PSEUDO-UMBILICAL CR-SUBMANIFOLD OF AN ALMOST HERMITIAN MANIFOLD

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Abstract. In this paper, we firstly study differentiable functions on $M$, where $M$ is a pseudo-umbilical CR-submanifold of an almost Hermitian manifold, then give a theorem which concerns the geodesic character of $M$, and extend Bejancu and Chen B.Y.'s conclusions.

1. Introduction

Let $M$ be a real differentiable manifold. An almost complex structure on $M$ is a tensor field $J$ of type $(1, 1)$ on $M$ such that at every point $x \in M$ we have $J^2 = -I$, where $I$ denotes the identify transformation of $T_xM$. A manifold $M$ endowed with an almost complex structure is called an almost complex manifold.

A Hermitian metric on an almost complex manifold $M$ is a Riemannian metric $g$ satisfying

$$g(JX, JY) = g(X, Y)$$

for any $X, Y \in \Gamma(TM)$.

An almost Hermitian manifold $M$ with Levi-Civita connection $\nabla$ is called a Kaehlerian manifold if we have $\nabla X J = 0$ for any $X \in \Gamma(TM)$.

Let $M$ be an $m$-dimensional Riemannian submanifold of an $n$-dimensional Riemannian manifold $\overline{M}$. We denote by $TM^\perp$ the normal bundle to $M$ and by $g$ both metric on $M$ and $\overline{M}$. Also, by $\nabla$ we denote the Levi-Civita connection on $\overline{M}$, by $\nabla^\perp$ denote the induced connection on $M$, by $\nabla^\perp$ and denote the induced normal connection on $M$.

Then, for any $X, Y \in \Gamma(TM)$, we have

$$\nabla_X Y = \nabla_X Y + h(X, Y),$$

where $h: \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.2) is called the Gauss formula and $h$ is called the second fundamental form of $M$.

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Now, for any \( X \in \Gamma(TM) \) and \( V \in \Gamma(TM^\perp) \) by \(-AVX\) and \( \nabla_X^\perp V\) we denote the tangent part and normal part of \( \nabla_X V\), respectively. Then we have

\[
\nabla_X V = -AVX + \nabla_X^\perp V.
\]

Thus, for any \( V \in \Gamma(TM^\perp) \), we have a linear operator, satisfying

\[
g(AVX, Y) = g(X, AVY) = g(h(X, Y), V).
\]

The equation (1.3) is called the Weingarten formula.

**Definition 1.1 ([1])**

Let \( M \) be a real \( n \)-dimensional almost Hermitian manifold with almost complex structure \( J \) and with Hermitian metric \( g \). Let \( M \) be a real \( m \)-dimensional Riemannian manifold isometrically immersed in \( M \). Then \( M \) is called a CR-submanifold of \( M \) if there exists a differentiable distribution \( D: x \mapsto D_x \subset T_x M \), on \( M \) satisfying the following conditions:

1. \( D \) is holomorphic, that is, \( J(D_x) = D_x \) for each \( x \in M \),
2. the complementary orthogonal distribution \( D^\perp: x \mapsto D^\perp_x \subset T_x M \),

is anti-invariant, that is, \( J(D^\perp_x) \subset T_x M^\perp \) for each \( x \in M \).

Let \( M \) be a CR-submanifold of an almost Hermitian manifold \( M \), then we have the orthogonal decomposition

\[
TM^\perp = JD^\perp \oplus \nu.
\]

By \( r \) denote the complex dimension of \( \nu_x (x \in M) \). Since \( \nu \) is a holomorphic vector bundle, we can take a local field of orthonormal frames on \( TM^\perp \)

\[
\{JE_1, JE_2, \cdots, JE_q, V_1, V_2, \cdots, V_r, V_{r+1} = JV_1, V_{r+2} = JV_2, \cdots, V_{2r} = JV_r\}
\]

where \( \{E_1, E_2, \cdots, E_q\} \) is a local field of orthonormal frames on \( D^\perp \). Then we let

\[
A_i = A_{JE_i}, \quad A_\alpha = A_{V_\alpha}, \quad A_\alpha^* = A_{V_\alpha^*},
\]

where

\[
i, j, k, \cdots = 1, \cdots, q; \alpha, \beta, \gamma, \cdots = 1, \cdots, r; \alpha^*, \beta^*, \gamma^* \cdots = r + 1, \cdots, 2r.
\]

**Definition 1.2 ([1])**

The CR-submanifold \( M \) is said to be pseudo-umbilical if the fundamental tensors of Weingarten are given by

\[
A_i X = a_i X + b_i g(X, E_i) E_i,
\]

\[
A_\alpha X = a_\alpha X + \sum_{i=1}^q b^i_\alpha g(X, E_i) E_i,
\]

\[
A_\alpha^* X = a_\alpha^* X + \sum_{i=1}^q b^{i*}_\alpha g(X, E_i) E_i,
\]

where \( a_i, b_i, a_\alpha, a_\alpha^*, b^i_\alpha, b^{i*}_\alpha \) are differential functions on \( M \) and \( X \in \Gamma(TM) \).

Now let \( M \) be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold \( M \). For each vector field \( X \) tangent to \( M \), we put

\[
JX = \phi X + \omega X,
\]
where $\phi X$ and $\omega X$ are the tangent part and the normal part of $JX$, respectively. Also, for each vector field $V$ normal to $M$, we put
\begin{equation}
J V = BV + CV,
\end{equation}
where $BV$ and $CV$ are the tangent part and the normal part of $JV$, respectively.

The covariant derivative of $B$, $C$, respectively, is defined by
\begin{equation}
(\nabla_X B)V = \nabla^\perp_X BV - B \nabla^\perp_X V,
\end{equation}
\begin{equation}
(\nabla_X C)V = \nabla^\perp_X CV - C \nabla^\perp_X V
\end{equation}
for all $X \in \Gamma(TM), V \in \Gamma(TM^\perp)$.

A CR-submanifold $M$ of an almost Hermitian manifold $\overline{M}$ is $D$-geodesic if we have
\begin{equation}
\nonumber h(X, Y) = 0
\end{equation}
for any $X, Y \in \Gamma(D)$. $M$ is mixed geodesic if we have
\begin{equation}
\nonumber h(X, Y) = 0
\end{equation}
for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$.

2. Main Results

**Theorem 2.1** ([1]). Let $M$ be a CR-submanifold of an almost Hermitian manifold $\overline{M}$, then $M$ is mixed geodesic if and only if
\begin{align*}
A_V X &\in \Gamma(D), \quad A_V U \in \Gamma(D^\perp) \\
A_V X &\in \Gamma(D), \quad A_V U \in \Gamma(TM)
\end{align*}
for each $X \in \Gamma(D), U \in \Gamma(D^\perp), V \in \Gamma(TM)$.

**Theorem 2.2.** Let $M$ be a pseudo-umbilical CR-submanifold of an almost Hermitian manifold $\overline{M}$, then $M$ is mixed geodesic.

**Proof.** For each $X \in \Gamma(D), Y \in \Gamma(D^\perp)$, according to the Definition 1.2 we get
\begin{align*}
A_i X &= a_i X \in \Gamma(D), \quad A_\alpha X = a_\alpha X \in \Gamma(D), \quad A_\alpha^* X = a_\alpha^* X \in \Gamma(D) \\
A_i Y &= a_i Y + b_i g(Y, E_i) E_i \in \Gamma(D^\perp), \\
A_\alpha Y &= a_\alpha Y + \sum_{i=1}^q b^\alpha_{i, g}(Y, E_i) E_i \in \Gamma(D^\perp), \\
A_\alpha^* Y &= a_\alpha^* Y + \sum_{i=1}^q b^\alpha_{i* g}(Y, E_i) E_i \in \Gamma(D^\perp).
\end{align*}
The assertion follows from Theorem 2.1.

Since a Kaehlerian manifold is an almost Hermitian manifold, we obtain the following corollary.

**Corollary 2.1** ([1]). Any pseudo-umbilical CR-submanifold of a Kaehlerian manifold is mixed geodesic.
Lemma 2.1. Let \( M \) be a pseudo-umbilical CR-submanifold of an almost Hermitian manifold \( \mathcal{M} \), then

\[
g(A_{JV}X - JA_{V}X + (\nabla_X J)V, Z) = 0
\]

for all \( X, Z \in \Gamma(D), V \in \Gamma(\nu) \).

Proof. Let \( X, Z \in \Gamma(D), V \in \Gamma(\nu) \). From Weingarten formula and (1.1), we get

\[
g(A_{JV}X - JA_{V}X, Z) = g(-\nabla_X JV, Z) + g(A_{V}X, JZ)
\]

(2.1)

\[
= -g(\nabla_X JV, Z) + g(J\nabla_X V, Z)
\]

(2.2)

\[
= -g((\nabla_X J)V, Z).
\]

The proof is now complete from (2.2). \( \square \)

Lemma 2.2. Let \( M \) be a CR-submanifold of an almost Hermitian manifold \( \mathcal{M} \). Then we have

\[
(\nabla_X B)V = \nabla_X BV - B\nabla_X^\perp V
\]

(2.3)

\[
= A_{CV}X - \phi A_{V}X + ((\nabla_X J)V)^\perp
\]

for all \( X \in \Gamma(TM), V \in \Gamma(TM^\perp) \).

Proof. Let \( X \in \Gamma(TM), V \in \Gamma(TM^\perp) \). From (1.10) and Weingarten formula, we obtain

\[
(\nabla_X J)V = \nabla_X JV - J\nabla_X
\]

(2.4)

\[
= \nabla_X (BV + CV) + J(A_{V}X - \nabla_X^\perp V).
\]

By using the Gauss formula, we get

\[
\nabla_X (BV + CV) = \nabla_X BV + h(X, BV) - A_{CV}X + \nabla_X^\perp CV.
\]

(2.5)

Taking account of (1.9) and (1.10), we have

\[
J(A_{V}X - \nabla_X^\perp V) = \phi A_{V}X + \omega A_{V}X - B\nabla_X^\perp V - C\nabla_X^\perp V.
\]

(2.6)

From (2.5), (2.6), (1.11) and (1.12), (2.4) can become

\[
(\nabla_X J)V = (\nabla_X B)V + h(X, BV) - A_{CV}X
\]

(2.7)

\[
+ (\nabla_X^\perp C)V + \phi A_{V}X + \omega A_{V}X
\]

By comparing to the tangent part in (2.7), (2.3) is satisfied. \( \square \)

Theorem 2.3. Let \( M \) be a pseudo-umbilical proper CR-submanifold of an almost Hermitian manifold \( \mathcal{M} \). If \( q > 1 \), then we have \( A_{E_i}E_i = A_{\alpha X}X = A_{\alpha, X}X = 0 \) for all \( X \in \Gamma(D), i \neq j \).

Proof. From (1.4) and (1.6), we obtain

\[
g(A_{E_i}E_i, E_i) = g(A_{E_j}E_j, E_i) = 0,
\]

thus \( A_{E_i}E_i \in \Gamma(D) \). On the other hand, \( A_{E_i}E_i = a_{j}E_i + b_{i}g(E_i, E_j)E_j = a_{j}E_i \in \Gamma(D^\perp) \), hence \( A_{E_i} = 0 \).
For a unit vector $X \in \Gamma(D)$, by using (1.7), (1.1), (2.1) and (1.8) we have
\[(2.8)\]
\[a_{\alpha} = g(a_{\alpha}X, X) = g(A_{\alpha}X, X) = g(A_{\alpha}X + (\nabla_X J)V_{\alpha}, JX) = g(a_{\alpha}X + (\nabla_X J)V_{\alpha}, JX) = a_{\alpha}X + g(\nabla_X J)V_{\alpha}, JX) = g((\nabla_X J)V_{\alpha})^T, JX).\]

Taking (2.3) into account, (2.9) can become
\[(2.10)\]
\[a_{\alpha} = g(-A_{\alpha}V_{\alpha}X + \phi A_{\alpha}V_{\alpha}X, JX) = g(-a_{\alpha}X, JX) + g(A_{\alpha}X, X).\]

From (2.8) and (2.10), we have
\[(2.11)\]
\[g(-a_{\alpha}X, JX) = 0,\]
thus $A_{\alpha}X \in \Gamma(D^\perp)$. On the other hand, $A_{\alpha}X = a_{\alpha}X + \sum_{i=1}^q b_{\alpha}^i g(X, E_i)E_i = a_{\alpha}X \in \Gamma(D)$, hence $A_{\alpha}X = 0$.

In a similar way we get $A_{\alpha}X = 0$.

For $E_i \in \Gamma(D^\perp)$ and a unit vector field $X \in \Gamma(D)$, from $a_i = g(A_i E_i, E_i)$, $a_{\alpha} = g(A_{\alpha}X, X)$ and (2.8), according to the Theorem 2.3, we have the following theorem.

**Theorem 2.4.** Let $M$ be a pseudo-umbilical proper CR-submanifold of an almost Hermitian manifold $M$. If $q > 1$, then $a_j = a_{\alpha} = a_{\alpha}^* = 0$.

Since a Kaehlerian manifold is an almost Hermitian manifold, we obtain

**Corollary 2.2 ([1]).** Let $M$ be a pseudo-umbilical proper CR-submanifold of a Kaehlerian manifold $M$. If $q > 1$, then the functions $a_j, a_{\alpha}, a_{\alpha}^*$ vanish identically on $M$.

**Theorem 2.5.** Let $M$ be a pseudo-umbilical proper CR-submanifold of an almost Hermitian manifold $\overline{M}$. If $q > 1$, then $M$ is $D$-geodesic.

**Proof.** Taking account of Definition 1.2 and Theorem 2.4, we get
\[(2.12)\]
\[g(h(X, Y), \sum_{i=1}^q J E_i + \sum_{\alpha=1}^r V_{\alpha} + \sum_{\alpha^*=r+1}^{2r} V_{\alpha^*}) = \sum_{i=1}^q g(A_{\alpha}E_i, X, Y) + \sum_{\alpha=1}^r g(A_{\alpha}V_{\alpha}, X, Y) + \sum_{\alpha^*=r+1}^{2r} g(A_{\alpha^*}X, Y) = \sum_{i=1}^q b_{\alpha}^i g(X, E_i)g(Y, E_i) + \sum_{\alpha=1}^r \sum_{i=1}^q b_{\alpha}^i g(X, E_i)g(Y, E_i) + \sum_{\alpha^*=r+1}^{2r} \sum_{i=1}^q b_{\alpha}^i g(X, E_i)g(Y, E_i).\]
for all $X, Y \in \Gamma(D)$.

From (2.12), we have

$$g(h(X, Y), \sum_{i=1}^{q} JE_i + \sum_{\alpha=1}^{r} V_{\alpha} + \sum_{\alpha^* = r+1}^{2r} V_{\alpha^*}) = 0,$$

so $h(X, Y) = 0$, i.e., $M$ is $D$-geodesic.

References


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