Acta Math. Univ. Comenianae
Vol. XCIV, 3 (2025), pp. 161–180

A COUPLED SYSTEM OF RIEMANN-LIOUVILE TYPE FRACTIONAL LANGEVIN EQUATIONS WITH NONLOCAL MULTI-POINT AND MULTI-STRIP COUPLED BOUNDARY CONDITIONS

A. ALSAEDI, H. A. SAEED, B. AHMAD AND S. K. NTOUYAS

Abstract. In this paper, we investigate the existence and uniqueness of solutions for a coupled system of nonlinear Riemann-Liouvile type fractional Langevin equations equipped with nonlocal multi-point and multi-strip coupled boundary conditions. We make use of Leray-Schauder's alternative and Banach's fixed point theorem to derive the desired results, which are well-illustrated with examples. Our results are useful in the given configuration and enrich the literature on boundary value problems for fractional Langevin equations.

1. Introduction

Langevin in [**34**] applied Newtonian dynamics to a Brownian particle and presented an analytical approach to deal with the random processes. In fact, he invented Newton's second law of motion for stochastic physics, which is known as the Langevin equation. Langevin (1872–1946) was a French physicist and contemporary of Einstein. In his own words, his approach to Brownian motion was "infinitely more simple" than the one offered by Einstein.

However, it was found later that Langevin equation failed to describe the complex systems. This finding led to some generalizations of Langevin equation to formulate the physical phenomena in disordered regions [**28**], fluctuation-dissipation configuration [**29**], statistical physics [**30**], etc. The fractional analogue of Langevin equation was proposed by replacing the ordinary derivative in it by fractional order derivative in [**18**], while the Langevin equation in terms of two different fractional orders was considered in [**36**]. Other variants of Langevin equation can be found in the papers [**19, 45**].

The topic of initial and boundary value problems is an important and interesting area of research as such problems constitute mathematical models associated with real-world problems. One can observe an overwhelming interest in the investigation

Received October 9, 2024; revised July 1, 2025.
2020 *Mathematics Subject Classification.* Primary 34A08, 34B10, 34B15.
Key words and phrases. Fractional Langevin equation; system; nonlocal multi-point and multi-strip boundary conditions; existence; fixed point.

of boundary value problems involving different kinds of fractional derivatives and boundary conditions. For some recent works on the theoretical aspects of fractional boundary value problems, we refer the reader to the books [**4, 1**] and articles [**16, 22, 25, 46, 31, 2, 39, 27, 3, 32, 40**]. One can find some recent results on boundary value problems of systems of fractional differential equations in [**5, 6, 21, 7, 24, 8, 23, 37**].

The study of boundary value problems for fractional order Langevin equation, initiated in [**9**], received considerable attention in the later years, for instance, see [**10, 11, 44, 42, 35, 48, 13, 12, 17, 47, 41**]. Systems of nonlinear fractional order Langevin equations equipped with boundary conditions have also been discussed by many researchers. The authors in [**43**] studied a coupled system of Riemann-Liouville type fractional Langevin equations complemented with uncoupled generalized nonlocal integral boundary conditions. In [**33**], a coupled system of nonlinear fractional Langevin equations with nonlocal and non-separated boundary conditions was investigated. In [**38, 15**], the authors studied systems of generalized Sturm-Liouville and Langevin fractional differential equations. The authors in [**14**] analyzed a coupled system of fractional order Langevin differential equations associated with anti-periodic boundary conditions.

Motivated by aforementioned works on systems of fractional order Langevin equations, in this paper, we investigate the existence and uniqueness of solutions for a coupled system of Riemann-Liouville type fractional Langevin equations with nonlocal multi-point and multi-strip coupled boundary conditions given by

$$(1)\qquad \begin{cases} D^{\alpha_1}(D^{\beta_1}+\lambda_1)x(t)=f_1(t,x(t),y(t)), & t\in\mathcal{J},\\ D^{\alpha_2}(D^{\beta_2}+\lambda_2)y(t)=f_2(t,x(t),y(t)), & t\in\mathcal{J},\\ x(0)=0, \quad x'(1)=\displaystyle\sum_{i=1}^{m}\mu_i y(\eta_i), \quad x(1)=\sum_{j=1}^{n}\rho_j\int_{\zeta_j}^{\xi_j}y(s)\mathrm{d}s, \\ y(0)=0, \quad y'(1)=\displaystyle\sum_{i=1}^{m}\sigma_i x(\eta_i), \quad y(1)=\sum_{j=1}^{n}\omega_j\int_{\zeta_j}^{\xi_j}x(s)\mathrm{d}s, \end{cases}$$

where D^{α_1}, D^{α_2}, D^{β_1} and D^{β_2} denote the Riemann–Liouvile fractional derivatives of order $\alpha_1,\alpha_2,\beta_1,\beta_2$, respectively, with $0<\alpha_1,\alpha_2<1$, $1<\beta_1,\beta_2<2$, $\lambda_1,\lambda_2>0$, $\mu_i,\sigma_i,\rho_j,\omega_j\in\mathbb{R}$, $\mathcal{J}=[0,1]$, $0<\eta_i<\zeta_j<\xi_j<1$ and $f_1,f_2\in C(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

We apply Leray-Schauder's alternative and Banach's contraction mapping principle to establish the existence and uniqueness of solutions to the problem (1), respectively. Our results are novel and enrich the literature on boundary value problem for systems of fractional order Langevin equations. Moreover, some new results follow from the present ones as special cases by fixing the values of the parameters involved in the boundary conditions (see the Conclusions section).

We arrange the rest of the paper as follows. We collect some preliminary definitions and solve a linear version of the problem (1) in Section 2. The main results are accomplished in Section 3. Illustrative examples for the obtained results are offered in Section 4. Section 5 contains the concluding remarks and indicates some special cases arising from the present work.

2. A subsidiary result

Let us begin this section with some basic concepts of fractional calculus [**26**].

Definition 2.1. For $\psi \in L_1[a,b]$, the (left) Riemann–Liouville fractional integral $I_{a^+}^{\alpha}$ of order $\alpha \in \mathbb{R}^+$ is defined as

$$I_{a^+}^{\alpha}\psi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\psi(s)\mathrm{d}s,$$

where Γ denotes the Euler gamma function.

Definition 2.2. Let $\psi, \psi^{(m)} \in L_1[a,b]$ for $-\infty \le a < t < b \le +\infty$. The Riemann–Liouville fractional derivative $D_{a^+}^{\alpha}$ of order order $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, is defined as

$$D_{a^+}^{\alpha}\psi(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} I_{a^+}^{1-\alpha}\psi(t) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_a^t (t-s)^{m-1-\alpha}\psi(s)\mathrm{d}s.$$

In the present work, we denote the Riemann-Liouville fractional integral and derivative operators $I_{a^+}^{q}$ and $D_{a^+}^{q}$ with $a=0$ by I^q and D^q, respectively.

Now, we prove an auxiliary lemma dealing with linear variant of the problem (1).

Lemma 2.3. *For $g_1, g_2 \in C([0,1]) \cap L([0,1])$, the unique solution of the linear system*

$$\text{(2)} \qquad \begin{cases} D^{\alpha_1}(D^{\beta_1}+\lambda_1)x(t) = g_1(t)), & t \in \mathcal{J}, \\ D^{\alpha_2}(D^{\beta_2}+\lambda_2)y(t) = g_2(t), & t \in \mathcal{J}, \end{cases}$$

subject to the boundary conditions in (1)*, is given by a pair of integral equations*

$$\begin{aligned}
\text{(3)} \quad x(t) = {} & \int_0^t \Big[\frac{(t-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}g_1(s) - \lambda_1\frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s \\
& + \phi_1(t)\Big\{\sum_{i=1}^{m}\mu_i\int_0^{\eta_i}\Big[\frac{(\eta_i-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}g_2(s) - \lambda_2\frac{(\eta_i-s)^{\beta_2-1}}{\Gamma(\beta_2)}y(s)\Big]\mathrm{d}s \\
& \quad - \int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-2}}{\Gamma(\beta_1+\alpha_1-1)}g_1(s) - \lambda_1\frac{(1-s)^{\beta_1-2}}{\Gamma(\beta_1-1)}x(s)\Big]\mathrm{d}s\Big\} \\
& + \phi_2(t)\Big\{\sum_{j=1}^{n}\rho_j\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}g_2(u) - \lambda_2\frac{(s-u)^{\beta_2-1}}{\Gamma(\beta_2)}y(u)\Big]\mathrm{d}u\mathrm{d}s \\
& \quad - \int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}g_1(s) - \lambda_1\frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s\Big\} \\
& + \phi_3(t)\Big\{\sum_{i=1}^{m}\sigma_i\int_0^{\eta_i}\Big[\frac{\eta_i-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}g_1(s) - \lambda_1\frac{(\eta_i-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s \\
& \quad - \int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-2}}{\Gamma(\beta_2+\alpha_2-1)}g_2(s) - \lambda_2\frac{(1-s)^{\beta_2-2}}{\Gamma(\beta_2-1)}y(s)\Big]\mathrm{d}s\Big\}
\end{aligned}$$

$$+\phi_4(t)\Big\{\sum_{j=1}^{n}\omega_j\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}g_1(u)-\lambda_1\frac{(s-u)^{\beta_1-1}}{\Gamma(\beta_1)}x(u)\Big]\mathrm{d}u\mathrm{d}s$$
$$-\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}g_2(s)-\lambda_2\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}y(s)\Big]\mathrm{d}s\Big\}$$

and

$$y(t)=\int_0^t\Big[\frac{(t-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}g_2(s)-\lambda_2\frac{(t-s)^{\beta_2-1}}{\Gamma(\beta_2)}y(s)\Big]\mathrm{d}s \tag{4}$$
$$+\theta_1(t)\Big\{\sum_{i=1}^{m}\mu_i\int_0^{\eta_i}\Big[\frac{(\eta_i-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}g_2(s)-\lambda_2\frac{(\eta_i-s)^{\beta_2-1}}{\Gamma(\beta_2)}y(s)\Big]\mathrm{d}s$$
$$-\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-2}}{\Gamma(\beta_1+\alpha_1-1)}g_1(s)-\lambda_1\frac{(1-s)^{\beta_1-2}}{\Gamma(\beta_1-1)}x(s)\Big]\mathrm{d}s\Big\}$$
$$+\theta_2(t)\Big\{\sum_{j=1}^{n}\rho_j\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}g_2(u)-\lambda_2\frac{(s-u)^{\beta_2-1}}{\Gamma(\beta_2)}y(u)\Big]\mathrm{d}u\mathrm{d}s$$
$$-\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}g_1(s)-\lambda_1\frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s\Big\}$$
$$+\theta_3(t)\Big\{\sum_{i=1}^{m}\sigma_i\int_0^{\eta_i}\Big[\frac{\eta_i-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}g_1(s)-\lambda_1\frac{(\eta_i-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s$$
$$-\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-2}}{\Gamma(\beta_2+\alpha_2-1)}g_2(s)-\lambda_2\frac{(1-s)^{\beta_2-2}}{\Gamma(\beta_2-1)}y(s)\Big]\mathrm{d}s\Big\}$$
$$+\theta_4(t)\Big\{\sum_{j=1}^{n}\omega_j\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}g_1(u)-\lambda_1\frac{(s-u)^{\beta_1-1}}{\Gamma(\beta_1)}x(u)\Big]\mathrm{d}u\mathrm{d}s$$
$$-\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}g_2(s)-\lambda_2\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}y(s)\Big]\mathrm{d}s\Big\},$$

where

$$\phi_1(t)=\Big(\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)}r_4t^{\alpha_1}+r_3\Big)t^{\beta_1-1},\quad \phi_2(t)=\Big(\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)}z_4t^{\alpha_1}+z_3\Big)t^{\beta_1-1}, \tag{5}$$
$$\phi_3(t)=\Big(\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)}u_4t^{\alpha_1}+u_3\Big)t^{\beta_1-1},\quad \phi_4(t)=\Big(\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)}s_4t^{\alpha_1}+s_3\Big)t^{\beta_1-1},$$
$$\theta_1(t)=\Big(\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2)}r_2t^{\alpha_2}+r_1\Big)t^{\beta_2-1},\quad \theta_2(t)=\Big(\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2)}z_2t^{\alpha_2}+z_1\Big)t^{\beta_2-1},$$
$$\theta_3(t)=\Big(\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2)}u_2t^{\alpha_2}+u_1\Big)t^{\beta_2-1},\quad \theta_4(t)=\Big(\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2)}s_2t^{\alpha_2}+s_1\Big)t^{\beta_2-1},$$
$$e_1=B_2-\frac{A_2B_1}{A_1},\quad e_2=\frac{A_3B_1}{A_1}-B_3,\quad e_3=\frac{A_4B_1}{A_1}-B_4,\quad e_4=\frac{A_2E_1}{A_1}-E_2,$$
$$e_5=E_3-\frac{A_3E_1}{A_1},\quad e_6=E_4-\frac{A_4E_1}{A_1},\ e_7=\frac{A_2L_1}{A_1}-L2,\ e_8=L_3-\frac{A_3L_1}{A_1},$$
$$e_9=L_4-\frac{A_4L_1}{A_1},\quad e_{10}=\frac{B_1}{A_1},\quad e_{11}=\frac{-E_1}{A_1},\quad e_{12}=\frac{-L_1}{A_1},\quad e_{13}=k_4-\frac{k_2k_3}{k_1},$$
$$e_{14}=\frac{1}{e_1},\quad k_1=e_5-e_2e_4e_{14},\quad k_2=e_6-e_3e_4e_{14},\quad k_3=e_8-e_2e_7e_{14},$$

$$k_4 = e_9 - e_3e_7e_{14}, \quad r_1 = \frac{1}{e_{13}}\Big[\frac{k_3}{k_1}\big(e_{11} - e_4e_{10}e_{14}\big) - e_{12} + e_7e_{10}e_{14}\Big],$$

$$r_2 = \frac{-1}{k_1}\big(e_{11} + k_2r_1 - e_4e_{10}e_{14}\big), \quad r_3 = -e_{14}\big(e_{10} + e_3r_1 + e_2r_2\big),$$

$$r_4 = \frac{1}{A_1}\big(-A_2r_3 + A_4r_1 + A_3r_2 + 1\big), \quad z_1 = \frac{-1}{e_{13}}\big(e_7e_{14} - \frac{k_3e_4e_{14}}{k_1}\big),$$

$$z_2 = \frac{-1}{k_1}\big(e_4e_{14+k_2z_1}\big), \quad z_3 = e_{14}\big(-e_3z_1 - e_2z_2 + 1\big),$$

$$z_4 = \frac{-1}{A_1}\big(A_2z_3 - A_3z_2 - A_4z_1\big), \quad u_1 = \frac{-k_3}{k_1e_{13}}, \quad u_2 = \frac{1}{k_1}\big(1 - k_2u_1\big),$$

$$u_3 = -e_{14}\big(e_2u_2 + e_3u_1\big), \quad u_4 = \frac{1}{A_1}\Big[-A_2u_3 + A_3u_2 + A_4u_1\Big], \quad s_1 = \frac{1}{e_{13}},$$

$$s_2 = \frac{-k_2s_1}{k_1}, \quad s_3 = -e_{14}\Big(e_3s_1 + e_2s_2\Big), \quad s_4 = \frac{1}{A_1}\Big(A_4s_1 - A_2s_3 + A_3s_2\Big),$$

$$A_1 = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1-1)}, \quad A_2 = (\beta_1 - 1), \quad A_3 = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{m}\mu_i\eta_i^{\alpha_2+\beta_2-1},$$

$$A_4 = \sum_{i=1}^{m}\mu_i\eta_i^{\beta_2-1}, \quad B_1 = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)}, \quad B_2 = 1,$$

$$B_3 = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2+1)}\sum_{j=1}^{n}\rho_j(\xi_j^{\alpha_2+\beta_2} - \zeta_j^{\alpha_2+\beta_2}), \quad B_4 = \frac{1}{\beta_2}\sum_{j=1}^{n}\rho_j(\xi_j^{\beta_2} - \zeta_j^{\beta_2}),$$

$$E_1 = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)}\sum_{i=1}^{m}\sigma_i\eta_i^{\alpha_1+\beta_1-1}, \quad E_2 = \sum_{i=1}^{m}\sigma_i\eta_i^{\beta_1-1},$$

$$E_3 = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2-1)}, \quad E_4 = (\beta_2 - 1),$$

$$L_1 = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{n}\omega_j(\xi_j^{\alpha_1+\beta_1} - \zeta_j^{\alpha_1+\beta_1}),$$

$$L_2 = \frac{1}{\beta_1}\sum_{j=1}^{n}\omega_j(\xi_j^{\beta_1} - \zeta_j^{\beta_1}), \quad L_3 = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2)}, \quad L_4 = 1.$$

Proof. Applying the integral operators I^{α_1} and I^{α_2} to the first and second equations in (2), respectively, we get

$$\begin{cases} (D^{\beta_1} + \lambda_1)x(t) = I^{\alpha_1}g_1(t) + t^{\alpha_1-1}c_1, \\ (D^{\beta_2} + \lambda_2)y(t) = I^{\alpha_2}g_2(t) + t^{\alpha_2-1}d_1, \end{cases} \tag{6}$$

where $c_1, d_1 \in \mathbb{R}$ are unknown arbitrary constants.

Now, applying the integral operators I^{β_1} and I^{β_2} to the first and second equations in (6), respectively, we obtain

(7)
$$x(t) = I^{\alpha_1+\beta_1}g_1(t) - \lambda_1 I^{\beta_1}x(t) + \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)}t^{\alpha_1+\beta_1-1}c_1 + t^{\beta_1-1}c_2 + t^{\beta_1-2}c_3,$$

$$y(t) = I^{\alpha_2+\beta_2}g_2(t) - \lambda_2 I^{\beta_2}y(t) + \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2)}t^{\alpha_2+\beta_2-1}d_1 + t^{\beta_2-1}d_2 + t^{\beta_2-2}d_3,$$

where c_2, d_2, c_3 and d_3 are unknown arbitrary real constants. Combining the conditions $x(0) = 0 = y(0)$ with (7), we find that $c_3 = d_3 = 0$ as $1 < \beta_1, \beta_2 < 2$. Then, the system (7) takes the form

$$
\begin{aligned}
x(t) &= I^{\alpha_1+\beta_1} g_1(t) - \lambda_1 I^{\beta_1} x(t) + \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)} t^{\alpha_1+\beta_1-1} c_1 + t^{\beta_1-1} c_2, \\
y(t) &= I^{\alpha_2+\beta_2} g_2(t) - \lambda_2 I^{\beta_2} y(t) + \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2)} t^{\alpha_2+\beta_2-1} d_1 + t^{\beta_2-1} d_2.
\end{aligned} \tag{8}
$$

From (8), we have

$$
\begin{aligned}
x'(t) &= I^{\alpha_1+\beta_1-1} g_1(t) - \lambda_1 I^{\beta_1-1} x(t) + \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1-1)} t^{\alpha_1+\beta_1-2} c_1 \\
&\quad + (\beta_1-1) t^{\beta_1-2} c_2, \\
y'(t) &= I^{\alpha_2+\beta_2-1} g_2(t) - \lambda_2 I^{\beta_2-1} y(t) + \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta_2-1)} t^{\alpha_2+\beta_2-2} d_1 \\
&\quad + (\beta_2-1) t^{\beta_2-2} d_2.
\end{aligned} \tag{9}
$$

Now, using (8) and (9) in the remaining boundary conditions of (1), we get

$$
\begin{aligned}
A_1 c_1 + A_2 c_2 - A_3 d_1 - A_4 d_2 &= J_1, \\
B_1 c_1 + B_2 c_2 - B_3 d_1 - B_4 d_2 &= J_2, \\
-E_1 c_1 - E_2 c_2 + E_3 d_1 + E_4 d_2 &= J_3, \\
-L_1 c_1 - L_2 c_2 + L_3 d_1 + L_4 d_2 &= J_4,
\end{aligned} \tag{10}
$$

where A_p, B_p, E_p, L_p, $p = 1, 2, 3, 4$, are given in (5), and

$$
\begin{aligned}
J_1 &= \sum_{i=1}^{m} \mu_i \Big[I^{\alpha_2+\beta_2} g_2(\eta_i) - \lambda_2 I^{\beta_2} y(\eta_i) \Big] - \Big[I^{\alpha_1+\beta_1-1} g_1(1) - \lambda_1 I^{\beta_1-1} x(1) \Big], \\
J_2 &= \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\xi_j} \Big[I^{\alpha_2+\beta_2} g_2(s) - \lambda_2 I^{\beta_2} y(s) \Big] \mathrm{d}s - \Big[I^{\alpha_1+\beta_1} g_1(1) - \lambda_1 I^{\beta_1} x(1) \Big], \\
J_3 &= \sum_{i=1}^{m} \sigma_i \Big[I^{\alpha_1+\beta_1} g_1(\eta_i) - \lambda_1 I^{\beta_1} x(\eta_i) \Big] - \Big[I^{\alpha_2+\beta_2-1} g_2(1) - \lambda_2 I^{\beta_2-1} y(1) \Big], \\
J_4 &= \sum_{j=1}^{n} \omega_j \int_{\zeta_j}^{\xi_j} \Big[I^{\alpha_1+\beta_1} g_1(s) - \lambda_1 I^{\beta_1} x(s) \Big] \mathrm{d}s - \Big[I^{\alpha_2+\beta_2} g_2(1) - \lambda_2 I^{\beta_2} y(1) \Big].
\end{aligned} \tag{11}
$$

Solving the system (10) for c_1, c_2, d_1 and d_2, we find that

$$
\begin{aligned}
c_1 &= r_4 J_1 + z_4 J_2 + u_4 J_3 + s_4 J_4, \\
c_2 &= r_3 J_1 + z_3 J_2 + u_3 J_3 + s_3 J_4, \\
d_1 &= r_2 J_1 + z_2 J_2 + u_2 J_3 + s_2 J_4, \\
d_2 &= r_1 J_1 + z_1 J_2 + u_1 J_3 + s_1 J_4.
\end{aligned} \tag{12}
$$

Inserting the above values of c_1, c_2, d_1 and d_2 in (8) together with (5) and (11), we obtain the solution (3) and (4). By direct computation, one can obtain the converse of the lemma. □

3. Main results

Let Y denote the Banach space of all continuous functions from $\mathcal{J} \to \mathbb{R}$ endowed with the supremum norm $\|x\| = \sup_{t\in\mathcal{J}} |x(t)|$. Then, the product space $Y \times Y$ is also a Banach space endowed with the norm $\|(x, y)\| = \|x\| + \|y\|, (x, y) \in Y \times Y$. In view of Lemma 2.3, we can transform the problem (1) into a fixed point problem $(x, y) = \mathcal{T}(x, y)$, where $\mathcal{T} : Y \times Y \to Y \times Y$ is an operator defined by

$$\mathcal{T}(x,y)(t) = \begin{pmatrix} \mathcal{T}_1(x,y)(t) \\ \mathcal{T}_2(x,y)(t) \end{pmatrix}, \tag{13}$$

with

(14) $\mathcal{T}_1(x,y)(t)$

$$\begin{aligned}
&= \int_0^t \Big[\frac{(t-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)} f_1(s,x(s),y(s)) - \lambda_1 \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} x(s)\Big] \mathrm{d}s \\
&+ \phi_1(t) \Big\{ \sum_{i=1}^m \mu_i \int_0^{\eta_i} \Big[\frac{(\eta_i-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)} f_2(s,x(s),y(s)) - \lambda_2 \frac{(\eta_i-s)^{\beta_2-1}}{\Gamma(\beta_2)} y(s)\Big] \mathrm{d}s \\
&\qquad - \int_0^1 \Big[\frac{(1-s)^{\beta_1+\alpha_1-2}}{\Gamma(\beta_1+\alpha_1-1)} f_1(s,x(s),y(s)) - \lambda_1 \frac{(1-s)^{\beta_1-2}}{\Gamma(\beta_1-1)} x(s)\Big] \mathrm{d}s \Big\} \\
&+ \phi_2(t) \Big\{ \sum_{j=1}^n \rho_j \int_{\zeta_j}^{\xi_j} \int_0^s \Big[\frac{(s-u)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)} f_2(u,x(u),y(u)) - \lambda_2 \frac{(s-u)^{\beta_2-1}}{\Gamma(\beta_2)} y(u)\Big] \mathrm{d}u \mathrm{d}s \\
&\qquad - \int_0^1 \Big[\frac{(1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)} f_1(s,x(s),y(s)) - \lambda_1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} x(s)\Big] \mathrm{d}s \Big\} \\
&+ \phi_3(t) \Big\{ \sum_{i=1}^m \sigma_i \int_0^{\eta_i} \Big[\frac{\eta_i-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)} f_1(s,x(s),y(s)) - \lambda_1 \frac{(\eta_i-s)^{\beta_1-1}}{\Gamma(\beta_1)} x(s)\Big] \mathrm{d}s \\
&\qquad - \int_0^1 \Big[\frac{(1-s)^{\beta_2+\alpha_2-2}}{\Gamma(\beta_2+\alpha_2-1)} f_2(s,x(s),y(s)) - \lambda_2 \frac{(1-s)^{\beta_2-2}}{\Gamma(\beta_2-1)} y(s)\Big] \mathrm{d}s \Big\} \\
&+ \phi_4(t) \Big\{ \sum_{j=1}^n \omega_j \int_{\zeta_j}^{\xi_j} \int_0^s \Big[\frac{(s-u)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)} f_1(u,x(u),y(u)) - \lambda_1 \frac{(s-u)^{\beta_1-1}}{\Gamma(\beta_1)} x(u)\Big] \mathrm{d}u \mathrm{d}s \\
&\qquad - \int_0^1 \Big[\frac{(1-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)} f_2(s,x(s),y(s)) - \lambda_2 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} y(s)\Big] \mathrm{d}s \Big\}
\end{aligned}$$

and

(15) $\mathcal{T}_2(x,y)(t)$

$$\begin{aligned}
&= \int_0^t \Big[\frac{(t-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)} f_2(s,x(s),y(s)) - \lambda_2 \frac{(t-s)^{\beta_2-1}}{\Gamma(\beta_2)} y(s)\Big] \mathrm{d}s \\
&+ \theta_1(t) \Big\{ \sum_{i=1}^m \mu_i \int_0^{\eta_i} \Big[\frac{(\eta_i-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)} f_2(s,x(s),y(s)) - \lambda_2 \frac{(\eta_i-s)^{\beta_2-1}}{\Gamma(\beta_2)} y(s)\Big] \mathrm{d}s \\
&\qquad - \int_0^1 \Big[\frac{(1-s)^{\beta_1+\alpha_1-2}}{\Gamma(\beta_1+\alpha_1-1)} f_1(s,x(s),y(s)) - \lambda_1 \frac{(1-s)^{\beta_1-2}}{\Gamma(\beta_1-1)} x(s)\Big] \mathrm{d}s \Big\}
\end{aligned}$$

$$
\begin{aligned}
&+\theta_2(t)\Big\{\sum_{j=1}^{n}\rho_j\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}f_2(u,x(u),y(u))-\lambda_2\frac{(s-u)^{\beta_2-1}}{\Gamma(\beta_2)}y(u)\Big]\mathrm{d}u\mathrm{d}s\\
&\qquad-\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}f_1(s,x(s),y(s))-\lambda_1\frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s\Big\}\\
&+\theta_3(t)\Big\{\sum_{i=1}^{m}\sigma_i\int_0^{\eta_i}\Big[\frac{(\eta_i-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}f_1(s,x(s),y(s))-\lambda_1\frac{(\eta_i-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s\\
&\qquad-\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-2}}{\Gamma(\beta_2+\alpha_2-1)}f_2(s,x(s),y(s))-\lambda_2\frac{(1-s)^{\beta_2-2}}{\Gamma(\beta_2-1)}y(s)\Big]\mathrm{d}s\Big\}\\
&+\theta_4(t)\Big\{\sum_{j=1}^{n}\omega_j\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}f_1(u,x(u),y(u))-\lambda_1\frac{(s-u)^{\beta_1-1}}{\Gamma(\beta_1)}x(u)\Big]\mathrm{d}u\mathrm{d}s\\
&\qquad-\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}f_2(s,x(s),y(s))-\lambda_2\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}y(s)\Big]\mathrm{d}s\Big\}.
\end{aligned}
$$

Observe that the fixed points of the operator $\mathcal{T}$ are solutions to the problem (1).

In the forthcoming analysis, we need the following assumptions:

(H_1) There exist real constants $m_p, n_p \geq 0$, $p = 1, 2$, and $m_0, n_0 > 0$ such that

$$
\begin{aligned}
|f_1(t,x,y)| &\leq m_0 + m_1|x| + m_2|y|,\\
|f_2(t,x,y)| &\leq n_0 + n_1|x| + n_2|y|,
\end{aligned}
$$

for all $x, y \in \mathbb{R}$;

(H_2) There exist constants ℓ_1 and ℓ_2 such that for all $t \in \mathcal{J}$, $x_p, y_p \in \mathbb{R}$, $p = 1, 2$,

$$
\begin{aligned}
|f_1(t,x_2,y_2) - f_1(t,x_1,y_1| &\leq \ell_1(|x_2 - x_1| + |y_2 - y_1|),\\
|f_2(t,x_2,y_2) - f_2(t,x_1,y_1| &\leq \ell_2(|x_2 - x_1| + |y_2 - y_1|).
\end{aligned}
$$

For the sake of computational convenience, we introduce the notations:

$$
\begin{aligned}
\mathcal{Z}_1 &= \frac{1}{\Gamma(\beta_1+\alpha_1+1)} + \overline{\phi}_1\frac{1}{\Gamma(\beta_1+\alpha_1)} + \overline{\phi}_2\frac{1}{\Gamma(\beta_1+\alpha_1+1)}\\
&\quad + \overline{\phi}_3\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\beta_1+\alpha_1}}{\Gamma(\beta_1+\alpha_1+1)} + \overline{\phi}_4\sum_{j=1}^{n}\frac{|\omega_j|}{\Gamma(\beta_1+\alpha_1+2)}\left(\xi_j^{\beta_1+\alpha_1+1} - \zeta_j^{\beta_1+\alpha_1+1}\right),\\
\mathcal{Z}_2 &= \overline{\phi}_1\sum_{i=1}^{m}|\mu_i|\frac{\eta_i^{\beta_2+\alpha_2}}{\Gamma(\beta_2+\alpha_2+1)} + \overline{\phi}_2\sum_{j=1}^{n}\frac{|\rho_j|}{\Gamma(\beta_2+\alpha_2+2)}\left(\xi_j^{\beta_2+\alpha_2+1} - \zeta_j^{\beta_2+\alpha_2+1}\right)\\
&\quad + \overline{\phi}_3\frac{1}{\Gamma(\beta_2+\alpha_2)} + \overline{\phi}_4\frac{1}{\Gamma(\beta_2+\alpha_2+1)},\\
\mathcal{Z}_3 &= \lambda_1\Big[\frac{1}{\Gamma(\beta_1+1)} + \overline{\phi}_1\frac{1}{\Gamma(\beta_1)} + \overline{\phi}_2\frac{1}{\Gamma(\beta_1+1)} + \overline{\phi}_3\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\beta_1}}{\Gamma(\beta_1+1)}\\
&\quad + \overline{\phi}_4\sum_{j=1}^{n}\frac{|\omega_j|}{\Gamma(\beta_1+2)}\left(\xi_j^{\beta_1+1} - \zeta_j^{\beta_1+1}\right)\Big],\\
\mathcal{Z}_4 &= \lambda_2\Big[\overline{\phi}_1\sum_{i=1}^{m}|\mu_i|\frac{\eta_i^{\beta_2}}{\Gamma(\beta_2+1)} + \overline{\phi}_2\sum_{j=1}^{n}\frac{|\rho_j|}{\Gamma(\beta_2+2)}\left(\xi_j^{\beta_2+1} - \zeta_j^{\beta_2+1}\right)\\
&\quad + \overline{\phi}_3\frac{1}{\Gamma(\beta_2)} + \overline{\phi}_4\frac{1}{\Gamma(\beta_2+1)}\Big],
\end{aligned}
\tag{16}
$$

$$\mathcal{N}_1 = \overline{\theta}_1 \frac{1}{\Gamma(\beta_1+\alpha_1)} + \overline{\theta}_2 \frac{1}{\Gamma(\beta_1+\alpha_1+1)} + \overline{\theta}_3 \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\beta_1+\alpha_1}}{\Gamma(\beta_1+\alpha_1+1)}$$
$$+ \overline{\theta}_4 \sum_{j=1}^{n} \frac{|\omega_j|}{\Gamma(\beta_1+\alpha_1+2)} \left(\xi_j^{\beta_1+\alpha_1+1} - \zeta_j^{\beta_1+\alpha_1+1}\right),$$
$$\mathcal{N}_2 = \frac{1}{\Gamma(\beta_2+\alpha_2+1)} + \overline{\theta}_1 \sum_{i=1}^{m} |\mu_i| \frac{\eta_i^{\beta_2+\alpha_2}}{\Gamma(\beta_2+\alpha_2+1)}$$
$$+ \overline{\theta}_2 \sum_{j=1}^{n} \frac{|\rho_j|}{\Gamma(\beta_2+\alpha_2+2)} \left(\xi_j^{\beta_2+\alpha_2+1} - \zeta_j^{\beta_2+\alpha_2+1}\right) + \overline{\theta}_3 \frac{1}{\Gamma(\beta_2+\alpha_2)}$$
$$+ \overline{\theta}_4 \frac{1}{\Gamma(\beta_2+\alpha_2+1)},$$
$$\mathcal{N}_3 = \lambda_1 \Bigg[\overline{\theta}_1 \frac{1}{\Gamma(\beta_1)} + \overline{\theta}_2 \frac{1}{\Gamma(\beta_1+1)} + \overline{\theta}_3 \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\beta_1}}{\Gamma(\beta_1+1)}$$
$$+ \overline{\theta}_4 \sum_{j=1}^{n} \frac{|\omega_j|}{\Gamma(\beta_1+2)} \left(\xi_j^{\beta_1+1} - \zeta_j^{\beta_1+1}\right)\Bigg],$$
$$\mathcal{N}_4 = \lambda_2 \Bigg[\frac{1}{\Gamma(\beta_2+1)} + \overline{\theta}_1 \sum_{i=1}^{m} |\mu_i| \frac{\eta_i^{\beta_2}}{\Gamma(\beta_2+1)} + \overline{\theta}_2 \sum_{j=1}^{n} \frac{|\rho_j|}{\Gamma(\beta_2+2)} \left(\xi_j^{\beta_2+1} - \zeta_j^{\beta_2+1}\right)$$
$$+ \overline{\theta}_3 \frac{1}{\Gamma(\beta_2)} + \overline{\theta}_4 \frac{1}{\Gamma(\beta_2+1)}\Bigg],$$

where

$$\overline{\phi}_p = \sup_{t\in\mathcal{J}} |\phi_p(t)|, \quad \overline{\theta}_p = \sup_{t\in\mathcal{J}} |\theta_p(t)|, \qquad p = 1,2,3,4.$$

Now, we are in a position to present our main results. Our first result, dealing with the existence of solutions for the problem (1), relies on Leray-Schauder's alternative [**20**].

Theorem 3.1. *If the condition* (H_1) *is satisfied, then the problem* (1) *has at least one solution on* $\mathcal{J}$*, provided that* $\max\{\mathcal{W}_1, \mathcal{W}_2\} < 1$*, where*

$$\begin{aligned} \mathcal{W}_1 &= m_1(\mathcal{Z}_1+\mathcal{N}_1) + n_1(\mathcal{Z}_2+\mathcal{N}_2) + (\mathcal{Z}_3+\mathcal{N}_3), \\ \mathcal{W}_2 &= m_2(\mathcal{Z}_1+\mathcal{N}_1) + n_2(\mathcal{Z}_2+\mathcal{N}_2) + (\mathcal{Z}_4+\mathcal{N}_4), \end{aligned} \tag{17}$$

$\mathcal{Z}_p$ *and* $\mathcal{N}_p$*,* $p = 1,2,3,4$*, are given in* (16).

Proof. Let us first establish that the operator $\mathcal{T}\colon Y\times Y \to Y\times Y$ defined by (13) is completely continuous. Observe that the operator $\mathcal{T}$ is continuous in view of continuity of functions f_1 and f_2. Let $B_\rho = \{(x,y)\in Y\times Y : \|(x,y)\| \le \rho\}$ with

$$\rho \ge \frac{m_0(\mathcal{Z}_1+\mathcal{N}_1) + n_0(\mathcal{Z}_2+\mathcal{N}_2)}{1-\max\{\mathcal{W}_1,\mathcal{W}_2\}}, \tag{18}$$

where $\mathcal{W}_1$ and $\mathcal{W}_2$ are given in (17). For any $(x,y)\in B_\rho$, let

$$|f_1(t,x,y)| \le m_0 + m_1\|x\| + m_2\|y\| = K_1,$$
$$|f_2(t,x,y)| \le n_0 + n_1\|x\| + n_2\|y\| = K_2.$$

Then, for any $(x,y)\in B_\rho$, we obtain

$$
\begin{aligned}
&\|\mathcal{T}_1(x,y)\|\\
&\le \sup_{t\in\mathcal{J}}\Bigg\{\int_0^t\Big[\frac{(t-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}|f_1(s,x(s),y(s))|+\lambda_1\frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big]\mathrm{d}s\\
&+|\phi_1(t)|\Big\{\sum_{i=1}^m|\mu_i|\int_0^{\eta_i}\Big[\frac{(\eta_i-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}|f_2(s,x(s),y(s))|+\lambda_2\frac{(\eta_i-s)^{\beta_2-1}}{\Gamma(\beta_2)}|y(s)|\Big]\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-2}}{\Gamma(\beta_1+\alpha_1-1)}|f_1(s,x(s),y(s))|+\lambda_1\frac{(1-s)^{\beta_1-2}}{\Gamma(\beta_1-1)}|x(s)|\Big]\mathrm{d}s\Big\}\\
&+|\phi_2(t)|\Big\{\sum_{j=1}^n|\rho_j|\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}|f_2(u,x(u),y(u))|+\lambda_2\frac{(s-u)^{\beta_2-1}}{\Gamma(\beta_2)}|y(u)|\Big]\mathrm{d}u\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}|f_1(s,x(s),y(s))|+\lambda_1\frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big]\mathrm{d}s\Big\}\\
&+|\phi_3(t)|\Big\{\sum_{i=1}^m|\sigma_i|\int_0^{\eta_i}\Big[\frac{\eta_i-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}|f_1(s,x(s),y(s))|+\lambda_1\frac{(\eta_i-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big]\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-2}}{\Gamma(\beta_2+\alpha_2-1)}|f_2(s,x(s),y(s))|+\lambda_2\frac{(1-s)^{\beta_2-2}}{\Gamma(\beta_2-1)}|y(s)|\Big]\mathrm{d}s\Big\}\\
&+|\phi_4(t)|\Big\{\sum_{j=1}^n|\omega_j|\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}|f_1(u,x(u),y(u))|+\lambda_1\frac{(s-u)^{\beta_1-1}}{\Gamma(\beta_1)}|x(u)|\Big]\mathrm{d}u\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}|f_2(s,x(s),y(s))|+\lambda_2\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}|y(s)|\Big]\mathrm{d}s\Big\}\Bigg\}\\
&\le K_1\mathcal{Z}_1+K_2\mathcal{Z}_2+\mathcal{Z}_3\|x\|+\mathcal{Z}_4\|y\|,
\end{aligned}
$$

where $\mathcal{Z}_p,\ p=1,2,3,4$, are given in (16). Hence

$$\|\mathcal{T}_1(x,y)\|\le K_1\mathcal{Z}_1+K_2\mathcal{Z}_2+\mathcal{Z}_3\|x\|+\mathcal{Z}_4\|y\|. \tag{19}$$

Similarly, we can find that

$$\|\mathcal{T}_2(x,y)\|\le K_1\mathcal{N}_1+K_2\mathcal{N}_2+\mathcal{N}_3\|x\|+\mathcal{N}_4\|y\|, \tag{20}$$

where $\mathcal{N}_p,\ p=1,2,3,4$, are given in (16).

Inserting the values of K_1 and K_2 into (19) and (20) and using the definition of norm, we get

$$\|\mathcal{T}(x,y)\|\le m_0(\mathcal{Z}_1+\mathcal{N}_1)+n_0(\mathcal{Z}_2+\mathcal{N}_2)+\max(\mathcal{W}_1,\mathcal{W}_2)\rho\le\rho,$$

where have used (18). This shows that $\mathcal{T}(B_\rho)$ is uniformly bounded.

Next, we show that $\mathcal{T}(B_\rho)$ is equicontinuous. For that, we take $t_1,t_2\in\mathcal{J}$ with $t_1<t_2$ and $(x,y)\in B_\rho$. Then, we obtain

$$
\begin{aligned}
&|\mathcal{T}_1(x,y)(t_2)-\mathcal{T}_1(x,y)(t_1)|\\
&\le\Big|\int_0^{t_2}\Big[\frac{(t_2-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}f_1(s,x(s),y(s))-\lambda_1\frac{(t_2-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s\\
&\quad-\int_0^{t_1}\Big[\frac{(t_1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}f_1(s,x(s),y(s))-\lambda_1\frac{(t_1-s)^{\beta_1-1}}{\Gamma(\beta_1)}x(s)\Big]\mathrm{d}s\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+ |\phi_1(t_2) - \phi_1(t_1)| \big|\tilde{J}_1\big| + |\phi_2(t_2) - \phi_2(t_1)| \big|\tilde{J}_2\big| \\
&+ |\phi_3(t_2) - \phi_3(t_1)| \big|\tilde{J}_3\big| + |\phi_4(t_2) - \phi_4(t_1)| \big|\tilde{J}_4\big| \\
&\leq \frac{[m_0 + (m_1+m_2)\rho]}{\Gamma(\beta_1+\alpha_1+1)} \Big[2(t_2-t_1)^{\beta_1+\alpha_1} + |t_2^{\beta_1+\alpha_1} - t_1^{\beta_1+\alpha_1}|\Big] \\
&+ \frac{\lambda_1}{\Gamma(\beta_1+1} \Big[2(t_2-t_1)^{\beta_1} + |t_2^{\beta_1} - t_1^{\beta_1}|\Big]\rho \\
&+ \Big[\frac{\Gamma(\alpha_1)|r_4|}{\Gamma(\beta_1+\alpha_1)} |t_2^{\beta_1+\alpha_1-1} - t_1^{\beta_1+\alpha_1-1}| + |r_3||t_2^{\beta_1-1} - t_1^{\beta_1-1}|\Big] \big|\tilde{J}_1\big| \\
&+ \Big[\frac{\Gamma(\alpha_1)|z_4|}{\Gamma(\beta_1+\alpha_1)} |t_2^{\beta_1+\alpha_1-1} - t_1^{\beta_1+\alpha_1-1}| + |z_3||t_2^{\beta_1-1} - t_1^{\beta_1-1}|\Big] \big|\tilde{J}_2\big| \\
&+ \Big[\frac{\Gamma(\alpha_1)|u_4|}{\Gamma(\beta_1+\alpha_1)} |t_2^{\beta_1+\alpha_1-1} - t_1^{\beta_1+\alpha_1-1}| + |u_3||t_2^{\beta_1-1} - t_1^{\beta_1-1}|\Big] \big|\tilde{J}_3\big| \\
&+ \Big[\frac{\Gamma(\alpha_1)|s_4|}{\Gamma(\beta_1+\alpha_1)} |t_2^{\beta_1+\alpha_1-1} - t_1^{\beta_1+\alpha_1-1}| + |s_3||t_2^{\beta_1-1} - t_1^{\beta_1-1}|\Big] \big|\tilde{J}_4\big| \\
&\longrightarrow 0 \text{ as } (t_2 - t_1) \to 0 \text{ independently of } (x,y) \in B_\rho,
\end{aligned}
$$

where $\tilde{J}_1$ and $\tilde{J}_2$ are obtained from (11) by replacing g_1 and g_2 with f_1 and f_2, respectively, and

$$
\begin{aligned}
\big|\tilde{J}_1\big| &\leq n_0 \sum_{i=1}^{m} |\mu_i| \frac{\eta_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{m_0}{\Gamma(\alpha_1+\beta_1)} \\
&+ \Big[\big(n_1+n_2\big) \sum_{i=1}^{m} |\mu_i| \frac{\eta_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{m_1+m_2}{\Gamma(\alpha_1+\beta_1)} + \frac{\lambda_1}{\Gamma(\beta_1)} + \lambda_2 \sum_{i=1}^{m} |\mu_i| \frac{\eta_i^{\beta_2}}{\Gamma(\beta_2+1)}\Big]\rho, \\
\big|\tilde{J}_2\big| &\leq n_0 \sum_{j=1}^{n} |\rho_j| \frac{\big(\xi_j^{\alpha_2+\beta_2+1} - \zeta_j^{\alpha_2+\beta_2+1}\big)}{\Gamma(\alpha_2+\beta_2+2)} + \frac{m_0}{\Gamma(\alpha_1+\beta_1+1)} \\
&+ \Big[\big(n_1+n_2\big) \sum_{j=1}^{n} |\rho_j| \frac{\big(\xi_j^{\alpha_2+\beta_2+1} - \zeta_j^{\alpha_2+\beta_2+1}\big)}{\Gamma(\alpha_2+\beta_2+2)} + \frac{m_1+m_2}{\Gamma(\alpha_1+\beta_1+1)} + \frac{\lambda_1}{\Gamma(\beta_1+1)} \\
&+ \lambda_2 \sum_{j=1}^{n} |\rho_j| \frac{\big(\xi_j^{\beta_2+1} - \zeta_j^{\beta_2+1}\big)}{\Gamma(\beta_2+2)}\Big]\rho, \\
\big|\tilde{J}_3\big| &\leq m_0 \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{n_0}{\Gamma(\alpha_2+\beta_2)} \\
&+ \Big[\big(m_1+m_2\big) \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{n_1+n_2}{\Gamma(\alpha_2+\beta_2)} + \lambda_1 \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\beta_1}}{\Gamma(\beta_1+1)} + \frac{\lambda_2}{\Gamma(\beta_2)}\Big]\rho, \\
\big|\tilde{J}_4\big| &\leq m_0 \sum_{j=1}^{n} |\omega_j| \frac{\big(\xi_j^{\alpha_1+\beta_1+1} - \zeta_j^{\alpha_1+\beta_1+1}\big)}{\Gamma(\alpha_1+\beta_1+2)} + \frac{n_0}{\Gamma(\alpha_2+\beta_2+1)} \\
&+ \Big[\big(m_1+m_2\big) \sum_{j=1}^{n} |\omega_j| \frac{\big(\xi_j^{\alpha_1+\beta_1+1} - \zeta_j^{\alpha_1+\beta_1+1}\big)}{\Gamma(\alpha_1+\beta_1+2)} + \frac{n_1+n_2}{\Gamma(\alpha_2+\beta_2+1)} \\
&+ \lambda_1 \sum_{j=1}^{n} |\omega_j| \frac{\big(\xi_j^{\beta_1+1} - \zeta_j^{\beta_1+1}\big)}{\Gamma(\beta_1+2)} + \frac{\lambda_2}{\Gamma(\beta_2+1)}\Big]\rho,
\end{aligned}
$$

Similarly, it can be shown that

$$|\mathcal{T}_2(x,y)(t_2)-\mathcal{T}_2(x,y)(t_1)| \longrightarrow 0 \text{ as } (t_2-t_1)\to 0 \text{ independently of } (x,y)\in B_\rho.$$

Thus, $\mathcal{T}_1(B_\rho)$ and $\mathcal{T}_2(B_\rho)$ are equicontinuous and hence $\mathcal{T}(B_\rho)$ is equicontinuous. Therefore, by Arzelà-Ascoli theorem, $\mathcal{T}$ is completely continuous.

In the final step, we consider a set $\Omega=\{(x,y)\in Y\times Y:(x,y)=\tau\mathcal{T}(x,y),\ 0<\tau\le 1\}$ and show that it is bounded. Let $(x,y)\in\Omega$. Then $(x,y)=\tau\mathcal{T}(x,y)$ implies that $x(t)=\tau\mathcal{T}_1(x,y)(t)$ and $y(t)=\tau\mathcal{T}_2(x,y)(t)$ for $t\in\mathcal{J}$. Then, by the assumption (H_1), we have

$$\begin{aligned}
\|x\| &= \sup_{t\in\mathcal{J}}|x(t)| \le \sup_{t\in\mathcal{J}}|\mathcal{T}_1(x,y)(t)| \\
&\le \sup_{t\in\mathcal{J}}\Bigg\{\int_0^t\Big[\frac{(t-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}\Big(m_0+m_1|x|+m_2|y|\Big)+\lambda_1\frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big]\mathrm{d}s \\
&\quad+|\phi_1(t)|\Big\{\sum_{i=1}^m|\mu_i|\int_0^{\eta_i}\Big[\frac{(\eta_i-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}\Big(n_0+n_1|x|+n_2|y|\Big)\lambda_2\frac{(\eta_i-s)^{\beta_2-1}}{\Gamma(\beta_2)}|y(s)|\Big]\mathrm{d}s \\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-2}}{\Gamma(\beta_1+\alpha_1-1)}\Big(m_0+m_1|x|+m_2|y|\Big)+\lambda_1\frac{(1-s)^{\beta_1-2}}{\Gamma(\beta_1-1)}|x(s)|\Big]\mathrm{d}s\Big\} \\
&\quad+|\phi_2(t)|\Big\{\sum_{j=1}^n|\rho_j|\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}\Big(n_0+n_1|x|+n_2|y|\Big) \\
&\qquad+\lambda_2\frac{(s-u)^{\beta_2-1}}{\Gamma(\beta_2)}|y(u)|\Big]\mathrm{d}u\mathrm{d}s \\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}\Big(m_0+m_1|x|+m_2|y|\Big)+\lambda_1\frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big]\mathrm{d}s\Big\} \\
&\quad+|\phi_3(t)|\Big\{\sum_{i=1}^m|\sigma_i|\int_0^{\eta_i}\Big[\frac{\eta_i-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}\Big(m_0+m_1|x|+m_2|y|\Big) \\
&\qquad+\lambda_1\frac{(\eta_i-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big]\mathrm{d}s \\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-2}}{\Gamma(\beta_2+\alpha_2-1)}\Big(n_0+n_1|x|+n_2|y|\Big)+\lambda_2\frac{(1-s)^{\beta_2-2}}{\Gamma(\beta_2-1)}|y(s)|\Big]\mathrm{d}s\Big\} \\
&\quad+|\phi_4(t)|\Big\{\sum_{j=1}^n|\omega_j|\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}\Big(m_0+m_1|x|+m_2|y|\Big) \\
&\qquad+\lambda_1\frac{(s-u)^{\beta_1-1}}{\Gamma(\beta_1)}|x(u)|\Big]\mathrm{d}u\mathrm{d}s \\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}\Big(n_0+n_1|x|+n_2|y|\Big)+\lambda_2\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}|y(s)|\Big]\mathrm{d}s\Big\}\Bigg\},
\end{aligned}$$

which implies that

$$\|x\|\le(m_0\mathcal{Z}_1+n_0\mathcal{Z}_2)+(m_1\mathcal{Z}_1+n_1\mathcal{Z}_2+\mathcal{Z}_3)\|x\|+(m_2\mathcal{Z}_1+n_2\mathcal{Z}_2+\mathcal{Z}_4)\|y\|. \tag{21}$$

In a similar manner, we can find that

$$\|y\|\le(m_0\mathcal{N}_1+n_0\mathcal{N}_2)+(m_1\mathcal{N}_1+n_1\mathcal{N}_2+\mathcal{N}_3)\|x\|+(m_2\mathcal{N}_1+n_2\mathcal{N}_2+\mathcal{N}_4)\|y\|. \tag{22}$$

From (21) and (22), it follows that

$$\|x\| + \|y\| \le m_0(\mathcal{Z}_1 + \mathcal{N}_1) + n_0(\mathcal{Z}_2 + \mathcal{N}_2) + \max(\mathcal{W}_1, \mathcal{W}_2)\big(\|x\| + \|y\|\big),$$

where $\mathcal{W}_1$ and $\mathcal{W}_2$ are given in (17). The above inequality can alternatively be written as

$$\|(x, y)\| \le \frac{1}{G_0}[m_0(\mathcal{Z}_1 + \mathcal{N}_1) + n_0(\mathcal{Z}_2 + \mathcal{N}_2)],$$

where

$$G_0 = 1 - \max(\mathcal{W}_1, \mathcal{W}_2).$$

Therefore, the set Ω is bounded. In consequence, we deduce by the Leray-Schauder alternative [**20**] that there exists at least one fixed point for the operator $\mathcal{T}$. Hence, the problem (1) admits at least one solution on $\mathcal{J}$. □

Now, we accomplish a uniqueness result for the problem (1) by means of a fixed point theorem due to Banach.

Theorem 3.2. *Suppose that the condition* (H_2) *is satisfied. Then the problem* (1) *has a unique solution on* $\mathcal{J}$*, provided that*

$$\ell_1(\mathcal{Z}_1 + \mathcal{N}_1) + \ell_2(\mathcal{Z}_2 + \mathcal{N}_2) + \max(\mathcal{Z}_3 + \mathcal{N}_3, \mathcal{Z}_4 + \mathcal{N}_4) < 1, \tag{23}$$

where $\mathcal{Z}_p$ *and* $\mathcal{N}_p$, $p = 1, 2, 3, 4$, *are given in* (16).

Proof. Define a closed ball $B_r = \{(x, y) \in Y \times Y : \|(x, y)\| \le r\}$ with

$$r \ge \frac{\mathcal{M}_1(\mathcal{Z}_1 + \mathcal{N}_1) + \mathcal{M}_2(\mathcal{Z}_2 + \mathcal{N}2)}{1 - \big[\ell_1(\mathcal{Z}_1 + \mathcal{N}_1) + \ell_2(\mathcal{Z}_2 + \mathcal{N}_2)\big] - \max(\mathcal{Z}_3 + \mathcal{N}_3, \mathcal{Z}_4 + \mathcal{N}_4)}, \tag{24}$$

where $\sup_{t\in\mathcal{J}} |f_1(t, 0, 0)| = \mathcal{M}_1 < \infty$ and $\sup_{t\in\mathcal{J}} |f_2(t, 0, 0)| = \mathcal{M}_2 < \infty$. Now, we show that $\mathcal{T}B_r \subset B_r$, where $\mathcal{T} : B_r \longrightarrow Y \times Y$ is defined by (13). By (H_2), we have

$$|f_1(t, x(t), y(t))| \le |f_1(t, x(t), y(t)) - f_1(t, 0, 0,)| + |f_1(t, 0, 0,)| \le \ell_1 r + \mathcal{M}_1,$$
$$|f_2(t, x(t), y(t))| \le |f_2(t, x(t), y(t)) - f_2(t, 0, 0,)| + |f_2(t, 0, 0,)| \le \ell_2 r + \mathcal{M}_2.$$

For $(x, y) \in B_r$, we obtain

$$\begin{aligned}
&\|\mathcal{T}_1(x, y)\| \\
&\le \sup_{t\in\mathcal{J}} \Bigg\{ \int_0^t \Big[\frac{(t-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}\Big(\ell_1 r + \mathcal{M}_1\Big) + \lambda_1 \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big] \mathrm{d}s \\
&\quad + |\phi_1(t)| \Bigg\{ \sum_{i=1}^m |\mu_i| \int_0^{\eta_i} \Big[\frac{(\eta_i - s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}\Big(\ell_2 r + \mathcal{M}_2\Big) + \lambda_2 \frac{(\eta_i - s)^{\beta_2-1}}{\Gamma(\beta_2)}|y(s)|\Big] \mathrm{d}s \\
&\qquad + \int_0^1 \Big[\frac{(1-s)^{\beta_1+\alpha_1-2}}{\Gamma(\beta_1+\alpha_1-1)}\Big(\ell_1 r + \mathcal{M}_1\Big) + \lambda_1 \frac{(1-s)^{\beta_1-2}}{\Gamma(\beta_1-1)}|x(s)|\Big] \mathrm{d}s \Bigg\} \\
&\quad + |\phi_2(t)| \Bigg\{ \sum_{j=1}^n |\rho_j| \int_{\zeta_j}^{\xi_j} \int_0^s \Big[\frac{(s-u)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}\Big(\ell_2 r + \mathcal{M}_2\Big) + \lambda_2 \frac{(s-u)^{\beta_2-1}}{\Gamma(\beta_2)}|y(u)|\Big] \mathrm{d}u\mathrm{d}s \\
&\qquad + \int_0^1 \Big[\frac{(1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}(\ell_1 r + \mathcal{M}_1\Big) + \lambda_1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big] \mathrm{d}s \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+|\phi_3(t)|\Big\{\sum_{i=1}^{m}|\sigma_i|\int_0^{\eta_i}\Big[\frac{\eta_i-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}\Big(\ell_1 r+\mathcal{M}_1\Big)+\lambda_1\frac{(\eta_i-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x(s)|\Big]\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-2}}{\Gamma(\beta_2+\alpha_2-1)}\Big(\ell_2 r+\mathcal{M}_2\Big)+\lambda_2\frac{(1-s)^{\beta_2-2}}{\Gamma(\beta_2-1)}|y(s)|\Big]\mathrm{d}s\Big\}\\
&+|\phi_4(t)|\Big\{\sum_{j=1}^{n}|\omega_j|\int_{\zeta_j}^{\xi_j}\int_0^{s}\Big[\frac{(s-u)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}\Big(\ell_1 r+\mathcal{M}_1\Big)+\lambda_1\frac{(s-u)^{\beta_1-1}}{\Gamma(\beta_1)}|x(u)|\Big]\mathrm{d}u\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}\Big(\ell_2 r+\mathcal{M}_2\Big)+\lambda_2\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}|y(s)|\Big]\mathrm{d}s\Big\}\Big\}\\
&\le\big(\ell_1 r+\mathcal{M}_1\big)\mathcal{Z}_1+\mathcal{Z}_3\|x\|+\big(\ell_2 r+\mathcal{M}_2\big)\mathcal{Z}_2+\mathcal{Z}_4\|y\|.
\end{aligned}
$$

Likewise, we can find that

$$
\|\mathcal{T}_2(x,y)\|\le\big(\ell_1 r+\mathcal{M}_1\big)\mathcal{N}_1+\mathcal{N}_3\|x\|+\big(\ell_2 r+\mathcal{M}_2\big)\mathcal{N}_2+\mathcal{N}_4\|y\|.
$$

Using the above inequalities together with (24), we get

$$
\begin{aligned}
\|\mathcal{T}(x,y)\|&=\|\mathcal{T}_1(x,y)\|+\|\mathcal{T}_2(x,y)\|\\
&\le\big[\ell_1(\mathcal{Z}_1+\mathcal{N}_1)+\ell_2(\mathcal{Z}_2+\mathcal{N}_2)\big]r+\mathcal{M}_1(\mathcal{Z}_1+\mathcal{N}_1)+\mathcal{M}_2(\mathcal{Z}_2+\mathcal{N}_2)\\
&\quad+\max\left(\mathcal{Z}_3+\mathcal{N}_3,\mathcal{Z}_4+\mathcal{N}_4\right)r\le r,
\end{aligned}
$$

which shows that $\mathcal{T}(x,y)\in B_r$. Hence, $\mathcal{T}B_r\subset B_r$ as $(x,y)\in B_r$ is an arbitrary element.

Next, it will be established that the operator $\mathcal{T}$ is a contraction. For that, let $(x_1,y_1),(x_2,y_2)\in Y\times Y$. Then, for any $t\in\mathcal{J}$, we obtain

$$
\begin{aligned}
&\|\mathcal{T}_1(x_2,y_2)-\mathcal{T}_1(x_1,y_1)\|\\
&\le\sup_{t\in\mathcal{J}}\Big\{\int_0^t\Big[\frac{(t-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|\\
&\qquad+\lambda_1\frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x_2(s)-x_1(s)|\Big]\mathrm{d}s\\
&\quad+|\phi_1(t)|\Big\{\sum_{i=1}^{m}|\mu_i|\int_0^{\eta_i}\Big[\frac{(\eta_i-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}|f_2(s,x_2(s),y_2(s))-f_2(s,x_1(s),y_1(s))|\\
&\qquad+\lambda_2\frac{(\eta_i-s)^{\beta_2-1}}{\Gamma(\beta_2)}|y_2(s)-y_1(s)|\Big]\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-2}}{\Gamma(\beta_1+\alpha_1-1)}|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|\\
&\qquad+\lambda_1\frac{(1-s)^{\beta_1-2}}{\Gamma(\beta_1-1)}|x_2(s)-x_1(s)|\Big]\mathrm{d}s\Big\}\\
&\quad+|\phi_2(t)|\Big\{\sum_{j=1}^{n}|\rho_j|\int_{\zeta_j}^{\xi_j}\int_0^{s}\Big[\frac{(s-u)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}\Big|f_2(u,x_2(u),y_2(u))-f_2(u,x_1(u),y_1(u))|\\
&\qquad+\lambda_2\frac{(s-u)^{\beta_2-1}}{\Gamma(\beta_2)}|y_2(u)-y_1(u)|\Big]\mathrm{d}u\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|\\
&\qquad+\lambda_1\frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x_2(s)-x_1(s)|\Big]\mathrm{d}s\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+|\phi_3(t)|\Big\{\sum_{i=1}^{m}|\sigma_i|\int_0^{\eta_i}\Big[\frac{\eta_i-s)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|\\
&\qquad+\lambda_1\frac{(\eta_i-s)^{\beta_1-1}}{\Gamma(\beta_1)}|x_2(s)-x_1(s)|\Big]\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-2}}{\Gamma(\beta_2+\alpha_2-1)}|f_2(s,x_2(s),y_2(s))-f_2(s,x_1(s),y_1(s))|\\
&\qquad+\lambda_2\frac{(1-s)^{\beta_2-2}}{\Gamma(\beta_2-1)}|y_2(s)-y_1(s))|\Big]\mathrm{d}s\Big\}\\
&+|\phi_4(t)|\Big\{\sum_{j=1}^{n}|\omega_j|\int_{\zeta_j}^{\xi_j}\int_0^s\Big[\frac{(s-u)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)}|f_1(u,x_2(u),y_2(u))-f_1(u,x_1(u),y_1(u))|\\
&\qquad+\lambda_1\frac{(s-u)^{\beta_1-1}}{\Gamma(\beta_1)}|x_2(u)-x_1(u)|\Big]\mathrm{d}u\mathrm{d}s\\
&\qquad+\int_0^1\Big[\frac{(1-s)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)}\Big|f_2(s,x_2(s),y_2(s))-f_2(s,x_1(s),y_1(s))|\\
&\qquad+\lambda_2\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}|y_2(s)-y_1(s)|\Big]\mathrm{d}s\Big\}\Big\}\\
&\le\big(\ell_1\mathcal{Z}_1+\ell_2\mathcal{Z}_2\big)(\|x_2-x_1\|+\|y_2-y_1\|)+\mathcal{Z}_3\|x_2-x_1\|+\mathcal{Z}_4\|y_2-y_1\|.
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
&\|\mathcal{T}_2(x_2,y_2)-\mathcal{T}_2(x_1,y_1)\|\\
&\le\big(\ell_1\mathcal{N}_1+\ell_2\mathcal{N}_2\big)(\|x_2-x_1\|+\|y_2-y_1\|)+\mathcal{N}_3\|x_2-x_1\|+\mathcal{N}_4\|y_2-y_1\|.
\end{aligned}
$$

From the foregoing inequalities, we have

$$
\begin{aligned}
&\|\mathcal{T}(x_2,y_2)-\mathcal{T}(x_1,y_1)\|\\
&\le\big[\ell_1(\mathcal{Z}_1+\mathcal{N}_1)+\ell_2(\mathcal{Z}_2+\mathcal{N}_2)+\max\left(\mathcal{Z}_3+\mathcal{N}_3,\mathcal{Z}_4+\mathcal{N}_4\right)\big]\\
&\quad\times(\|x_2-x_1\|+\|y_2-y_1\|),
\end{aligned}
$$

which, in view of (23), implies that $\mathcal{T}$ is a contraction. Thus, the conclusion of Banach's fixed point theorem applies and the operator $\mathcal{T}$ has a unique fixed point. Therefore, the problem (1) has a unique solution on $\mathcal{J}$. □

4. Examples

Example 4.1. Consider a coupled system of nonlinear Langevin equations:

$$
\left\{
\begin{array}{l}
D^{\alpha_1}(D^{\beta_1}+\lambda_1)x(t)=f_1(t,x(t),y(t)),\ t\in\mathcal{J},\\
D^{\alpha_2}(D^{\beta_2}+\lambda_2)y(t)=f_2(t,x(t),y(t)),\ t\in\mathcal{J},\\
x(0)=0,\quad x'(1)=\displaystyle\sum_{i=1}^{m}\mu_i y(\eta_i),\quad x(1)=\sum_{j=1}^{n}\rho_j\int_{\zeta_j}^{\xi_j}y(s)\mathrm{d}s,\\
y(0)=0,\quad y'(1)=\displaystyle\sum_{i=1}^{m}\sigma_i x(\eta_i),\quad y(1)=\sum_{j=1}^{n}\omega_j\int_{\zeta_j}^{\xi_j}x(s)\mathrm{d}s,
\end{array}
\right. \tag{25}
$$

where $\alpha_1 = 3/4$, $\alpha_2 = 1/2$, $\beta_1 = 3/2$, $\beta_2 = 4/3$, $\eta_1 = 1/28$, $\eta_2 = 1/21$, $\eta_3 = 1/14$, $\mu_1 = 1/15$, $\mu_2 = 1/10$, $\mu_3 = 1/5$, $\sigma_1 = 1/15$, $\sigma_2 = 1/10$, $\sigma_3 = 1/5$, $\zeta_1 = 1/9$, $\zeta_2 = 3/9$, $\zeta_3 = 5/9$, $\xi_1 = 2/9$, $\xi_2 = 4/9$, $\xi_3 = 6/9$, $\lambda_1 = 1/20$, $\lambda_2 = 1/4$, $\rho_1 = 1/11$, $\rho_2 = 1/5$, $\rho_3 = 1/3$, $\omega_1 = 1/12$, $\omega_2 = 1/6$, $\omega_3 = 1/4$, $m = n = 3$, $\mathcal{J} = [0, 1]$.

Using the given values, we find that $Z_1 \approx 0.78606595$, $Z_2 \approx 0.08344012$, $Z_3 \approx 0.07533521$, $Z_4 \approx 0.02825324$, $N_1 \approx 0.04145488$, $N_2 \approx 1.16257045$, $N_3 \approx 0.00364788$, and $N_4 \approx 0.42076590$. $\big(\mathcal{Z}_p$ and $\mathcal{N}_p$, $p = 1, 2, 3, 4$, are given in (16)$\big)$. In order to illustrate Theorem 3.1, we take the following nonlinear functions in (25):

$$
\begin{aligned}
f_1(t, x(t), y(t)) &= \frac{1}{12} + \frac{|x(t)|}{210(1 + |x(t)|)} + \frac{|y(t)|}{(t+2)^3(1 + |y(t)|)}, \\
f_2(t, x(t), y(t)) &= \frac{1}{6} + \frac{|x(t)|}{2(1 + |x(t)|)} + \frac{|y(t)|}{(t+3)^3(1 + |y(t)|)}.
\end{aligned} \tag{26}
$$

For each $t \in [0, 1]$ and $x, y \in \mathbb{R}$, we have

$$
\begin{aligned}
|f_1(t, x(t), y(t))| &\leq \frac{1}{12} + \frac{1}{210}|x(t)| + \frac{1}{8}|y(t)|, \\
|f_2(t, x(t), y(t))| &\leq \frac{1}{6} + \frac{1}{2}|x(t)| + \frac{1}{27}|y(t)|.
\end{aligned}
$$

Hence, the assumption (H_1) holds with $m_0 = 1/12$, $m_1 = 1/210$, $m_2 = 1/8$, $n_0 = 1/6$, $n_1 = 1/2$, $n_2 = 1/27$. Moreover, we find that

$$
\mathcal{W}_1 \approx 0.70592896 < 1 \quad \text{and} \quad \mathcal{W}_2 \approx 0.59860778 < 1.
$$

Thus, the hypothesis of Theorem 3.1 is verified. Therefore, by the conclusion of Theorem 3.1, the Langevin system (25), with the nonlinear functions f_1 and f_2 given by (26) has at least one solution on $\mathcal{J}$.

Example 4.2. Consider a coupled fractional boundary value problem of

$$
\begin{cases}
D^{\alpha_1}(D^{\beta_1} + \lambda_1)x(t) = f_1(t, x(t), y(t)), \ t \in \mathcal{J}, \\
D^{\alpha_2}(D^{\beta_2} + \lambda_2)y(t) = f_2(t, x(t), y(t)), \ t \in \mathcal{J}, \\
x(0) = 0, \quad x'(1) = \displaystyle\sum_{i=1}^{m} \mu_i y(\eta_i), \quad x(1) = \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\xi_j} y(s)\mathrm{d}s, \\
y(0) = 0, \quad y'(1) = \displaystyle\sum_{i=1}^{m} \sigma_i x(\eta_i), \quad y(1) = \sum_{j=1}^{n} \omega_j \int_{\zeta_j}^{\xi_j} x(s)\mathrm{d}s,
\end{cases} \tag{27}
$$

where $\alpha_1 = 3/4$, $\alpha_2 = 1/2$, $\beta_1 = 3/2$, $\beta_2 = 4/3$, $\eta_1 = 1/28$, $\eta_2 = 1/21$, $\eta_3 = 1/14$, $\mu_1 = 1/15$, $\mu_2 = 1/10$, $\mu_3 = 1/5$, $\sigma_1 = 1/15$, $\sigma_2 = 1/10$, $\sigma_3 = 1/5$, $\zeta_1 = 1/9$, $\zeta_2 = 3/9$, $\zeta_3 = 5/9$, $\xi_1 = 2/9$, $\xi_2 = 4/9$, $\xi_3 = 6/9$, $\lambda_1 = 1/20$, $\lambda_2 = 1/4$, $\rho_1 = 1/11$, $\rho_2 = 1/5$, $\rho_3 = 1/3$, $\omega_1 = 1/12$, $\omega_2 = 1/6$, $\omega_3 = 1/4$, $m = n = 3$, $\mathcal{J} = [0, 1]$.

Making use of the given data, we find that: $Z_1 \approx 0.78606595$, $Z_2 \approx 0.08344012$, $Z_3 \approx 0.07533521$, $Z_4 \approx 0.02825324$, $N_1 \approx 0.04145488$, $N_2 \approx 1.16257045$, $N_3 \approx 0.00364788$, $N_4 \approx 0.42076590$. $\big(\mathcal{Z}_p$ and $\mathcal{N}_p$, $p = 1, 2, 3, 4$, are given in (16)$\big)$.

For demonstrating the application of Theorem 3.2, we take the nonlinear functions in (27) as

$$\begin{aligned} f_1(t,x(t),y(t)) &= \frac{1}{100}\tan^{-1}x(t) + \frac{1}{4(t+5)^2(1+|y(t)|)} + 2, \\ f_2(t,x(t),y(t)) &= \frac{1}{3\sqrt{t^3+144}}\Big(\cos x(t) + \frac{|y(t)|}{1+|y(t)|}\Big) + 5\mathrm{e}^t. \end{aligned} \tag{28}$$

For each $t \in \mathcal{J}$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}$, we notice that

$$\begin{aligned} |f_1(t,x_2,y_2) - f_1(t,x_1,y_1)| &\le \frac{1}{100}(|x_2-x_1| + |y_2-y_1|), \\ |f_2(t,x_2,y_2) - f_2(t,x_1,y_1)| &\le \frac{1}{36}(|x_2-x_1| + |y_2-y_1|). \end{aligned}$$

Clearly, the condition (H_2) holds true with $\ell_1 = 1/100$ and $\ell_2 = 1/36$. Further, we have

$$\ell_1(\mathcal{Z}_1 + \mathcal{N}_1) + \ell_2(\mathcal{Z}_2 + \mathcal{N}_2) + \max(\mathcal{Z}_3 + \mathcal{N}_3, \mathcal{Z}_4 + \mathcal{N}_4) \approx 0.49190575 < 1.$$

Since the assumptions of Theorem 3.2 are satisfied, therefore, it follows by its conclusion that the problem (27) with f_1 and f_2 given in (28) has a unique solution on $\mathcal{J}$.

5. Conclusion

In this study, we have established the existence and uniqueness results for a coupled system of Riemann–Liouville type nonlinear fractional Langevin equations equipped with nonlocal multi-point and multi-strip coupled boundary conditions. Our results are novel in the given configuration and contribute significantly to the related literature on fractional Langevin equations. Some new results can be obtained from the present ones by fixing the parameters involved in boundary data of the given problem. For example, by taking $\mu_i = \sigma_i = 0$ for $i = 1, \ldots, m$, our results correspond to the ones equipped with multi-strip coupled boundary conditions

$$\begin{aligned} &x(0) = 0, \quad y(0) = 0, \quad x'(1) = 0, \quad y'(1) = 0, \\ &x(1) = \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\xi_j} y(s)\mathrm{d}s, \quad y(1) = \sum_{j=1}^{n} \omega_j \int_{\zeta_j}^{\xi_j} x(s)\mathrm{d}s. \end{aligned}$$

By taking $\rho_j = \omega_j = 0$ for $j = 1, \ldots, n$, in the present results, we obtain the ones associated with multi-point coupled boundary conditions

$$\begin{aligned} &x(0) = 0, \quad x(1) = 0, \quad y(0) = 0, \quad y(1) = 0, \\ &x'(1) = \sum_{i=1}^{m} \mu_i y(\eta_i), \quad y'(1) = \sum_{i=1}^{m} \sigma_i x(\eta_i). \end{aligned}$$

References

1. Abbas S., Ahmad B., Benchohra M. and Salim A., *Fractional Difference, Differential Equations, and Inclusions: Analysis and Stability*, Elsevier, 2024.

2. Agarwal R., Hristova S. and O'Regan D., *Integral presentations of the solution of a boundary value problem for impulsive fractional integro-differential equations with Riemann–Liouville derivatives*, AIMS Math. **7** (2022), 2973–2988.

3. Agarwal R. P., Assolami A., Alsaedi A. and Ahmad B., *Existence results and Ulam-Hyers stability for a fully coupled system of nonlinear sequential Hilfer fractional differential equations and integro-multistrip-multipoint boundary conditions*, Qual. Theory Dyn. Syst. **21** (2022), Paper No. 125.

4. Ahmad B. and Ntouyas S. K., *Nonlocal Nonlinear Fractional-Order Boundary Value Problems*, World Scientific: Singapore, 2021.

5. Ahmad B., Henderson J. and Luca R., *Boundary Value Problems for Fractional Differential Equations and Systems*, World Scientific, Singapore, 2021.

6. Ahmad B. and Ntouyas S. K., *Nonlinear Systems of Fractional Differential Equations*, Springer Nature Switzerland, 2024.

7. Ahmad B., Hamdan S., Alsaedi A. and Ntouyas S. K., *On a nonlinear mixed-order coupled fractional differential system with new integral boundary conditions*, AIMS Math. **6** (2021), 5801–5816.

8. Ahmad B., Alsaedi A. and Aljahdali A. S., *On a coupled system of nonlinear generalized fractional differential equations with nonlocal coupled Riemann-Stieltjes boundary conditions*, Qual. Theory Dyn. Syst. **23** (2024), Art. No. 215.

9. Ahmad B. and Nieto J. J., *Solvability of nonlinear Langevin equation involving two fractional orders with Dirichlet boundary conditions*, Int. J. Differ. Equ. **2010** (2010), Art. No. 649486.

10. Ahmad B., Nieto J. J., Alsaedi A. and El-Shahed M., *A study of nonlinear Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal. Real World Appl. **13** (2012), 599–606.

11. Ahmad B. and Ntouyas S. K., *New existence results for differential inclusions involving Langevin equation with two indices*, J. Nonlinear Convex Anal. **14** (2013), 437–450.

12. Ahmad B., Ahmed A. and Salem S., *On a nonlocal integral boundary value problem of nonlinear Langevin equation with different fractional orders*, Adv. Difference Equ. **2019** (2019), Art. No. 57.

13. Baghani H. and Nieto J. J., *On fractional Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal. Model. Control **24** (2019), 884–897.

14. Baghani H., Alzabut J. and Nieto J. J., *A coupled system of Langevin differential equations of fractional order and associated to antiperiodic boundary conditions*, Math. Methods Appl. Sci. **47** (2024), 10900–10910.

15. Berhail A., Tabouche N., Matar M. M. and Alzabut J., *Boundary value problem defined by system of generalized Sturm-Liouville and Langevin Hadamard fractional differential equations*, Math. Methods Appl. Sci. **47** (2024), 10786–10798.

16. Cen Z., Liu L.-B. and Huang J., *A posteriori error estimation in maximum norm for a two-point boundary value problem with a Riemann-Liouville fractional derivative*, Appl. Math. Lett. **102** (2020), Art. No. 106086.

17. Dien N. M. and Viet T. Q., *On mild solutions of the p-Laplacian fractional Langevin equations with anti-periodic type boundary conditions*, Int. J. Comput. Math. **99** (2022), 1823–1848.

18. Eab C. H. and Lim S. C., *Fractional generalized Langevin equation approach to single-file diffusion*, Physica A **389** (2010) 2510–2521.

19. Eule S., Friedrich R., Jenko F. and Kleinhans D., *Langevin approach to fractional diffusion equations including inertial effects*, J. Phys. Chem. B **111** (2007), 11474–11477.

20. Granas A. and Dugundji J., *Fixed Point Theory*, Springer-Verlag, New York, 2003.

21. Guendouz C., Lazreg J. E., Nieto J. J. and Ouahab A., *Existence and compactness results for a system of fractional differential equations*, J. Funct. Spaces (2020), Art. No. 5735140.

22. Iskenderoglu G. and Kaya D., *Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense*, Chaos Solitons Fractals **134** (2020), Art. No. 109684.

23. Kamsrisuk N., Ntouyas S. K., Ahmad B., Samadi A. and Tariboon J., *Existence results for a coupled system of (k, φ)-Hilfer fractional differential equations with nonlocal integro-multipoint boundary conditions*, AIMS Math. **8** (2023), 4079–4097.

24. Kang P., *Positive solutions for a singular system of nonlinear fractional differential equations*, Studia Sci. Math. Hungar. **59** (2022), 183–195.

25. Kiataramkul C., Ntouyas S. K. and Tariboon J., *Existence results for ψ-Hilfer fractional integro-differential hybrid boundary value problems for differential equations and inclusions*, Adv. Math. Phys. **2021** (2021), 1–12.

26. Kilbas A. A., Srivastava H. M. and Trujillo J. J., *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, Vol. 204, Elsevier Science B. V., 2006.

27. Kirane M. and Abdeljabbar A., *Nonexistence of global solutions of systems of time fractional differential equations posed on the Heisenberg group*, Math. Methods Appl. Sci. **45** (2022), 7336–7345.

28. Klages R., Radons G. and Sokolov M. (eds.), *Anomalous Transport: Foundations and Applications*, Wiley VCH, Weinheim, Germany, 2008.

29. Kubo R., *The fluctuation-dissipation theorem*, Rep. Prog. Phys. **29** (1966), 255–284.

30. Kubo R., Toda M. and Hashitsume N., *Statistical physics II: nonequilibrium statistical mechanics*. Vol. 2., Springer Science Business Media, 1991.

31. Kucche K. D. and Mali A. D., *On the nonlinear (k, ψ)-Hilfer fractional differential equations*, Chaos Solitons Fractals **152** (2021), Art. No. 111335.

32. Laadjal Z. and Jarad F., *Existence, uniqueness and stability of solutions for generalized proportional fractional hybrid integro-differential equations with Dirichlet boundary conditions*, AIMS Math. **8** (2023), 1172–1194.

33. Laadjal Z., Al-Mdallal Q. M. and Jarad F., *Analysis of a coupled system of nonlinear fractional Langevin equations with certain nonlocal and non-separated boundary conditions*, J. Math. (2021), Art. ID 3058414.

34. Langevin P., *Sur la théorie du mouvement brownien*, CR Acad. Sci. Paris **146** (1908), 530–533.

35. Li B., Sun S. and Sun Y., *Existence of solutions for fractional Langevin equation with infinite-point boundary conditions*, J. Appl. Math. Comput. **53** (2017), 683–692.

36. Lim S. C., Li M. and Teo L. P., *Langevin equation with two fractional orders*, Phys. Lett. A **372** (2008), 6309–6320.

37. Ma L. and Wu B., *On the fractional Lyapunov exponent for Hadamard-type fractional differential system*, Chaos **33** (2023), Art. No. 013117.

38. Muensawat T., Ntouyas S. K. and Tariboon J., *Systems of generalized Sturm-Liouville and Langevin fractional differential equations*, Adv. Difference Equ. **2017** (2017), Art. No. 63.

39. Nieto J. J., *Fractional Euler numbers and generalized proportional fractional logistic differential equation*, Fract. Calc. Appl. Anal. **25** (2022), 876–886.

40. Nyamoradi N. and Ahmad B., *Hadamard fractional differential equations on an unbounded domain with integro-initial conditions*, Qual. Theory Dyn. Syst. **23** (2024), Art. No. 183.

41. Shah S. O., Rizwan R., Xia Y. and Zada A., *Existence, uniqueness, and stability analysis of fractional Langevin equations with anti-periodic boundary conditions*, Math. Methods Appl. Sci. **46** (2023), 17941–17961.

42. Sudsutad W., Ahmad B., Ntouyas S. K. and Tariboon J., *Impulsively hybrid fractional quantum Langevin equation with boundary conditions involving Caputo q_k-fractional derivatives*, Chaos Solitons Fractals **91** (2016), 47–62.

43. Thaiprayoon C., Ntouyas S. K. and Tariboon J., *On systems of fractional Langevin equations of Riemann-Liouville type with generalized nonlocal fractional integral boundary conditions*, J. Comput. Anal. Appl. **27** (2019), 727–737.

44. Torres C., *Existence of solution for fractional Langevin equation: variational approach*, Electron. J. Qual. Theory Differ. Equ. **2014** (2014), Art. No. 54.

45. West B. J. and Latka M., *Fractional Langevin model of gait variability*, J. Neuroeng. Rehabil. **2** (2005), Art. No. 24.

46. Wongcharoen A., Ntouyas S. K., Wongsantisuk P. and Tariboon J., *Existence results for a nonlocal coupled system of sequential fractional differential equations involving ψ-Hilfer fractional derivatives*, Adv. Math. Phys. **2021** (2021), Art. 5554619.

47. Zhang W. and Ni J., *Qualitative analysis of tripled system of fractional Langevin equations with cyclic anti-periodic boundary conditions*, Fract. Calc. Appl. Anal. **26** (2023), 2392–2420.

48. Zhou Z. and Qiao Y., *Solutions for a class of fractional Langevin equations with integral and anti-periodic boundary conditions*, Bound. Value Probl. **2018** (2018), Art. No. 152.

A. Alsaedi, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia,
e-mail: aalsaedi@hotmail.com

H. A. Saeed, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia,
e-mail: hafed2006@gmail.com

B. Ahmad, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia,
e-mail: bashirahmad_qau@yahoo.com

S. K. Ntouyas, Department of Mathematics, University of Ioannina, 451 10, Ioannina, Greece,
e-mail: sntouyas@uoi.gr