufficient conditions from the theorem that are convenient to apply.

For the function $\hat{g}(x) := \left[\frac{\pi^2}{\ln^2\left(\frac{l}{\varepsilon}+1\right)} + \frac{1}{4}\right] \frac{1}{(x+\varepsilon)^2}$ on the right-hand side of the inequality in Lemma 2.10, it can be seen that $\hat{g}(x) \to \frac{1}{4x^2}$ when $\varepsilon \to 0$. This implies that the condition $g(x) < \frac{1}{4x^2}$ for all $x \in (0, l]$, is a sufficient condition for existence and uniqueness. This result can be expressed in terms of input parameters as follows.

Corollary 3.2. *If* $4q(x) - p^2(x) - 2p'(x) < \frac{1}{x^2}$ *for all* $x \in (0, l]$, *then BVP* (1) *has a solution and it is unique (regardless of what the values of A and B are).*

It can also be seen that if $\varepsilon \to \infty$, then

$$\begin{aligned}
\hat{g}(x) &= \frac{\pi^2}{\ln^2\left(\frac{l}{\varepsilon}+1\right)} \frac{1}{(x+\varepsilon)^2} + \frac{1}{4} \frac{1}{(x+\varepsilon)^2} \approx \frac{\pi^2}{\left[(x+\varepsilon)\ln\left(1+\frac{l}{\varepsilon}\right)\right]^2} + 0 \\
&= \frac{\pi^2}{\left[(x+\varepsilon)\frac{l}{\varepsilon}\ln\left(1+\frac{l}{\varepsilon}\right)^{\frac{\varepsilon}{l}}\right]^2} \approx \frac{\pi^2}{\left[l\left(\frac{x}{\varepsilon}+1\right)\ln \mathrm{e}\right]^2} \to \frac{\pi^2}{[l \cdot 1 \cdot 1]^2}.
\end{aligned}$$

From here, we get $\hat{g}(x) \to \frac{\pi^2}{l^2}$. It can then be seen that the condition $g(x) < \frac{\pi^2}{l^2}$ for all $x \in [0, l]$, is also a sufficient condition for existence and uniqueness. The corresponding proposition is as follows.

Corollary 3.3. *If* $4q(x) - p^2(x) - 2p'(x) < \frac{4\pi^2}{l^2}$ *for all* $x \in [0, l]$, *then BVP* (1) *has a solution and it is unique (regardless of what the values of A and B are).*

Notice that Corollary 3.3 is the same as Lemma 2.6.

None of Corollaries 3.2 and 3.3 is stronger than the other. This statement can be justified by the examples below.

Let us consider the equation $y'' + \frac{1}{8}y = 0$. Take $l = 3$. The condition of Corollary 3.2 (i.e., the condition $\frac{1}{2} < \frac{1}{x^2} \iff x^2 < 2$) is not satisfied if the entire interval $(0, l]$ is considered. But the condition of Corollary 3.3 (the condition $\frac{1}{2} < \frac{4\pi^2}{9}$) is satisfied. Therefore, based on Corollary 3.2, it cannot be decided about the solution of BVP for the given equation. However, based on Corollary 3.3, the existence and uniqueness of the solution can be established.

Now consider the equation $y'' + \frac{1}{4}\frac{1}{(x+1)^2}\, y = 0$ and take $l = 3\pi$. The condition of Corollary 3.2 (the condition $\frac{1}{(x+1)^2} < \frac{1}{x^2} \iff (x+1)^2 > x^2$) is satisfied, but the condition of Corollary 3.3 (the condition $\frac{1}{(x+1)^2} < \frac{4\pi^2}{9\pi^2} \iff (x+1)^2 > \frac{9}{4}$) is not satisfied on some part of the interval $(0, l] = (0, 3\pi]$.

Remark. For differential equations with constant coefficients, the condition of Corollary 3.3 becomes $4q - p^2 < \frac{4\pi^2}{l^2}$, which can also be expressed as a disjunction of two conditions: $p^2 - 4q \geq 0$ or $0 < 4q - p^2 < \frac{4\pi^2}{l^2}$. Therefore, Corollary 3.3 completely covers the first sufficient condition of Corollary 1.5 and partially covers its condition (a). In the case of Cauchy-Euler equations, a similar statement is valid for Theorem 3.1(ii) and Corollary 1.6.

4. Comparison with existing results and discussion

In this section, we demonstrate that the existing results on existence and uniqueness cover only a narrow class of BVPs, and, in contrast, the results obtained in this study allow us to expand this class significantly. For this purpose, we use BVPs (15) for linear differential equations with constant coefficients. In this case, the Lipschitz coefficients are $K = |q|$ and $L = |p|$. Also, to be specific, we take $l = b - a = 1$.

First, we compare the result by Lettenmeyer [**6**] with Coles and Sherman [**2**], i.e., the estimates (6) and (7), which can be represented as

$$|q| < \pi^2 - 4\,|p|$$

and

$$\begin{aligned} |q| &< 12 - 4\,|p| && \text{for } |q| \leq p^2, \\ |q| &< \frac{12 - 2\sqrt{3}\ |p|}{3 - \sqrt{3}} && \text{for } |q| > p^2, \end{aligned}$$

respectively. As can be seen from Figure 2, the estimate by Coles and Sherman (Figure 2(a)) is mainly better than that by Lettenmeyer (Figure 2(b), the inner region), except for a thin region that becomes visible when we zoom in on it (see Figure 2(c)).

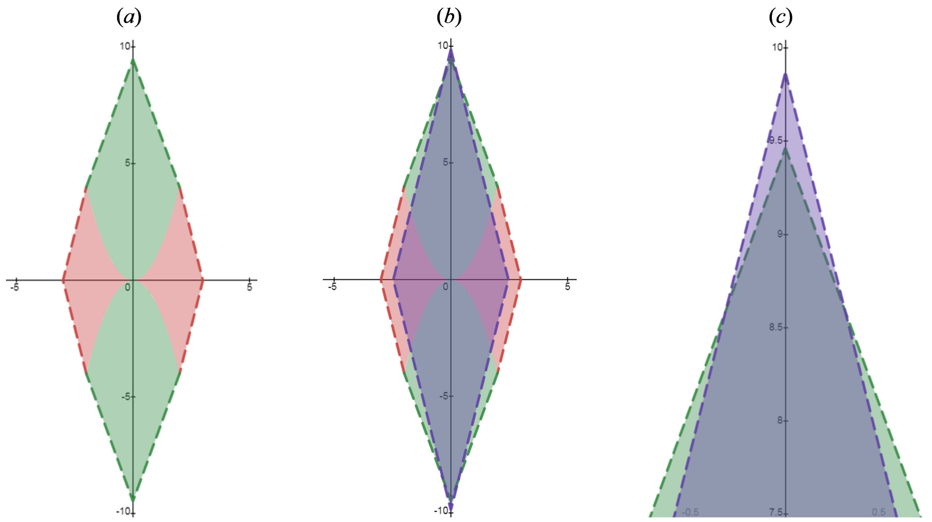

Figure 2. The regions in the pq-plane, where the solution of BVP (15) exists and is unique: (a) by Coles and Sherman, (b) by Lettenmeyer (the inner region), (c) the thin region where Lettenmeyer's estimate is better (the intermediate region at the top of the figure).

The estimate (8)–(9) by de la Vallee Poussin is better than all existing Picard-like estimates. To illustrate this circumstance, we make a comparison with the Lettenmeyer's estimate in Figure 3.

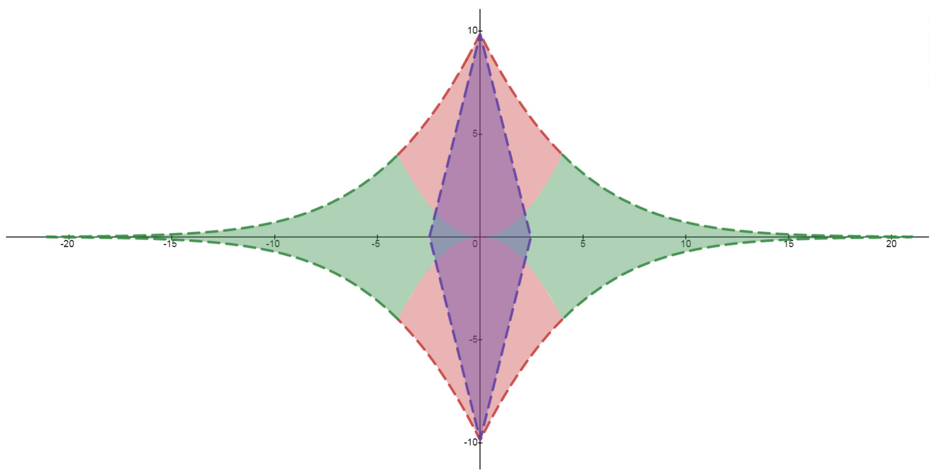

Figure 3. The regions in the pq-plane, where the solution of BVP (15) exists and is unique: by Lettenmeyer (the inner region) and by de la Vallee Poussin (the entire marked region including the inner part).

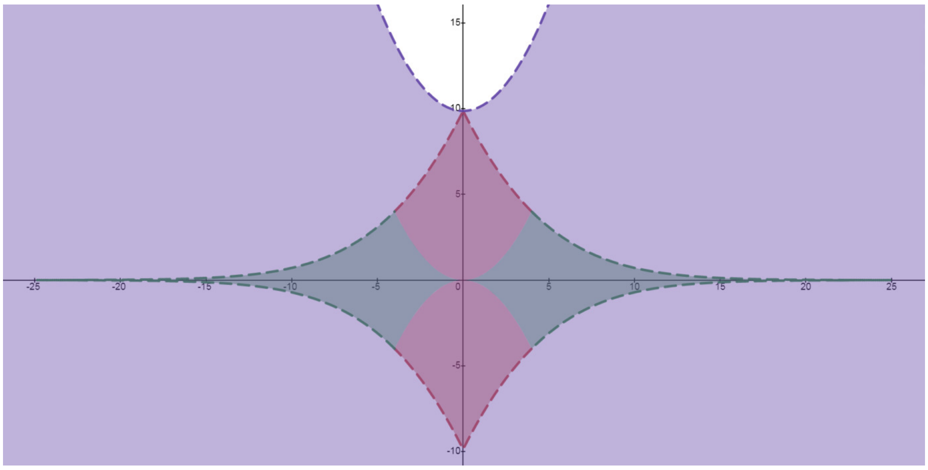

Figure 4. The regions in the pq-plane, where the solution of BVP (15) exists and is unique: by de la Vallee Poussin (the inner region) and by Corollary 3.3 (the entire marked region including the inner part).

Now let us compare our result with the most accurate of the estimates mentioned above, i.e., with the estimate by de la Vallee Poussin. For the comparison we use Corollary 3.3. For constant coefficient linear differential equations, it becomes as $4q - p^2 < \frac{4\pi^2}{l^2}$. Thus, we have the estimate $q < \frac{1}{4}p^2 + \pi^2$, when $l = 1$. Figure 4 shows that de la Vallee Poussin's estimate determines only a bounded region. The region according to Corollary 3.3 (see Figure 4) is unbounded and entirely includes the region determined by de la Vallee Poussin. Therefore, our analysis demonstrates that the result obtained in this study is significantly better than the ones obtained for non-linear BVPs.

We would like to emphasize that we used the Graphing Calculator – Desmos (desmos.com) when creating Figures 2–4.

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N. A. Gasilov, Department of Computer Engineering, Baskent University, Ankara, Turkey,
e-mail: `gasilov@baskent.edu.tr`