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PETROVIĆ-TYPE INEQUALITIES: GENERALIZATIONS AND EXTENSIONS

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Abstract. In this paper, we consider the classes of h-convex and m-convex functions and prove some continuous versions of Petrović-type inequalities. Then, as an application of these results, we provide some new properties for the generalized Caputo-type fractional integrals.

1. Introduction

Integral inequalities belong to the class of mathematical inequalities playing a significant leading role in different fields of science and technology, such as physical, biological and engineering sciences, approximation theory and spectral analysis, statistical analysis and theory of distributions, financial economics, and computer science. Hence, it is not surprising that in last years many researchers have paid attention to generalizations of classical integral inequalities with the purpose of using them as tools in the description of several new properties of integral operators linked with different types of fractional derivatives (see, e.g., [**6, 7, 12, 15, 23, 25, 29, 30, 31, 35, 36**]).

As is pointed out in [**34**], historically, both Liouville and Riemann introduced the notion of fractional integral and their formulations associated to the unsolved problem of the so-called complementary function. However, it was not until 1884 with the work of H. Laurent that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. Since then, there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types

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of fractional derivatives. Some of the inequalities studied are Gronwall, Chebyshev, Jensen, Hermite–Hadamard-type, Ostrowski-type, Opial-type, Grüss-type, Hardy-type, Gagliardo–Nirenberg-type, reverse Minkowski and reverse Hölder inequalities (see, e.g., [**12, 17, 15, 5, 19, 23, 25, 8, 29, 30, 31, 35, 36**]).

We refer the interested reader to the classical books [**14, 18, 21, 22, 28**], and [**13, 24**] and the references cited therein for a detailed exposition about the fundamentals of some recent trends of the research in this broad field.

Motivated by the recent papers [**20, 7**], in which the authors obtain, respectively, a new Petrović-type inequality (for convex functions) and a new Jensen-type (for m-convex functions), and apply these inequalities to generalized Caputo-type fractional integrals and generalized Riemann–Liouville-type integral operators, in the present work we consider a similar situation for the classes of h-convex and m-convex functions. In each case, we obtain new Petrović-type inequalities, and as a consequence, we derive some new properties for the generalized Caputo-type fractional integrals defined in [**6**], which include most of known Caputo-type fractional integrals [**9, 10, 11**].

The current paper is organized as follows. The first part of Section 2 provides some short background about the Caputo, Caputo–Fabrizio and generalized Caputo derivatives and their associated integral operators, the second one deals with the celebrated Petrović inequality and one of its recent analogs in the setting of h-convex functions [**32**]. The main results of this paper about h-harmonic and m-harmonic functions are given in Sections 3 and 4, respectively (see Theorems 3.1 and 4.3). Furthermore, as an application of our results, we obtain new properties for the general fractional integrals of Caputo-type (see Propositions 3.3 and 4.4). Finally, some concluding remarks are given in Section 5.

2. Preliminaries

Michele Caputo proposed a new definition of fractional derivative in [**9**]. This definition has an important property associated with the resolution of differential equations, since it is not necessary to define the initial conditions of fractional order. For multiple applications of the so-called Caputo differential operator, the interested reader is referred to [**10**].

The *Caputo derivative* of a differentiable function g of order $0 < \alpha < 1$ is defined as

$$ {}^{C}D_a^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{g'(s)}{(t-s)^{\alpha}}\, \mathrm{d}s. \tag{1}$$

An extension of ${}^{C}D^{\alpha}$ is the so-called *Caputo–Fabrizio derivative* (see [**3, 11**]), given by

$$ {}^{CF}D_a^{\alpha} g(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t g'(s) \mathrm{e}^{-\frac{\alpha(t-s)}{1-\alpha}}\, \mathrm{d}s, \tag{2}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

A more recent extension is the *Atangana–Baleanu derivative*, defined in [**1**] by

$$ {}^{ABC}D_a^\alpha g(t) = \frac{M(\alpha)}{1-\alpha}\int_a^t g'(s)E_\alpha\Big(-\frac{\alpha(t-s)^\alpha}{1-\alpha}\Big)\,\mathrm{d}s, \tag{3} $$

where

$$ E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} $$

is the *Mittag–Leffler function.* Note that $E_1(z) = \mathrm{e}^z$.

Definition 2.1. We say that F is an *admissible kernel* for the interval $[a,b]$ if $F\colon [0,b-a]\times(0,1)\to[0,\infty)$ is a non-negative continuous function such that

$$ \mathbb{F}(\alpha) = \int_0^{b-a}\frac{\mathrm{d}s}{F(s,\alpha)} < \infty $$

for each $\alpha\in(0,1)$. F is an *admissible kernel* for $[a,\infty)$ if it is admissible for $[a,b]$ for every $b>a$.

Next, we present the definition of the generalized Caputo derivative in [**6**].

Definition 2.2. Let $\alpha\in(0,1)$, F be an admissible kernel for $[a,b]$, $g\colon[a,b]\to\mathbb{R}$ be a differentiable function and $t\in[a,b]$. The *generalized Caputo derivative* of order α of the function g at the point t is

$$ {}^{C}D_{F,a}^\alpha g(t) = \int_a^t\frac{g'(s)}{F(t-s,\alpha)}\,\mathrm{d}s. \tag{4} $$

Remark 2.3. If $F(x,\alpha)=\Gamma(1-\alpha)\,x^\alpha$, then we obtain the classical Caputo derivative. Similarly, we can obtain the kernels for Caputo–Fabrizio and Atangana–Baleanu extensions. Hence, each result for ${}^{C}D_{F,a}^\alpha$ has as a consequence the same result for the classical Caputo derivative, and Caputo–Fabrizio and Atangana–Baleanu extensions. This shows that the generalized Caputo derivative is quite general and unifying.

The following integral operator is associated to the generalized Caputo derivative in a natural way.

Definition 2.4. Let $\alpha\in(0,1)$, F be an admissible kernel for $[a,b]$, $g\colon[a,b]\to\mathbb{R}$ be a differentiable function and $t\in[a,b]$. The *generalized Caputo integral operator* of order α of the function g at the point t is

$$ {}^{C}J_{F,a}^\alpha g(t) = \int_a^t\frac{g(s)}{F(t-s,\alpha)}\,\mathrm{d}s. $$

Hence,

$$ {}^{C}D_{F,a}^\alpha g(t) = {}^{C}J_{F,a}^\alpha g'(t). $$

We will use also the functional defined by

$$ {}^{C}J_{F}^\alpha(g) = {}^{C}J_{F,a}^\alpha g(b) = \int_a^b\frac{g(s)}{F(b-s,\alpha)}\,\mathrm{d}s. $$

It is clear that the above operators are non-local linear operators.

The following result summarizes some properties of the generalized Caputo derivative and its associated integral operator (see [**6**]).

Proposition 2.5. *Let $\alpha \in (0,1)$, $x \in [a,b]$, and let F be an admissible kernel for $[a,b]$. If g, g_1, g_2 are differentiable functions on $[a,x]$ and $\mathbb{F}_\alpha = \min_{y\in[0,x-a]} F(y,\alpha) > 0$, then*

$$\begin{aligned} \left\| {}^C J^{\alpha}_{F,a}(g) \right\|_{L^\infty[a,x]} &\le \frac{1}{\mathbb{F}_\alpha} \left\| g \right\|_{L^1[a,x]}, \\ \left\| {}^C D^{\alpha}_{F,a} g \right\|_{L^\infty[a,x]} &\le \frac{1}{\mathbb{F}_\alpha} \left\| g' \right\|_{L^1[a,x]}, \\ \left\| {}^C J^{\alpha}_{F,a}(g_1) - {}^C J^{\alpha}_{F,a}(g_2) \right\|_{L^\infty[a,x]} &\le \frac{1}{\mathbb{F}_\alpha} \left\| g_1 - g_2 \right\|_{L^1[a,x]}, \\ \left\| {}^C D^{\alpha}_{F,a} g_1 - {}^C D^{\alpha}_{F,a} g_2 \right\|_{L^\infty[a,x]} &\le \frac{1}{\mathbb{F}_\alpha} \left\| g_1' - g_2' \right\|_{L^1[a,x]}. \end{aligned}$$

On the other hand, the famous Petrović inequality [**26**] is stated as follows:

Theorem 2.6. *Let f be a convex function on $[0,a]$, and $p_1, \dots, p_n \ge 0$. If $x_1, \dots, x_n \in [0,a]$ satisfy $\sum_{k=1}^n x_k p_k \in (0,a]$, and*

$$x_j \le \sum_{k=1}^n x_k p_k, \qquad j = 1, \dots, n,$$

then

$$\sum_{k=1}^n f(x_k) p_k \le f\Big(\sum_{k=1}^n x_k p_k \Big) + \Big(\sum_{k=1}^n p_k - 1 \Big) f(0).$$

There are many generalizations of Petrović inequality (see, e.g., [**4, 16, 27, 32, 33**] and the references therein). We are mainly interested in establishing some continuous analog of Petrović inequality in the setting of the notion of h-convexity [**37**]. So, we begin formulating the well-known concept of h convex function and some recent results about a Petrović type inequality [**32**].

Definition 2.7. Let $[c,d]$ be an interval containing $(0,1)$ and $h\colon [c,d] \to \mathbb{R}$ be a non-negative function. A function $f\colon [a,b] \to \mathbb{R}$ is said to be an h-convex function if f is non-negative, and for all $x, y \in [a,b]$, $\alpha \in (0,1)$, the following inequality holds:

$$f(\alpha x + (1-\alpha) y) \le h(\alpha) f(x) + h(1-\alpha) f(y). \tag{5}$$

If the inequality (5) is reversed, then f is said to be h-concave.

The notion of h-convexity was introduced by S. Varošanec in [**37**]. The most important feature of this notion is that it allows the unified treatment of several varieties of convexity. For instance, when $h(\alpha) = \alpha$ in (5), we recover the classical definition of convex function. If $h(\alpha) = 1$, then (5) gives the definition of P-functions. And the definitions of s-convex functions in the second sense, Godunova–Levin functions, and s-Godunova–Levin functions in the second sense appear, respectively, when we take in (5) $h(\alpha) = \alpha^s$, $h(\alpha) = \frac{1}{\alpha}$ and $h(\alpha) = \frac{1}{\alpha^s}$.

Definition 2.8. A function $h\colon [c,d] \to \mathbb{R}$ is said to be a submultiplicative function if

$$h(xy) \le h(x)h(y) \quad \text{if } x, y, xy \in [c,d]. \tag{6}$$

If the inequality (6) is reversed, then h is said to be a *supermultiplicative* function. If (6) becomes into an equality, then h is said to be a *multiplicative* function.

Note that the examples of functions h before Definition 2.8 are multiplicative functions.

Regarding an analog of Petrović inequality in the setting of h-convex functions, we have the following recent result due to Rehman et al. [**32**, Theorem 2.1].

Theorem 2.9. *Let $x_1, \dots, x_n \ge 0$ and $p_1, \dots, p_n \ge 0$ such that $\sum_{k=1}^n p_k > 0$,*

$$0 \le c \le x_j \le \sum_{k=1}^n p_k x_k \le a \quad \text{for } j = 1, \dots, n.$$

Also let $h\colon [0,a] \to \mathbb{R}^+$ be a supermultiplicative function such that

$$h(\alpha) + h(1-\alpha) \le 1 \quad \text{for all } \alpha \in (0,1).$$

If $f\colon [0,a] \to \mathbb{R}$ is an h-convex function on $[0,a]$, then

$$\begin{aligned}\sum_{k=1}^n p_k f(x_k) \le{}& \frac{\sum_{k=1}^n p_k h(x_k - c)}{h\left(\sum_{k=1}^n p_k x_k - c\right)} f\left(\sum_{k=1}^n p_k x_k\right)\\ &+ \left(\sum_{k=1}^n p_k - \frac{\sum_{k=1}^n p_k h(x_k - c)}{h\left(\sum_{k=1}^n p_k x_k - c\right)}\right) f(c).\end{aligned}$$

3. Continuous versions of Petrović inequality for h-convex functions

We will show now a continuous version of Theorem 2.9.

Theorem 3.1. *Let μ be a finite measure on the space X and let $g\colon X \to [0,a]$ be a μ-integrable function such that*

$$0 \le c \le g(x) \le \int_X g \,\mathrm{d}\mu \le a \quad \text{for } \mu\text{-a.e. } x \in X. \tag{7}$$

Also let $h\colon [0,a] \to \mathbb{R}^+$ be a continuous supermultiplicative function such that

$$h(\alpha) + h(1-\alpha) \le 1 \quad \text{for all } \alpha \in (0,1).$$

If $f\colon [0,a] \to \mathbb{R}$ is a continuous h-convex function on $[0,a]$, then $f \circ g$ and $h(g-c)$ are μ-integrable and

$$\int_X f \circ g \,\mathrm{d}\mu \le \frac{\int_X h(g-c)\,\mathrm{d}\mu}{h\left(\int_X g\,\mathrm{d}\mu - c\right)} f\left(\int_X g\,\mathrm{d}\mu\right) + \left(\mu(X) - \frac{\int_X h(g-c)\,\mathrm{d}\mu}{h\left(\int_X g\,\mathrm{d}\mu - c\right)}\right) f(c). \tag{8}$$

Proof. Without loss of generality, we can assume that (7) holds for every $x \in X$, since if F is the set of points in X such that (7) does not hold, we can define $g = \int_X g \, \mathrm{d}\mu$ on F. This modification does not change the inequality (8).

Assume first that g is constant μ-a.e. $g = C_1$. Thus, Theorem 2.9 with $n = 1$, $x_1 = C_1$ and $p_1 = \mu(X)$ yields (8).

Assume now that g is not constant μ-a.e. Then, $c < \beta = \int_X g \, \mathrm{d}\mu$.

For each $n \geq 1$ and $1 \leq k < 2^n$, consider the sets

$$\begin{aligned} U_{n,k} &= \big(c + (k-1)2^{-n}(\beta - c),\, c + k2^{-n}(\beta - c)\big], \\ U_{n,2^n} &= \big(c + (2^n - 1)2^{-n}(\beta - c),\, \beta\big), \\ U_{n,0} &= \{c\}, \\ U_{n,2^n+1} &= \{\beta\}. \end{aligned}$$

Note that $\{U_{n,k}\}_{k=0}^{2^n+1}$ is a partition of $[c, \beta]$ for each $n \geq 1$.

For each $n \geq 1$ and $0 \leq k \leq 2^n + 1$, define the sets $E_{n,k} = g^{-1}\big(U_{n,k}\big)$ and choose constants $a_{n,k} \in U_{n,k}$ satisfying

$$a_{n,k}\, \mu\big(E_{n,k}\big) = \int_{E_{n,k}} g \, \mathrm{d}\mu.$$

Thus, $a_{n,0} = c$ and $a_{n,2^n+1} = \beta$.

Since g is a measurable function satisfying $0 \leq c \leq g \leq \beta$, we have that $\{E_{n,k}\}_{k=0}^{2^n+1}$ are pairwise disjoint measurable sets and $X = \cup_{k=0}^{2^n+1} E_{n,k}$ for each $n \geq 1$.

Recall that the characteristic function of a set E is defined as $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. If we define

$$g_n = \sum_{k=0}^{2^n+1} a_{n,k}\, \chi_{E_{n,k}},$$

then

$$\begin{aligned} \int_X g_n \, \mathrm{d}\mu &= \sum_{k=0}^{2^n+1} a_{n,k}\, \mu(E_{n,k}) \\ &= \sum_{k=0}^{2^n+1} \int_{E_{n,k}} g \, \mathrm{d}\mu = \int_X g \, \mathrm{d}\mu. \end{aligned}$$

It is clear that

$$\begin{aligned} 0 \leq c \leq a_{n,k} &\leq \int_X g \, \mathrm{d}\mu = \int_X g_n \, \mathrm{d}\mu, \\ 0 \leq c \leq g_n &\leq \int_X g \, \mathrm{d}\mu = \int_X g_n \, \mathrm{d}\mu, \\ |g_n - g| &\leq 2^{-n}(\beta - c), \end{aligned}$$

and so, g_n uniformly converges to g.

Since $\{E_{n,k}\}_{k=0}^{2^n+1}$ are pairwise disjoint sets and $X = \cup_{k=0}^{2^n+1} E_{n,k}$, we have

$$f \circ g_n = \sum_{k=0}^{2^n+1} f(a_{n,k})\chi_{E_{n,k}},$$
$$\int_X f \circ g_n \,\mathrm{d}\mu = \sum_{k=0}^{2^n+1} f(a_{n,k})\mu(E_{n,k}),$$
$$h(g_n - c) = \sum_{k=0}^{2^n+1} h(a_{n,k} - c)\chi_{E_{n,k}},$$
$$\int_X h(g_n - c)\,\mathrm{d}\mu = \sum_{k=0}^{2^n+1} h(a_{n,k} - c)\mu(E_{n,k}).$$

Since

$$c \le a_{n,j} \le \int_X g\,\mathrm{d}\mu = \int_X g_n\,\mathrm{d}\mu = \sum_{k=0}^{2^n+1} a_{n,k}\,\mu(E_{n,k}),$$

for $0 \le j \le 2^n + 1$, the assumptions in Theorem 2.9 are satisfied, and hence,

$$\begin{aligned}\sum_{k=0}^{2^n+1} f\left(a_{n,k}\right)\mu(E_{n,k}) \le{}& \frac{\sum_{k=0}^{2^n+1} h\left(a_{n,k}-c\right)\mu(E_{n,k})}{h\left(\sum_{k=0}^{2^n+1} a_{n,k}\mu(E_{n,k}) - c\right)} f\left(\sum_{k=0}^{2^n+1} a_{n,k}\mu(E_{n,k})\right)\\ &+\left(\sum_{k=0}^{2^n+1}\mu(E_{n,k}) - \frac{\sum_{k=0}^{2^n+1} h\left(a_{n,k}-c\right)\mu(E_{n,k})}{h\left(\sum_{k=0}^{2^n+1} a_{n,k}\mu(E_{n,k}) - c\right)}\right) f(c).\end{aligned} \tag{9}$$

Then, since $\int_X g_n\,\mathrm{d}\mu = \int_X g\,\mathrm{d}\mu$ for every $n \ge 1$, the right-hand side of (9) is equal to

$$\frac{\int_X h(g_n - c)\,\mathrm{d}\mu}{h\left(\int_X g\,\mathrm{d}\mu - c\right)} f\left(\int_X g\,\mathrm{d}\mu\right) + \left(\mu(X) - \frac{\int_X h(g_n - c)\,\mathrm{d}\mu}{h\left(\int_X g\,\mathrm{d}\mu - c\right)}\right) f(c).$$

Since $0 \le c \le g_n \le \beta = \int_X g\,\mathrm{d}\mu \le a$ for every n, $\lim_{n\to\infty} g_n = g$ and f, h are continuous functions on $[0, \beta] \subseteq [0, a]$, we have $\lim_{n\to\infty} f \circ g_n = f \circ g$ and $\lim_{n\to\infty} h(g_n - c) = h(g - c)$.

Since f, h are continuous functions on $[0, \beta]$, there exists a constant M with $|f|, |h| \le M$ on $[0, \beta]$, and so, $|f \circ g_n|, |h(g_n - c)| \le M$ for every n.

Since μ is a finite measure, $M \in L^1(X, \mu)$ and dominated convergence theorem gives that $f \circ g, h(g - c)$ are μ-integrable and

$$\lim_{n\to\infty} \int_X f \circ g_n\,\mathrm{d}\mu = \int_X f \circ g\,\mathrm{d}\mu,$$
$$\lim_{n\to\infty} \int_X h(g_n - c)\,\mathrm{d}\mu = \int_X h(g - c)\,\mathrm{d}\mu.$$

Therefore, the right-hand side of (9) has limit

$$\frac{\int_X h(g-c)\,\mathrm{d}\mu}{h\left(\int_X g\,\mathrm{d}\mu - c\right)}\, f\left(\int_X g\,\mathrm{d}\mu\right) + \left(\mu(X) - \frac{\int_X h(g-c)\,\mathrm{d}\mu}{h\left(\int_X g\,\mathrm{d}\mu - c\right)}\right) f(c).$$

Also, the left-hand side of (9) has limit

$$\lim_{n\to\infty} \sum_{k=0}^{2^n+1} f(a_{n,k})\mu(E_{n,k}) = \lim_{n\to\infty} \int_X f\circ g_n\,\mathrm{d}\mu = \int_X f\circ g\,\mathrm{d}\mu.$$

These facts finish the proof. □

The following result is a straightforward consequence of Theorem 3.1 just by taking $c = 0$. It can be considered as a continuous version of Petrović inequality for h-convex functions.

Theorem 3.2. *Let μ be a finite measure on the space X and let $g\colon X \to [0,a]$ be a μ-integrable function such that*

$$g(x) \le \int_X g\,\mathrm{d}\mu \le a \quad \text{for } \mu\text{-a.e. } x \in X. \tag{10}$$

Also let $h\colon [0,a] \to \mathbb{R}^+$ be a continuous supermultiplicative function such that

$$h(\alpha) + h(1-\alpha) \le 1 \quad \text{for all } \alpha \in (0,1).$$

If $f\colon [0,a] \to \mathbb{R}$ is a continuous h-convex function on $[0,a]$, then $f\circ g$ and $h\circ g$ are μ-integrable and

$$\int_X f\circ g\,\mathrm{d}\mu \le \frac{\int_X h\circ g\,\mathrm{d}\mu}{h\left(\int_X g\,\mathrm{d}\mu\right)}\, f\left(\int_X g\,\mathrm{d}\mu\right) + \left(\mu(X) - \frac{\int_X h\circ g\,\mathrm{d}\mu}{h\left(\int_X g\,\mathrm{d}\mu\right)}\right) f(0). \tag{11}$$

Theorem 3.1 has the following direct consequence for the general fractional integrals of Caputo-type.

Proposition 3.3. *Let $a > 0$, $a_1 < b_1$, $\alpha \in (0,1)$ and $c \ge 0$. Let $g\colon [a_1,b_1] \to [0,a]$ and $h\colon [0,a] \to \mathbb{R}^+$ be measurable functions, and let F be an admissible kernel for the interval $[a_1,b_1]$ such that:*

(1) *h is a continuous supermultiplicative function satisfying*

$$h(t) + h(1-t) \le 1 \quad \text{for all } t \in (0,1).$$

(2) *The following inequalities hold*

$$0 \le c \le g(x) \le \int_{a_1}^{b_1} \frac{g(s)}{F(b_1-s,\alpha)}\,\mathrm{d}s \le a \quad \text{for a.e. } x \in [a_1,b_1].$$

If $f\colon [0,a] \to \mathbb{R}$ is a continuous h-convex function on $[0,a]$, then $f(g(s))/F(b_1 - s,\alpha) \in L^1[a_1,b_1]$ and

$$\int_{a_1}^{b_1} \frac{f(g(s))}{F(b_1-s,\alpha)}\,\mathrm{d}s \le A\, f\left(\int_{a_1}^{b_1} \frac{g(s)}{F(b_1-s,\alpha)}\,\mathrm{d}s\right) + (\mathbb{F}(\alpha) - A)\, f(c),$$

i.e.,

$$ {}^{C}J_F^{\alpha}\,(f\circ g) \le A\, f\left({}^{C}J_F^{\alpha}(g)\right) + (\mathbb{F}(\alpha)-A)\, f(c), $$

with

$$ \mathbb{F}(\alpha) = \int_0^{b_1-a_1} \frac{ds}{F(s,\alpha)} $$

and

$$ A = \frac{\int_{a_1}^{b_1} \frac{h(g(s)-c)}{F(b_1-s,\alpha)}\,\mathrm{d}s}{h\left(\int_{a_1}^{b_1} \frac{g(s)}{F(b_1-s,\alpha)}\,\mathrm{d}s - c\right)}. $$

4. Continuous versions of Petrović-type inequalities for m-convex functions

Definition 4.1. Let $I \subseteq \mathbb{R}$ be an interval containing 0, and let $m \in (0,1]$. A function $f\colon I \to \mathbb{R}$ is said to be m-convex if the inequality

$$ f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y) \tag{12} $$

holds for every pair of points $x, y \in I$ and every coefficient $t \in [0,1]$.

If $m \in (0,1)$, then the hypothesis $0 \in I$ guarantees that $tx + m(1-t)y \in I$.

It is clear, by taking $m = 1$ in Definition 4.1, that we recover the concept of classical convex functions on I. Note that in this case it is not necessary the hypothesis $0 \in I$, since $tx + (1-t)y \in I$ for every $x, y \in I$.

Note that if we choose the coefficient $t = 0$ in (12), we get the inequality $f(my) \le mf(y)$. Also, Definition 4.1 is equivalent to

$$ f(mtx + (1-t)y) \le mtf(x) + (1-t)f(y) \tag{13} $$

for all $x, y \in I$ and $t \in [0,1]$.

The following discrete Petrović-type inequality for m-convex functions can be deduced from [**2**, Corollary 3.4]:

Theorem 4.2. *Let f be an m-convex function on $[0,a]$ with $m \in (0,1]$. Let $p_1, \dots, p_n \ge 0$ and $x_1, \dots, x_n \in [0,a]$ satisfy $\sum_{k=1}^n p_k x_k/m \in (0,a]$, and*

$$ x_j \le \sum_{k=1}^n p_k x_k, \qquad j = 1, \dots, n, $$

then

$$ \sum_{k=1}^n p_k f(x_k) \le \min\left\{ mf\left(\frac{\sum_{k=1}^n p_k x_k}{m}\right) + \left(\sum_{k=1}^n p_k - 1\right) f(0), \right. $$
$$ \left. f\left(\sum_{k=1}^n p_k x_k\right) + m\left(\sum_{k=1}^n p_k - 1\right) f(0) \right\}. $$

Next, we present a continuous version of the above Petrović-type inequality.

Theorem 4.3. *Let μ be a finite measure on the space X, let f be a continuous m-convex function on $[0,a]$ and let $g\colon X \to [0,a]$ be a μ-integrable function with $\int_X g\,\mathrm{d}\mu/m \in (0,a]$. Then $f\circ g$ is a μ-integrable function. If*

$$g(x) \le \int_X g\,\mathrm{d}\mu \tag{14}$$

for μ-a.e. $x \in X$, then

$$\begin{aligned}\int_X f\circ g\,\mathrm{d}\mu \le \min\Bigg\{ & m f\left(\frac{\int_X g\,\mathrm{d}\mu}{m}\right) + (\mu(X)-1)\,f(0),\\ & f\left(\int_X g\,\mathrm{d}\mu\right) + m\,(\mu(X)-1)\,f(0)\Bigg\}.\end{aligned} \tag{15}$$

Proof. Without loss of generality, we can assume that (14) holds for every $x \in X$, since if F is the set of points in X such that (14) does not hold, we can define $g = \int_X g\,\mathrm{d}\mu$ on F. This modification does not change the inequality (15).

Assume first that g is constant μ-almost everywhere $g = C_1$. Thus, Theorem 4.2 with $n = 1$, $x_1 = C_1$ and $p_1 = \mu(X)$ gives the conclusion.

Assume now that g is not constant μ-a.e. We have

$$0 \le g(x) \le \alpha = \int_X g\,\mathrm{d}\mu$$

for μ-a.e. $x \in X$ and $0 < \alpha = \int_X g\,\mathrm{d}\mu$, since g is not constant μ-a.e. Since f is a continuous function on $[0,a]$, $f\circ g$ is a bounded function on X. Since μ is a finite measure, $f\circ g$ is a μ-integrable function.

For each $n \ge 1$ and $1 \le k < 2^n$, let be the sets

$$\begin{aligned} I_{n,0} &= \{0\},\\ I_{n,k} &= ((k-1)2^{-n}\alpha, k2^{-n}\alpha]\\ I_{n,2^n} &= ((2^n-1)2^{-n}\alpha, \alpha)\\ I_{n,2^n+1} &= \{\alpha\}.\end{aligned}$$

Note that $\{I_{n,k}\}_{k=0}^{2^n+1}$ is a partition of $[0,\alpha]$ for each $n \ge 1$.

For each $n \ge 1$ and $0 \le k \le 2^n+1$, define the sets

$$S_{n,k} = g^{-1}(I_{n,k})$$

and choose constants $a_{n,k} \in I_{n,k}$ satisfying

$$a_{n,k}\,\mu(S_{n,k}) = \int_{S_{n,k}} g\,\mathrm{d}\mu.$$

It is clear that $a_{n,0} = 0$ and $a_{n,2^n+1} = \alpha$.

Since g is a measurable function satisfying $0 \le g \le \alpha$, we have that $\{S_{n,k}\}_{k=0}^{2^n+1}$ are pairwise disjoint measurable sets and $X = \cup_{k=0}^{2^n+1} S_{n,k}$ for each $n \ge 1$.

Recall that χ_E denotes the characteristic function of a set E. If we define

$$g_n = \sum_{k=0}^{2^n+1} a_{n,k}\,\chi_{S_{n,k}},$$

then g_n are simple functions and

$$\int_X g_n \,\mathrm{d}\mu = \sum_{k=0}^{2^n+1} a_{n,k}\,\mu(S_{n,k}) = \sum_{k=0}^{2^n+1} \int_{S_{n,k}} g \,\mathrm{d}\mu = \int_X g \,\mathrm{d}\mu.$$

Therefore, for each $n \geq 1$ and $0 \leq k \leq 2^n + 1$,

$$0 \leq a_{n,k} \leq \int_X g \,\mathrm{d}\mu = \int_X g_n \,\mathrm{d}\mu, \qquad 0 \leq g_n \leq \int_X g \,\mathrm{d}\mu, \qquad |g_n - g| \leq 2^{-n}\alpha.$$

Hence, g_n uniformly converges to g and

$$0 \leq g_n \leq \int_X g \,\mathrm{d}\mu = \int_X g_n \,\mathrm{d}\mu.$$

Since $\{S_{n,k}\}_{k=0}^{2^n+1}$ are pairwise disjoint sets and $X = \cup_{k=0}^{2^n+1} S_{n,k}$, we have that $f \circ g_n$ is a μ-integrable function for every $n \geq 1$ and

$$f \circ g_n = \sum_{k=0}^{2^n+1} f(a_{n,k})\chi_{S_{n,k}},$$

$$\int_X f \circ g_n \,\mathrm{d}\mu = \sum_{k=0}^{2^n+1} f(a_{n,k})\mu(S_{n,k}).$$

Since

$$0 \leq a_{n,j} \leq \int_X g \,\mathrm{d}\mu = \int_X g_n \,\mathrm{d}\mu = \sum_{k=0}^{2^n+1} a_{n,k}\,\mu(S_{n,k})$$

for $n \geq 1$ and $0 \leq j \leq 2^n + 1$, by applying the discrete Petrović-type inequality in Theorem 4.2, we obtain

(16)

$$\begin{aligned}\sum_{k=0}^{2^n+1} f(a_{n,k})\mu(S_{n,k}) &\leq \min\Bigg\{ mf\left(\frac{\sum_{k=0}^{2^n+1} a_{n,k}\,\mu(S_{n,k})}{m}\right) + (\mu(X)-1)\,f(0),\\ &\qquad f\Bigg(\sum_{k=0}^{2^n+1} a_{n,k}\,\mu(S_{n,k})\Bigg) + m\,(\mu(X)-1)\,f(0)\Bigg\}\\ &= \min\left\{ mf\left(\frac{\int_X g\,\mathrm{d}\mu}{m}\right) + (\mu(X)-1)\,f(0),\right.\\ &\qquad \left. f\left(\int_X g\,\mathrm{d}\mu\right) + m\,(\mu(X)-1)\,f(0)\right\}.\end{aligned}$$

Hence, it suffices to prove that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n+1} f(a_{n,k})\mu(S_{n,k}) = \int_X f \circ g \,\mathrm{d}\mu.$$

Since $0 \leq g_n \leq \alpha = \int_X g\,\mathrm{d}\mu$ for every n, $\lim_{n\to\infty} g_n = g$ and f is a continuous function on $[0,\alpha]$, we have $\lim_{n\to\infty} f \circ g_n = f \circ g$.

Since f is a continuous function on $[0,\alpha]$, there exists a constant M with $|f| \le M$ on $[0,\alpha]$, and so, $|f \circ g_n| \le M$ on X for every n.

Since μ is finite measure, $M \in L^1(X,\mu)$ and dominated convergence theorem gives

$$\lim_{n\to\infty} \int_X f \circ g_n \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} f \circ g_n \,\mathrm{d}\mu = \int_X f \circ g \,\mathrm{d}\mu.$$

Therefore, we conclude

$$\lim_{n\to\infty} \sum_{k=0}^{2^n+1} f(a_{n,k})\mu(S_{n,k}) = \lim_{n\to\infty} \int_X f \circ g_n \,\mathrm{d}\mu = \int_X f \circ g \,\mathrm{d}\mu,$$

and this fact finishes the proof. □

Theorem 4.3 has the following direct consequence for the general fractional integrals of Caputo-type.

Proposition 4.4. *Let $a > 0$, $a_1 < b_1$, $\alpha \in (0,1)$ and $m \in (0,1]$. Assume that F is an admissible kernel for the interval $[a_1,b_1]$ and $g\colon [a_1,b_1] \to [0,a]$ is a measurable function satisfying the following inequalities:*

$$0 < \int_{a_1}^{b_1} \frac{g(s)}{mF(b_1-s,\alpha)} \,\mathrm{d}s \le a,$$

$$g(x) \le \int_{a_1}^{b_1} \frac{g(s)}{F(b_1-s,\alpha)} \,\mathrm{d}s \qquad \text{for } \mu\text{-a.e. } x \in [a_1,b_1].$$

If f is a continuous m-convex function on $[0,a]$, then $f(g(s))/F(b_1-s,\alpha) \in L^1[a_1,b_1]$ and

$$\int_{a_1}^{b_1} \frac{f(g(s))}{F(b_1-s,\alpha)} \,\mathrm{d}s \le \min\left\{ mf\left(\frac{A}{m}\right) + (\mathbb{F}(\alpha)-1)f(0), f(A) + m(\mathbb{F}(\alpha)-1)f(0) \right\},$$

i.e.,

$${}^{C}J_F^{\alpha}(f\circ g) \le \min\left\{ mf\left(\frac{{}^{C}J_F^{\alpha}(g)}{m}\right) + (\mathbb{F}(\alpha)-1)f(0), f\left({}^{C}J_F^{\alpha}(g)\right) + m(\mathbb{F}(\alpha)-1)f(0) \right\},$$

with

$$\mathbb{F}(\alpha) = \int_0^{b_1-a_1} \frac{ds}{F(s,\alpha)} \qquad \text{and} \qquad A = \int_{a_1}^{b_1} \frac{g(s)}{F(b_1-s,\alpha)} \,\mathrm{d}s = {}^{C}J_F^{\alpha}(g).$$

5. Concluding remarks

The main goal of our research is to obtain new Petrović-type inequalities for h-convex and m-convex functions, and apply them to the general fractional integrals of Caputo-type. In particular, we prove continuous versions of the discrete Petrović-type inequalities in [**2, 32**], in the setting of h-convexity and m-convexity. Furthermore, as an application of these inequalities, we provide some new properties for the general fractional integrals of Caputo-type.

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