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# GENERALIZED SASAKIAN SPACE FORMS WITH $m$-PROJECTIVE CURVATURE TENSOR

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Abstract. In the present paper, we study $\phi$-$m$-projectively flat generalized Sasakian space forms, $m$-projectively locally symmetric generalized Sasakian space forms and $m$-projectively locally $\phi$-symmetric generalized Sasakian space forms. Obtained results are supported by illustrative examples.

## 1. Introduction

Recently, P. Alegre, D. Blair and A. Carriazo [**1**] introduced and studied generalized Sasakian space forms. These space forms are defined as follows:
Given an almost contact metric manifold $M(\phi,\xi,\eta,g)$, we say that $M$ is generalized Sasakian space forms if there exist three functions $f_1, f_2, f_3$ on $M$ such that the curvature tensor $R$ is given by

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{1.1}$$

for any vector fields $X, Y, Z$ on $M$. In such a case, we denote the manifold as $M(f_1, f_2, f_3)$. These kinds of manifolds appear as a generalization of well-known Sasakian space forms which can be obtained as a particular case of generalized Sasakian space forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$. It may be noted that these are not merely generalization of Sasakian space forms but also contain a large class of almost contact manifolds. For example, it is well-known that [**2**] any three dimensional $(\alpha,\beta)$-trans Sasakian manifolds with $\alpha,\beta$ depending on $\xi$ is a generalized Sasakian space form. However, we can find generalized Sasakian space forms with non-constant functions and arbitrary dimensions. In [**1**], the authors cited several examples of generalized Sasakian space forms in terms of warped product space. In [**9**], U. K. Kim studied conformally flat generalized Sasakian space forms and locally symmetric generalized Sasakian space forms. Generalized Sasakian space forms studied by several authors, viz., [**5**], [**6**], [**7**]. In [**16**], the

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authors studied $\phi$-projectively flat generalized Sasakian space forms and obtained several interesting results.

In 1971, G. P. Pokhriyal and R. S. Mishra [**13**] defined a tensor field $W^*$ on a Riemannian manifold as

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}, \tag{1.2}$$

such a tensor field $W^*$ is known as $m$-projective curvature tensor.

The properties of the $m$-projective curvature tensor in Sasakian and Kahler manifolds were studied by R. H. Ojha ([**10**], [**11**]). He showed that it bridges the gap among the conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor. Recently this curvature tensor was studied by many geometers, viz., [**14**], [**15**], [**4**]. In [**17**], Venkatesha and B. Sumangala studied $m$-projective curvature tensor in generalized Sasakian space forms and showed that a $(2n+1)$ dimensional $(n > 1)$ generalized Sasakian space form is $m$-projectively flat if and only if $f_3 = \frac{3f-2}{1-2n}$. Motivated these studies, in the present paper, we made an attempt to study the properties of $\phi$-$m$-projectively flat generalized Sasakian space forms.

The present paper is organized as follows: In Section 2, we review some preliminary results. In Section 3, we study $\phi$-$m$-projectively flat generalized Sasakian space forms and prove that a generalized Sasakian space form of dimension greater than three is $\phi$-$m$-projectively flat if and only if it is $m$-projectively flat. Section 4 deals with $m$-projectively locally symmetric generalized Sasakian space forms and it is shown that a generalized Sasakian space form of dimension greater than three is $m$-projectively locally symmetric if and only if it is conformally flat. Section 5 is devoted to the study of $m$-projectively locally $\phi$-symmetric generalized Sasakian space forms. Here we find that an $m$-projectively locally $\phi$-symmetric generalized Sasakian space form of dimension greater than three is also conformally flat and hence $m$-projectively locally symmetric. The last section contains illustrative examples to ensure the validity of the obtained results.

## 2. PRELIMINARIES

In an almost contact metric manifold, we have [**3**]:

$$\phi^2(X) = -X + \eta(X)\xi, \qquad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad g(X,\xi) = \eta(X), \qquad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(\phi X, X) = 0, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \tag{2.5}$$

where $\phi$ is a $(1,1)$ tensor, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a Riemannian metric. The metric $g$ induces an inner product on the manifold.

Again, we know that [**1**] in a generalized Sasakian space form the curvature tensor $R$ of $M$ is given by the equation (1.1), where $f_1$, $f_2$, $f_3$ are smooth functions on the manifold. The Ricci operator $Q$, the Ricci tensor $S$, and the scalar curvature tensor $r$ of the manifold of dimension $(2n+1)$ are given by

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.6}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.7}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{2.8}$$

## 3. $\phi$-$m$-Projectively flat generalized Sasakian space forms

**Definition 3.1.** A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space forms is called $\phi$-$m$-projectively flat if it satisfies

$$\phi^2 W^*(\phi X, \phi Y)\phi Z = 0$$

for any vector fields $X$, $Y$, $Z$ on the manifolds [**12**].

From the definition, it follows that every $m$-projectively flat generalized Sasakian space forms is $\phi$-$m$-projectively flat. In this section, we prove that for a generalized Sasakian space form of dimension greater than three, the converse also holds.

Let us consider a $\phi$-$m$-projectively flat generalized Sasakian space form. Then by definition,

$$\phi^2 W^*(\phi X, \phi Y)\phi Z = 0.$$

In view of (1.2), the above equation yields

$$\begin{aligned}\phi^2[R(\phi X, \phi Y)\phi Z - \frac{1}{4n}\{&S(\phi Y, \phi Z)\phi X - S(\phi X, \phi Z)\phi Y \\ &+ g(\phi Y, \phi Z)Q\phi X - g(\phi X, \phi Z)Q\phi Y\}] = 0.\end{aligned} \tag{3.1}$$

Making use of (1.1), (2.6) and (2.7), we obtain

$$\begin{aligned}&\phi^2[f_1\{g(\phi Y, \phi Z)\phi X - g(\phi X, \phi Z)\phi Y\} \\ &\quad + f_2\{g(\phi X, \phi^2 Z)\phi^2 Y - g(\phi Y, \phi^2 Z)\phi X + 2g(\phi X, \phi^2 Y)\phi^2 Z\}] \\ &= \frac{2}{4n}(2nf_1 + 3f_2 - f_3)\phi^2\{g(\phi Y, \phi Z)\phi X - g(\phi X, \phi Z)\phi^2 Y\}.\end{aligned} \tag{3.2}$$

Applying (2.3), to the above equation we get

$$\begin{aligned}&\phi^2[f_1\{g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y\} \\ &\quad + f_2\{g(X, \phi Z)\phi^2 X - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z\}] \\ &= \frac{1}{2n}(2nf_1 + 3f_2 - f_3)\phi^2\{g(Y,Z)\phi X - \eta(Y)\eta(Z)QX \\ &\quad - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y\}.\end{aligned} \tag{3.3}$$

Making use of (2.1) and (2.2) in the equation (3.3), we have

$$\begin{aligned} & f_1\{g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y\} \\ & \quad + f_2\{g(X,\phi Z)\phi^2 X - g(Y,\phi Z)\phi^2 X + 2g(X,\phi Y)\phi^2 Z\} \\ & = \frac{1}{2n}(2nf_1 + 3f_2 - f_3)\{g(Y,Z)\phi X - \eta(Y)\eta(Z)QX \\ & \quad - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y\}. \end{aligned} \tag{3.4}$$

Taking inner product of both sides of the above equation with respect to an arbitrary vector field $U$, we get

$$\begin{aligned} & f_1\{g(Y,Z)g(\phi X,U) - \eta(Y)\eta(Z)g(\phi X,U) - g(X,Z)g(\phi Y,U) \\ & \quad + \eta(X)\eta(Z)g(\phi Y,U)\} + f_2\{g(X,\phi Z)g(\phi^2 X,U) \\ & \quad - g(Y,\phi Z)g(\phi^2 X,U) + 2g(X,\phi Y)g(\phi^2 Z,U)\} \\ & = \frac{1}{2n}(2nf_1 + 3f_2 - f_3)\{g(Y,Z)g(\phi X,U) - \eta(Y)\eta(Z)g(QX,U) \\ & \quad - g(X,Z)g(\phi Y,U) + \eta(X)\eta(Z)g(\phi Y,U)\}. \end{aligned} \tag{3.5}$$

Putting $Y = Z = e_i$ in the above equation, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, $i = 1, 2, \ldots, 2n+1$, we get

$$3f_2 \; g(X,\phi U) = \frac{3f_2 - f_3}{2n}(2n-1)g(X,\phi U).$$

The above equation is true for any vector fields $X$ and $U$. Let $U \neq X$. Then, it follows from the above definition that

$$3f_2 = \frac{3f_2 - f_3}{2n}(2n-1),$$

which after simplification gives

$$f_3 = \frac{3f_2}{1-2n}. \tag{3.6}$$

From [**17**], it is known that a generalized Sasakian space form of dimension greater than three is $m$-projectively flat if and only if $f_3 = \frac{3f_2}{1-2n}$. Hence, we see that a $\phi$-$m$-projectively flat generalized Sasakian space form is $m$-projectively flat.

Conversely, if the manifold is $m$-projectively flat, then $W^*(X,Y)Z = 0$. From that it trivially follows that $\phi^2 W^*(\phi X, \phi Y)\phi Z = 0$. Therefore, the manifold is $\phi$-$m$-projectively flat. Now, we can state the following theorem.

**Theorem 3.1.** *A* $(2n+1)$ *dimensional* $(n > 1)$ *generalized Sasakian space form is* $\phi$*-*$m$*-projectively flat if and only if it is* $m$*-projectively flat.*

It is known that [**17**] a generalized Sasakian space form of dimension greater than three is $m$-projectively flat if and only if it is Ricci-symmetric. Hence, we can state the following corollary.

**Corollary 3.1.** *A $(2n+1)$ dimensional $(n>1)$ generalized Sasakian space form is $\phi$-$m$-projectively flat if and only if it is Ricci-symmetric.*

If the manifold is flat, then it is $m$-projectively flat. If the manifold is $m$-projectively flat, then from the equations (1.2), (2.6), (2.7) and (3.6), we get

$$R(X,Y)Z = (f_1 - f_3)\{g(Y,Z)X - g(X,Z)Y\}.$$

Thus we obtain the following corollary.

**Corollary 3.2.** *Every flat generalized Sasakian space form is $m$-projectively flat, but the converse is true when $f_1 = f_3$.*

Now, in consequence of Theorem 3.1 and Corollary 3.2, we state the following corollary.

**Corollary 3.3.** *Every flat generalized Sasakian space form is $\phi$-$m$-projectively flat, but the converse is true when $f_1 = f_3$.*

Next we prove that the relation

$$f_3 = \frac{3f_2}{1-2n}$$

implies $f_2 = f_3 = 0$.
In view of (1.1), (2.6) and (2.7), we can write the equation (1.2) as

$$\begin{aligned} W^*(X,Y,Z,U) = {} & f_2\{g(X,\phi Z)g(\phi Y,U) - g(Y,\phi Z)g(\phi X,U) \\ & + 2g(X,\phi Y)g(\phi Z,U)\} + f_3\{\eta(X)\eta(Z)g(Y,U) \\ & - \eta(Y)\eta(Z)g(X,U) + g(X,Z)\eta(Y)\eta(U) \\ & - g(Y,Z)\eta(X)\eta(U) + g(Y,Z)g(X,U) \\ & - g(X,Z)g(Y,U)\}, \end{aligned} \tag{3.7}$$

where $W^*(X,Y,Z,U) = g(W^*(X,Y)Z,U)$.
Replacing $X$ by $\phi X$ and Y by $\phi Y$, we get

$$\begin{aligned} W^*(\phi X,\phi Y,Z,U) = {} & f_2\{g(\phi X,\phi Z)g(\phi^2 Y,U) - g(\phi Y,\phi Z)g(\phi^2 X,U) \\ & + 2g(\phi X,\phi^2 Y)g(\phi Z,U)\} + f_3\{g(\phi Y,Z)g(\phi X,U) \\ & - g(\phi X,Z)g(\phi Y,U)\}. \end{aligned} \tag{3.8}$$

Putting $Y = U = e_i$ in the above equation, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, $i = 1,2,\ldots,2n+1$, we have

$$\begin{aligned} \sum_{i=1}^{2n+1} W^*(\phi X,\phi e_i,Z,e_i) = {} & f_2\{-g(\phi X,\phi Z)g(\phi e_i,\phi e_i) \\ & + g(\phi^2 Z,\phi^2 X) + 2g(\phi^2 X,\phi^2 Z)\} \\ & - f_3 g(\phi Z,\phi X). \end{aligned} \tag{3.9}$$

Again putting $X = Z = e_i$ and taking summation over $i$, we get by virtue of (3.5), $f_2 = 0$ which in view of (3.5) yields $f_3 = 0$.

Conversely, $f_2 = f_3 = 0$ trivially implies $f_3 = \frac{3f_2}{1-2n}$ for $n > 1$.

**Theorem 3.2.** *A* $(2n+1)$ *dimensional* $(n > 1)$ *generalized Sasakian space form is* $\phi$*-m-projectively flat or m-projectively flat if and only if* $f_2 = f_3 = 0$.

It is known that [**9**] a generalized Sasakian space form of dimension greater than three is conformally flat if and only if $f_2 = 0$. Hence, we can say the following corollary.

**Corollary 3.4.** *A* $(2n + 1)$ *dimensional* $(n > 1)$ *generalized Sasakian space form is* $\phi$*-m-projectively flat or m-projectively flat if and only if it is conformally flat.*

## 4. $m$-projectively locally symmetric generalized Sasakian space forms

**Definition 4.1.** Sarkar and Akbar [**16**] defined a $(2n+1)$ dimensional $(n > 1)$ projectively locally symmetric generalized Sasakian space form as

$$(\nabla_U P)(X, Y)Z = 0$$

for all vector fields X,Y,Z orthogonal to $\xi$ and an arbitrary vector field $U$.

Analogous to this definition, we define a (2n+1) dimensional $(n > 1)$ $m$-projectively locally symmetric generalized Sasakian space form as

$$(\nabla_U W^*)(X, Y)Z = 0 \tag{4.1}$$

for all vector fields X,Y,Z orthogonal to $\xi$ and an arbitrary vector field $U$.

From (1.1) and (1.2), we have

$$\begin{aligned} W^*(X, Y)Z = {} & f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ & + 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} - \frac{1}{4n}[S(Y, Z)X - S(X, Z)Y \\ & + g(Y, Z)QX - g(X, Z)QY]. \end{aligned} \tag{4.2}$$

Taking covariant differentiation of both sides of the above equation with respect to an arbitrary vector field $U$, we get

$$\begin{aligned} & (\nabla_U W^*)(X, Y)Z \\ & = df_1(U)\{g(Y, Z)X - g(X, Z)Y\} + df_2(U)\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ & \quad + 2g(X, \phi Y)\phi Z\} + f_2\{g(X, \phi Z)(\nabla_U \phi)Y + g(X, (\nabla_U \phi)Z)\phi Y \\ & \quad - g(Y, \phi Z)(\nabla_U \phi)X - g(Y, (\nabla_U \phi)Z)\phi X + 2g(X, \phi)(\nabla_U \phi)Z \\ & \quad + 2g(X, (\nabla_U \phi)Y)\phi Z\} + df_3(U)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & \quad + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} + f_3\{(\nabla_U \eta)(X)\eta(Z)Y \\ & \quad + \eta(X)(\nabla_U \eta)(Z)Y - (\nabla_U \eta)(Y)\eta(Z)X - \eta(Y)(\nabla_U \eta)(Z)X \\ & \quad + g(X, Z)(\nabla_U \eta)(Y)\xi + g(X, Z)\eta(Y)(\nabla_U \xi) - g(Y, Z)(\nabla_U \eta)(X)\xi \\ & \quad - g(Y, Z)\eta(X)(\nabla_U \xi)\} - \frac{1}{4n}[(\nabla_U S)(Y, Z)X - (\nabla_U S)(X, Z)Y \\ & \quad + g(Y, Z)(\nabla_U Q)(X) - g(X, Z)(\nabla_U Q)(Y)], \end{aligned} \tag{4.3}$$

where $\nabla$ denotes the Riemannian connection on the manifold. Differentiating (2.7) with respect to an arbitrary vector field $U$, we get

$$
\begin{aligned}
&(\nabla_U S)(X,Y)\\
&= d(2nf_1+3f_2-f_3)(U)g(X,Y) - d(3f_2+(2n-1)f_3)(U)\eta(X)\eta(Y) \qquad (4.4)\\
&\quad - (3f_2+(2n-1)f_3)\{(\nabla_U\eta)(X)\eta(Y)+\eta(X)(\nabla_U\eta)(Y)\}.
\end{aligned}
$$

Again differentiating (2.6) covariantly with respect to an arbitrary vector field $U$, we get

$$
\begin{aligned}
&(\nabla_U Q)(X)\\
&= d(2nf_1+3f_2-f_3)(U)X - d(3f_2+(2n-1)f_3(U)\eta(X)\xi \qquad (4.5)\\
&\quad - (3f_2+(2n-1)f_3)\{(\nabla_U\eta)(X)\xi+\eta(X)(\nabla_U\xi)\}.
\end{aligned}
$$

In view of (4.3), (4.4) and (4.5), it follows that

$$
\begin{aligned}
&(\nabla_U W^*)(X,Y)Z\\
&= df_1(U)\{g(Y,Z)X-g(X,Z)Y\} + df_2(U)\{g(X,\phi Z)\phi Y\\
&\quad - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_2\{g(X,\phi Z)(\nabla_U\phi)Y\\
&\quad + g(X,(\nabla_U\phi)Z)\phi Y - g(Y,\phi Z)(\nabla_U\phi)X - g(Y,(\nabla_U\phi)Z)\phi X\\
&\quad + 2g(X,\phi)(\nabla_U\phi)Z + 2g(X,(\nabla_U\phi)Y)\phi Z\} + df_3(U)\{\eta(X)\eta(Z)Y\\
&\quad - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\
&\quad + f_3\{(\nabla_U\eta)(X)\eta Z)Y + \eta(X)(\nabla_U\eta)(Z)Y - (\nabla_U\eta)(Y)\eta Z)X\\
&\quad - \eta(Y)(\nabla_U\eta)(Z)X + g(X,Z)(\nabla_U\eta)(Y)\xi \qquad (4.6)\\
&\quad + g(X,Z)\eta(Y)(\nabla_U\xi) - g(Y,Z)(\nabla_U\eta)(X)\xi - g(Y,Z)\eta(X)(\nabla_U\xi)\}\\
&\quad - \frac{1}{4n}[2d(2nf_1+3f_2-f_3)(U)\{g(Y,Z)X-g(X,Z)Y\}\\
&\quad - d(3f_2+(2n-1)f_3)(U)\{\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\\
&\quad + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\}\\
&\quad - (3f_2+(2n-1)f_3)\{(\nabla_U\eta)(Y)\eta(Z)X + (\nabla_U\eta)(Z)\eta(Y)X\\
&\quad - (\nabla_U\eta)(X)\eta(Z)Y - (\nabla_U\eta)(Z)\eta(X)Y + (\nabla_U\eta)(X)g(Y,Z)\xi\\
&\quad - (\nabla_U\eta)(Y)g(X,Z)\xi + (\nabla_U\xi)\eta(X)g(Y,Z) - (\nabla_U\xi)\eta(Y)g(X,Z)\}].
\end{aligned}
$$

Taking $X, Y, Z$ orthogonal to $\xi$, from the previous equation, we get

$$
\begin{aligned}
&(\nabla_U W^*)(X,Y)Z \\
&= df_1(U)\{g(Y,Z)X - g(X,Z)Y\} \\
&\quad + df_2(U)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
&\quad + f_2\{g(X,\phi Z)(\nabla_U \phi)Y + g(X,(\nabla_U \phi)Z)\phi Y \\
&\quad - g(Y,\phi Z)(\nabla_U \phi)X - g(Y,(\nabla_U \phi)Z)\phi X \\
&\quad + 2g(X,\phi)(\nabla_U \phi)Z + 2g(X,(\nabla_U \phi)Y)\phi Z\} \\
&\quad - \frac{1}{2n} d(2nf_1 + 3f_2 - f_3)(U)\{g(Y,Z)X - g(X,Z)Y\}.
\end{aligned} \tag{4.7}
$$

If the manifold is $m$-projectively locally symmetric, then from the above equation, we get

$$
\begin{aligned}
&\frac{1}{2n} d(2nf_1 + 3f_2 - f_3)(U)\{g(Y,Z)X - g(X,Z)Y\} \\
&= df_1(U)\{g(Y,Z)X - g(X,Z)Y\} \\
&\quad + df_2(U)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
&\quad + f_2\{g(X,\phi Z)(\nabla_U \phi)Y + g(X,(\nabla_U \phi)Z)\phi Y \\
&\quad - g(Y,\phi Z)(\nabla_U \phi)X - g(Y,(\nabla_U \phi)Z)\phi X \\
&\quad + 2g(X,\phi)(\nabla_U \phi)Z + 2g(X,(\nabla_U \phi)Y)\phi Z\}.
\end{aligned} \tag{4.8}
$$

Taking inner product of both sides of the equation (4.8) with $V$, we have

$$
\begin{aligned}
&\frac{1}{2n} d(2nf_1 + 3f_2 - f_3)(U)\{g(Y,Z)g(X,V) - g(X,Z)g(Y,V)\} \\
&= df_1(U)\{g(Y,Z)g(X,V) - g(X,Z)g(Y,V)\} \\
&\quad + df_2(U)\{g(X,\phi Z)g(\phi Y,V) - g(Y,\phi Z)g(\phi X,V) \\
&\quad + 2g(X,\phi Y)g(\phi Z,V)\} \\
&\quad + f_2\{g(X,\phi Z)g((\nabla_U \phi)Y,V) + g(X,(\nabla_U \phi)Z)g(\phi Y,V) \\
&\quad - g(Y,\phi Z)g((\nabla_U \phi)X,V) - g(Y,(\nabla_U \phi)Z)g(\phi X,V) \\
&\quad + 2g(X,\phi)g((\nabla_U \phi)Z,V) + 2g(X,(\nabla_U \phi)Y)g(\phi Z,V)\}.
\end{aligned} \tag{4.9}
$$

Putting $V = Z = e_i$ in the above equation, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$,

$i = 1, 2, \ldots, 2n+1$, we get

$$\begin{aligned} &f_2\{-g(\phi X, (\nabla_U \phi)Y) + \sum_i g(X, (\nabla_U \phi)e_i)g(\phi Y, e_i) + g(\phi Y, (\nabla_U \phi)X) \\ &- \sum_i g((Y, (\nabla_U \phi)e_i)g(\phi X, e_i) + 2\sum_i g(X, \phi Y)g((\nabla_U \phi)e)i, e_i)\} = 0. \end{aligned} \tag{4.10}$$

For a Levi Civita connection $\nabla$, $(\nabla_U g)(X, Y) = 0$, which gives

$$\nabla_U g(X, Y) - g(\nabla_U X, Y) - g(X, \nabla_U Y) = 0.$$

Putting $X = e_i$ and $Y = e_i$ in the above equation, we obtain

$$-g(\nabla_U e_i, \phi e_i) - g(e_i, \nabla_U \phi e_i) = 0,$$

which can be written as

$$g(e_i, \phi \nabla_U e_i) - g(e_i, \nabla_U \phi e_i) = 0.$$

Thus we have

$$g(e_i, (\nabla_U \phi)e_i) = 0. \tag{4.11}$$

By the virtue of (4.11), (4.10) takes the form

$$\begin{aligned} &f_2\{-g(\phi X, (\nabla_U \phi)Y) + \sum_i g(X, (\nabla_U \phi)e_i)g(\phi Y, e_i) \\ &+ g(\phi Y, (\nabla_U \phi)X) - \sum_i g(Y, (\nabla_U \phi)e_i)g(\phi X, e_i). \end{aligned} \tag{4.12}$$

The above equation is true for any vector fields $X$, $Y$ on the manifold. For $X \neq Y$, the above equation yields $f_2 = 0$.

It is known that [**9**] a generalized Sasakian space form of dimension greater than three is conformally flat if and only if $f_2 = 0$. Hence the manifold under consideration is conformally flat.

Conversely, suppose that the manifold is conformally flat. Hence $f_2 = 0$. In addition, if we consider $X$, $Y$, $Z$ orthogonal to $\xi$, then (1.1) yields

$$R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\}.$$

The above equation gives

$$r = 2n(2n+1)f_1. \tag{4.13}$$

In view of (2.9) and (4.13), we obtain $f_3 = 0$. Hence from (4.7), we get

$$(\nabla_U W^*)(X, Y)Z = 0.$$

Therefore, the manifold is $m$-projectively locally symmetric. Now, we are in position to state the following theorem.

**Theorem 4.1.** *A $(2n+1)$ dimensional $(n > 1)$ generalized Sasakian space form is $m$-projectively locally symmetric if and only if it is conformally flat.*

## 5. $m$-projectively locally $\phi$-symmetric generalized Sasakian space forms

**Definition 5.1.** A generalized Sasakian space form of dimension greater than three is called $m$-projectively locally $\phi$-symmetric if it satisfies

$$\phi^2(\nabla_U W^*)(X,Y)Z = 0$$

for all vector fields $X$, $Y$, $Z$ orthogonal to $\xi$.

Let us consider a $m$-projectively locally $\phi$-symmetric generalized Sasakian space from of dimension greater than three. Then form the definition and (2.1), we get

$$-(\nabla_U W^*)(X,Y)Z + \eta((\nabla_U W^*)(X,Y)Z)\xi = 0. \tag{5.1}$$

Taking inner product of both sides of the above equation with respect to an arbitrary vector field $U$, we obtain

$$-g((\nabla_U W^*)(X,Y)Z,U) + \eta((\nabla_U W^*)(X,Y)Z)\eta(U) = 0. \tag{5.2}$$

If we take $U$ orthogonal to $\xi$, then the above equation yields

$$g((\nabla_U W^*)(X,Y)Z,U) = 0. \tag{5.3}$$

The Equation (5.3) is true for all $U$ orthogonal to $\xi$. If we choose $U \neq 0$ and not orthogonal to $(\nabla_U W^*)(X,Y)Z$, then it follows that

$$(\nabla_U W^*)(X,Y)Z = 0.$$

Hence, the manifold is $m$-projectively locally $\phi$-symmetric and so by theorem, it is conformally flat.

Conversely, let the manifold be conformally flat and hence $f_2 = 0$. Again for $X, Y, Z$ orthogonal to $\xi$, $f_2 = 0$ implies $f_3 = 0$, as before. From (4.7), we get $(\nabla_U W^*)(X,Y)Z = 0$, which implies that

$$\phi^2(\nabla_U W^*)(X,Y)Z = 0,$$

where $X, Y, Z$ are orthogonal to $\xi$. Therefore, the manifold is $m$-projectively locally $\phi$-symmetric.
This leads to the following theorem.

**Theorem 5.1.** *A $(2n+1)$ dimensional $(n > 1)$ generalized Sasakian space form is m-projectively locally $\phi$-symmetric if and only if it is conformally flat.*

Combining the results of Section 3, Section 4 and Section 5, we find the following corollary.

**Corollary 5.1.** *In a $(2n+1)$ dimensional $(n > 1)$ generalized Sasakian space form, the following conditions are equivalent:*
(i) *the manifold is m-projectively flat,*
(ii) *the manifold is $\phi$-m-projectively flat,*
(iii) *the manifold is conformally flat,*
(iv) *the manifold is m-projectively locally symmetric,*
(v) *the manifold is m-projectively locally $\phi$-symmetric.*

**Remark 5.1.** The notion of quarter-symmetric metric connection was introduced by S. Golab [**8**]. The torsion tensor of the quarter-symmetric metric connection is given by

$$T(X, Y) = \eta(Y)X - \eta(X)Y.$$

If $X, Y$ are orthogonal to $\xi$, then the torsion tensor vanishes and quarter-symmetric metric connection reduces to Levi-Civita connection. Therefore, all the results of the last two sections are of the same form with respect to quarter-symmetric metric connection and Levi-Civita connection.

**Example 5.1.** In[1], it is shown that $\mathbb{R} \times_f \mathbb{C}^m$ is a generalized Sasakian space form with

$$f_1 = -\frac{(f')^2}{f^2}, \qquad f_2 = 0, \qquad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$, $t \in \mathbb{R}$ and $f'$ denotes derivative of $f$ with respect to $t$. If we choose $m = 4$ and $f(t) = \mathrm{e}^t$, then $M$ is a 5-dimensional conformally flat generalized Sasakian space form because $f_2 = 0$. Consequently, we see that $f_3 = 0$. Therefore, by the results obtained in the present paper, $M$ is $m$-projectively flat, $\phi$-$m$-projectively flat, conformally flat, $m$-projectively locally symmetric and $m$-projectively locally $\phi$-symmetric.

**Example 5.2.** For a Sasakian space form of dimension greater than three and of constant $\phi$-sectional curvature 1, $f_1 = 0$, $f_2 = f_3 = 0$. So, by the results obtained in the present paper, the manifold is $m$-projectively flat, $\phi$-$m$-projectively flat, conformally flat, $m$-projectively locally symmetric and $m$-projectively locally $\phi$-symmetric.

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