Acta Math. Univ. Comenianae
Vol. LXXXIII, 2 (2014), pp. 311–316

PSEUDO-UMBILICAL CR-SUBMANIFOLD OF AN ALMOST HERMITIAN MANIFOLD

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Abstract. In this paper, we firstly study differentiable functions on M, where M is a pseudo-umbilical CR-submanifold of an almost Hermitian manifold, then give a theorem which concerns the geodesic character of M, and extend Bejancu and Chen B. Y.'s conclusions.

1. Introduction

Let $\overline{M}$ be a real differentiable manifold. An almost complex structure on $\overline{M}$ is a tensor field J of type (1, 1) on $\overline{M}$ such that at every point $x \in \overline{M}$ we have $J^2 = -I$, where I denotes the identify transformation of $T_x\overline{M}$. A manifold $\overline{M}$ endowed with an almost complex structure is called an almost complex manifold.

A Hermitian metric on an almost complex manifold $\overline{M}$ is a Riemannian metric g satisfying

$$g(JX, JY) = g(X, Y) \tag{1.1}$$

for any $X, Y \in \Gamma(T\overline{M})$.

An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a Kaehlerian manifold if we have$\overline{\nabla}_X J = 0$ for any $X \in \Gamma(T\overline{M})$.

Let M be an m-dimensional Riemannian submanifold of an n-dimensional Riemannian manifold $\overline{M}$. We denote by $TM^\perp$ the normal bundle to M and by g both metric on M and $\overline{M}$. Also, by $\overline{\nabla}$ we denote the Levi-Civita connection on $\overline{M}$, by ∇ denote the induced connection on M, by $\nabla^\perp$ and denote the induced normal connection on M.

Then, for any $X, Y \in \Gamma(TM)$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.2}$$

where $h\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.2) is called the Gauss formula and h is called the second fundamental form of M.

Received October 28, 2013.
2010 *Mathematics Subject Classification.* Primary 53C25, 58A30.
Key words and phrases. almost Hermitian manifold; CR-submanifold; distribution; connection; pseudo-umbilical.
Supported by Foundation of Department of Science and Technology of Hunan Province (No. 2010SK3023).

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$ by $-A_V X$ and $\nabla^\perp_X V$ we denote the tangent part and normal part of $\overline{\nabla}_X V$, respectively. Then we have

$$\overline{\nabla}_X V = -A_V X + \nabla^\perp_X V. \tag{1.3}$$

Thus, for any $V \in \Gamma(TM^\perp)$, we have a linear operator, satisfying

$$g(A_V X, Y) = g(X, A_V Y) = g(h(X,Y), V). \tag{1.4}$$

The equation (1.3) is called the Weingarten formula.

Definition 1.1 ([**1**])**.** Let $\overline{M}$ be a real n-dimensional almost Hermitian manifold with almost complex structure J and with Hermitian metric g. Let M be a real m-dimensional Riemannian manifold isometrically immersed in $\overline{M}$. Then M is called a CR-submanifold of $\overline{M}$ if there exists a differentiable distribution $D\colon x \to D_x \subset T_x M$, on M satisfying the following conditions:

(1) D is holomorphic, that is, $J(D_x) = D_x$ for each $x \in M$,

(2) the complementary orthogonal distribution $D^\perp\colon x \to D^\perp_x \subset T_x M$,

is anti-invariant, that is, $J(D^\perp_x) \subset T_x M^\perp$ for each $x \in M$.

Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$, then we have the orthogonal decomposition

$$TM^\perp = JD^\perp \oplus \nu. \tag{1.5}$$

By r denote the complex dimension of $\nu_x (x \in M)$, Since ν is a holomorphic vector bundle, we can take a local field of orthonormal frames on $TM^\perp$

$$\{JE_1, JE_2, \cdots, JE_q, V_1, V_2, \cdots, V_r, V_{r+1} = JV_1, V_{r+2} = JV_2, \cdots, V_{2r} = JV_r\}$$

where $\{E_1, E_2, \cdots, E_q\}$ is a local field of orthonormal frames on $D^\perp$. Then we let

$$A_i = A_{JE_i}, \quad A_\alpha = A_{V_\alpha}, \quad A_{\alpha^*} = A_{V_{\alpha^*}},$$

where

$$i, j, k, \cdots = 1, \cdots, q; \alpha, \beta, \gamma, \cdots = 1, \cdots, r; \alpha^*, \beta^*, \gamma^* \cdots = r+1, \cdots, 2r.$$

Definition 1.2 ([**1**])**.** The CR-submanifold M is said to be pseudo-umbilical if the fundamental tensors of Weingarten are given by

$$A_i X = a_i X + b_i g(X, E_i) E_i, \tag{1.6}$$

$$A_\alpha X = a_\alpha X + \sum_{i=1}^{q} b^i_\alpha g(X, E_i) E_i, \tag{1.7}$$

$$A_{\alpha^*} X = a_{\alpha^*} X + \sum_{i=1}^{q} b^i_{\alpha^*} g(X, E_i) E_i, \tag{1.8}$$

where $a_i, b_i, a_\alpha, a_{\alpha^*}, b^i_\alpha, b^i_{\alpha^*}$ are differential functions on M and $X \in \Gamma(TM)$.

Now let M be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold $\overline{M}$. For each vector field X tangent to M, we put

$$JX = \phi X + \omega X, \tag{1.9}$$

where ϕX and ωX are the tangent part and the normal part of JX, respectively. Also, for each vector field V normal to M, we put

$$JV = BV + CV, \tag{1.10}$$

where BV and CV are the tangent part and the normal part of JV, respectively.

The covariant derivative of B, C, respectively, is defined by

$$(\nabla_X B)V = \nabla_X^{\perp} BV - B\nabla_X^{\perp} V, \tag{1.11}$$

$$(\nabla_X C)V = \nabla_X^{\perp} CV - C\nabla_X^{\perp} V \tag{1.12}$$

for all $X \in \Gamma(TM), V \in \Gamma(TM^{\perp})$.

A CR-submanifold M of an almost Hermitian manifold $\overline{M}$ is D-geodesic if we have

$$h(X, Y) = 0 \tag{1.13}$$

for any $X, Y \in \Gamma(D)$. M is mixed geodesic if we have

$$h(X, Y) = 0 \tag{1.14}$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$.

2. Main Results

Theorem 2.1 ([**1**])**.** *Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$, then M is mixed geodesic if and only if*

$$A_V X \in \Gamma(D), \qquad A_V U \in \Gamma(D^{\perp})$$

for each $X \in \Gamma(D)$, $U \in \Gamma(D^{\perp})$, $V \in \Gamma(TM)$.

Theorem 2.2. *Let M be a pseudo-umbilical CR-submanifold of an almost Hermitian manifold $\overline{M}$, then M is mixed geodesic.*

Proof. For each $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$, according to the Definition 1.2 we get

$$A_i X = a_i X \in \Gamma(D), \qquad A_\alpha X = a_\alpha X \in \Gamma(D), \qquad A_{\alpha^*} X = a_{\alpha^*} X \in \Gamma(D)$$

and

$$A_i Y = a_i Y + b_i g(Y, E_i)E_i \in \Gamma(D^{\perp}),$$

$$A_\alpha Y = a_\alpha Y + \sum_{i=1}^{q} b_\alpha^i g(Y, E_i)E_i \in \Gamma(D^{\perp}),$$

$$A_{\alpha^*} Y = a_{\alpha^*} Y + \sum_{i=1}^{q} b_{\alpha^*}^i g(Y, E_i)E_i \in \Gamma(D^{\perp}).$$

The assertion follows from Theorem 2.1. □

Since a Kaehlerian manifold is an almost Hermitian manifold, we obtain the following corollary.

Corollary 2.1 ([**1**])**.** *Any pseudo-umbilical CR-submanifold of a Kaehlerian manifold is mixed geodesic.*

Lemma 2.1. *Let M be a pseudo-umbilical CR-submanifold of an almost Hermitian manifold $\overline{M}$, then*

$$g(A_{JV}X - JA_VX + (\overline{\nabla}_XJ)V, Z) = 0 \tag{2.1}$$

for all $X, Z \in \Gamma(D)$, $V \in \Gamma(\nu)$.

Proof. Let $X, Z \in \Gamma(D)$, $V \in \Gamma(\nu)$. From Weingarten formula and (1.1), we get

$$\begin{aligned} g(A_{JV}X - JA_VX, Z) &= g(-\overline{\nabla}_XJV, Z) + g(A_VX, JZ) \\ &= -g(\overline{\nabla}_XJV, Z) + g(J\overline{\nabla}_XV, Z) \\ &= -g((\overline{\nabla}_XJ)V, Z). \end{aligned} \tag{2.2}$$

The proof is now complete from (2.2). □

Lemma 2.2. *Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then we have*

$$\begin{aligned} (\nabla_XB)V &= \nabla_XBV - B\nabla^{\perp}_XV \\ &= A_{CV}X - \phi A_VX + ((\overline{\nabla}_XJ)V)^{\top} \end{aligned} \tag{2.3}$$

for all $X \in \Gamma(TM)$, $V \in \Gamma(TM^{\perp})$.

Proof. Let $X \in \Gamma(TM), V \in \Gamma(TM^{\perp})$. From (1.10) and Weingarten formula, we obtain

$$\begin{aligned} (\overline{\nabla}_XJ)V &= \overline{\nabla}_XJV - J\overline{\nabla}_X \\ &= \overline{\nabla}_X(BV + CV) + J(A_VX - \nabla^{\perp}_XV). \end{aligned} \tag{2.4}$$

By using the Gauss formula, we get

$$\overline{\nabla}_X(BV + CV) = \nabla_XBV + h(X, BV) - A_{CV}X + \nabla^{\perp}_XCV. \tag{2.5}$$

Taking account of (1.9) and (1.10), we have

$$J(A_VX - \nabla^{\perp}_XV) = \phi A_VX + \omega A_VX - B\nabla^{\perp}_XV - C\nabla^{\perp}_XV. \tag{2.6}$$

From (2.5), (2.6), (1.11) and (1.12), (2.4) can become

$$\begin{aligned} (\overline{\nabla}_XJ)V = \ &(\nabla_XB)V + h(X, BV) - A_{CV}X \\ &+ (\nabla^{\perp}_XC)V + \phi A_VX + \omega A_VX \end{aligned} \tag{2.7}$$

By comparing to the tangent part in (2.7), (2.3) is satisfied. □

Theorem 2.3. *Let M be a pseudo-umbilical proper CR-submanifold of an almost Hermitian manifold $\overline{M}$. If $q > 1$, then we have $A_jE_i = A_\alpha X = A_{\alpha^*}X = 0$ for all $X \in \Gamma(D)$, $i \neq j$.*

Proof. From (1.4) and (1.6), we obtain

$$g(A_{JE_j}E_j, E_i) = g(A_{JE_j}E_i, E_j) = 0,$$

thus $A_{JE_j}E_i \in \Gamma(D)$. On the other hand, $A_{JE_j}E_i = a_jE_i + b_jg(E_i, E_j)E_j = a_jE_i \in \Gamma(D^{\perp})$, hence $A_jE_i = 0$.

For a unit vector $X \in \Gamma(D)$,by using (1.7), (1.1), (2.1) and (1.8) we have

$$\begin{aligned} a_\alpha &= g(a_\alpha X, X) = g(A_\alpha X, X) \\ &= g(A_{\alpha^*} X + (\overline{\nabla}_X J)V_\alpha, JX) \\ &= g(a_{\alpha^*} X + (\overline{\nabla}_X J)V_\alpha, JX) \\ &= a_{\alpha^*} g(X, JX) + g((\overline{\nabla}_X J)V_\alpha, JX) \end{aligned} \tag{2.8}$$

$$= g(((\overline{\nabla}_X J)V_\alpha)^\top, JX). \tag{2.9}$$

Taking (2.3) into account, (2.9) can become

$$\begin{aligned} a_\alpha &= g(-A_{CV_\alpha} X + \phi A_{V_\alpha} X, JX) \\ &= g(-A_{\alpha^*} X, JX) + g(A_\alpha X, X). \end{aligned} \tag{2.10}$$

From (2.8) and (2.10), we have

$$g(-A_{\alpha^*} X, JX) = 0, \tag{2.11}$$

thus $A_{\alpha^*} X \in \Gamma(D^\perp)$. On the other hand, $A_{\alpha^*} X = a_{\alpha^*} X + \sum_{i=1}^q b^i_{\alpha^*} g(X, E_i) E_i = a_{\alpha^*} X \in \Gamma(D)$, hence $A_{\alpha^*} X = 0$.

In a similar way we get $A_\alpha X = 0$. □

For $E_i \in \Gamma(D^\perp)$ and a unit vector field $X \in \Gamma(D)$, from $a_i = g(A_i E_j, E_j)$, $a_{\alpha^*} = g(A_{\alpha^*} X, X)$ and (2.8), according to the Theorem 2.3, we have the following theorem.

Theorem 2.4. *Let M be a pseudo-umbilical proper CR-submanifold of an almost Hermitian manifold $\overline{M}$. If $q > 1$, then $a_j = a_\alpha = a_{\alpha^*} = 0$.*

Since a Kaehlerian manifold is an almost Hermitian manifold, we obtain

Corollary 2.2 ([**1**]). *Let M be a pseudo-umbilical proper CR-submanifold of a Kaehlerian manifold $\overline{M}$. If $q > 1$, then the functions a_j, a_α, a_{α^*} vanish identically on M.*

Theorem 2.5. *Let M be a pseudo-umbilical proper CR-submanifold of an almost Hermitian manifold $\overline{M}$. If $q > 1$, then M is D-geodesic.*

Proof. Taking account of Definition 1.2 and Theorem 2.4, we get

$$\begin{aligned} &g(h(X,Y), \sum_{i=1}^q JE_i + \sum_{\alpha=1}^r V_\alpha + \sum_{\alpha^*=r+1}^{2r} V_{\alpha^*}) \\ &\quad = \sum_{i=1}^q g(A_{JE_i} X, Y) + \sum_{\alpha=1}^r g(A_{V_\alpha} X, Y) + \sum_{\alpha^*=r+1}^{2r} g(A_{\alpha^*} X, Y) \\ &\quad = \sum_{i=1}^q b_i g(X, E_i) g(Y, E_i) + \sum_{\alpha=1}^r \sum_{i=1}^q b^i_\alpha g(X, E_i) g(Y, E_i) \\ &\qquad + \sum_{\alpha^*=r+1}^{2r} \sum_{i=1}^q b^i_{\alpha^*} g(X, E_i) g(Y, E_i) \end{aligned} \tag{2.12}$$

for all $X, Y \in \Gamma(D)$. From (2.12), we have

$$g(h(X,Y), \sum_{i=1}^{q} JE_i + \sum_{\alpha=1}^{r} V_\alpha + \sum_{\alpha^*=r+1}^{2r} V_{\alpha^*}) = 0,$$

so $h(X,Y) = 0$, i.e., M is D-geodesic. $\square$

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