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SOME RESULTS ON APPROXIMATE CHARACTER AMENABILITY OF BANACH ALGEBRAS

B. SHOJAEE AND A. BODAGHI

Abstract. In this paper, among many other things we prove that two Banach algebras are both approximately character amenable if and only if their direct sum is approximately character amenable. Some examples of approximately left character amenable Banach algebras which are not left character amenable are given.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then, the first dual of $\mathcal{X}$, denoted by $\mathcal{X}^*$, has a natural $\mathcal{A}$-bimodule structure defined by

$$\langle a \cdot x^*, x\rangle = \langle x^*, x \cdot a\rangle, \quad \langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle \quad (a \in \mathcal{A},\ x \in \mathcal{X},\ x^* \in \mathcal{X}^*).$$

A derivation $D\colon \mathcal{A} \to \mathcal{X}^*$ is a continuous linear map such that $D(ab) = a \cdot D(b) + D(a) \cdot b$ for $a, b \in \mathcal{A}$. Given $x^* \in \mathcal{X}^*$, the inner derivation $\delta_{x^*}\colon \mathcal{A} \to \mathcal{X}^*$ is defined by $\delta_{x^*}(a) = a \cdot x^* - x^* \cdot a$. A Banach algebra $\mathcal{A}$ is called amenable if every derivation from $\mathcal{A}$ into each dual Banach $\mathcal{A}$-bimodule $\mathcal{X}^*$ is inner. This concept for Banach algebras was first introduced by B. E. Johnson in [**9**]. A derivation $D\colon \mathcal{A} \to \mathcal{X}^*$ is called approximately inner if there exists a net $(x^*_\alpha) \subseteq \mathcal{X}^*$ such that $D(a) = \lim_\alpha \delta_{x^*_\alpha}(a) = \lim_\alpha a \cdot x^*_\alpha - x^*_\alpha \cdot a$ for all $a \in \mathcal{A}$. The notion of approximate amenability was presented and studied by Gahramani and Loy in [**5**] and followed in [**6**]. A Banach algebra $\mathcal{A}$ is said to be approximately amenable if every derivation $D\colon \mathcal{A} \to \mathcal{X}^*$ is approximately inner for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$.

Let $\mathcal{A}$ be a Banach algebra, and $\sigma(\mathcal{A})$ be the set of all non-zero multiplicative linear functionals on $\mathcal{A}$. Suppose that $\varphi \in \sigma(\mathcal{A})$, then $\mathcal{A}$ is said to be φ-amenable if there exists $m \in \mathcal{A}^{**}$ such that $m(\varphi) = 1$ and $m(a^* \cdot a) = \varphi(a)m(a^*)$ for all $a \in \mathcal{A}, a^* \in \mathcal{A}^*$. This concept was introduced by Kaniuth, Lau and Pym in [**10**]. After that they characterized φ-amenability in terms of cohomology groups and Hahn-Banach type extension property [**11**]. Monfared [**12**] introduced the concepts of left and right character amenability for Banach algebras and showed that for any locally compact group G, (right) character amenability of the group algebra $L^1(G)$ is equivalent to the amenability of G. Module character amenability of Banach algebras which defines the notion of invariant functional with respect

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to a Banach bimodule with compatible actions and application to the semigroup algebras of inverse semigroups is introduced in [**1**]; see also [**2**]. Precisely, a Banach algebra $\mathcal{A}$ is called left (right) φ-amenable if every derivation from $\mathcal{A}$ into $\mathcal{X}^*$ is inner for which left (right) module action of $\mathcal{X}$ is $a \cdot x = \varphi(a)x$ ($x \cdot a = \varphi(a)x$) for every $a \in \mathcal{A}$ and $x \in \mathcal{X}$, where $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$. We should mention that left φ-amenability is equivalent to φ-amenability in the sense of [**10**]. Moreover, $\mathcal{A}$ is called [approximately] left (right) character amenable if for any $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$, every derivation $D\colon \mathcal{A} \to \mathcal{X}^*$ for any Banach $\mathcal{A}$-bimodule $\mathcal{X}$ with left (right) module action defined by $a \cdot x = \varphi(a)x$ ($x \cdot a = \varphi(a)x$) for every $a \in \mathcal{A}$, $x \in \mathcal{X}$. The concepts of approximate left and right character amenability for Banach algebras introduced in [**14**].

In this paper, we obtain some results on [approximate] left (right) character amenability. Moreover, we prove that Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are both approximately character amenable if and only if $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately character amenable. Also, we provide some Banach algebras which are approximate character amenable.

2. Main results

Suppose that $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$ and $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. In view of [**12, 14**], we say that $\mathcal{X}$ is a $(\varphi, \mathcal{A})$-bimodule if the left action of $\mathcal{A}$ on $\mathcal{X}$ is given by

$$a \cdot x = \varphi(a)x \qquad (a \in \mathcal{A}, x \in \mathcal{X}).$$

So the right module action of $\mathcal{A}$ on $\mathcal{X}^*$ is $x^* \cdot a = \varphi(a)x^*$ for all $x^* \in \mathcal{X}^*$ and $a \in \mathcal{A}$. Similarly a Banach $(\mathcal{A}, \varphi)$-bimodule $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule in which the right action is given by

$$x \cdot a = \varphi(a)x \qquad (a \in \mathcal{A}, x \in \mathcal{X}).$$

The Banach algebra $\mathcal{A}$ is called approximately left (right) φ-amenable if every derivation $D\colon \mathcal{A} \to \mathcal{X}^*$ is approximately inner for all Banach $(\varphi, \mathcal{A})$-bimodule $\mathcal{X}$ ($(\mathcal{A}, \varphi)$-bimodule $\mathcal{X}$). Also, $\mathcal{A}$ is called approximately left (right) character amenable if for each $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$, every derivation $D\colon \mathcal{A} \to \mathcal{X}^*$ is approximately inner for all Banach $(\varphi, \mathcal{A})$-bimodules $\mathcal{X}$ ($(\mathcal{A}, \varphi)$-bimodule $\mathcal{X}$).

Recall that an $\mathcal{A}$-bimodule $\mathcal{X}$ is neo-unital if

$$\mathcal{X} = \mathcal{A} \cdot \mathcal{X} \cdot \mathcal{A} = \{a \cdot x \cdot b : a, b \in \mathcal{A}, x \in \mathcal{X}\}.$$

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and $\varphi \in \sigma(\mathcal{A})$. Then $\mathcal{A}$ is approximately left φ-amenable if and only if for every neo-unital Banach $(\varphi, \mathcal{A})$-bimodule $\mathcal{X}$, every derivation $D\colon \mathcal{A} \to \mathcal{X}^*$ is approximately inner.*

Proof. We only need to prove the necessary part since the other part is clear. Assume that $\mathcal{A}$ has a bounded approximate identity (e_α), $\mathcal{X}$ is a $(\varphi, \mathcal{A})$-bimodule and $D\colon \mathcal{A} \to \mathcal{X}^*$ is a derivation. Following the usual argument [**9**, Proposition 1.3], set $\mathcal{X}_1 = \{x \cdot a : a \in \mathcal{A}, x \in \mathcal{X}\}$. Considering the identification $\mathcal{X}^* = \mathcal{X}_1^* \oplus \mathcal{X}_1^\perp$ as a sum of closed submodules, we obtain that the maps PD and $(I - P)D$ are derivations, where P is the canonical projection from $\mathcal{X}^*$ to $\mathcal{X}_1^*$ and I is the identity map on $\mathcal{X}^*$. Since $\mathcal{A}$ has a bounded approximate identity, $\mathcal{X}_1$ is a neo-unital Banach $(\varphi, \mathcal{A})$-bimodule, and thus PD is approximately inner. Moreover, $\mathcal{A}$ has the trivial left action on $\mathcal{X}_1^{\perp}$, so we have that for any $a \in \mathcal{A}$,

$$\begin{aligned}(I-P)D(a) &= \lim_\alpha (I-P)D(e_\alpha a)\\ &= \lim_\alpha ((I-P)D(e_\alpha)\cdot a + (I-P)e_\alpha \cdot D(a)) = \lim_\alpha (I-P)D(e_\alpha)\cdot a.\end{aligned}$$

Hence $(I-P)D$ is also approximately inner, and thus D is approximately inner. □

Lemma 2.2. *Let $\mathcal{A}$ be a unital Banach algebra with the identity e. If $\mathcal{X}$ is a Banach $(\varphi, \mathcal{A})$-bimodule and $D\colon \mathcal{A} \to \mathcal{X}^*$ is a derivation, then there is a derivation $D_1\colon \mathcal{A} \to e\cdot\mathcal{X}^*$ and $\eta \in \mathcal{X}^*$ such that*

$$\|\eta\| \le 2\|D\| \quad \textit{and} \quad D = D_1 + \delta_\eta.$$

Proof. Put $Y_1 = e\cdot\mathcal{X}^*$, $Y_2 = (1-e)\cdot\mathcal{X}^* : = \mathcal{X}^* - e\cdot\mathcal{X}^*$. So $\mathcal{X}^* = Y_1 \oplus Y_2$. Let $\pi_j\colon \mathcal{X}^* \to Y_j$ $(j = 1, 2)$ be the associated projections. Then for the derivations $D_j = \pi_j \circ D\colon \mathcal{A} \to Y_j$ $(j = 1, 2)$, we have $D = D_1 + D_2$. On the other hand,

$$\begin{aligned}D_2(a) &= D_2(e)\cdot a + a\cdot D_2(e) = \varphi(a)D_2(e)\\ &= a\cdot(-D_2(e)) - \varphi(a)(-D_2(e)) = \delta_{-D_2(e)}(a).\end{aligned}$$

Note that in the above equalities, we have used the fact $a\cdot D_2(e) = a(1-e)\cdot x^* = 0$ where $x^* \in \mathcal{X}^*$. Since $e\cdot D_2(e) = 0$ and $D_1(e) = e\cdot D_1(e)$, we have

$$\begin{aligned}-D_2(e) &= D_1(e) - D(e) = D_1(e) - e\cdot D(e) + e\cdot D(e) - D(e)\\ &= D_1(e) - e\cdot(D_1(e) + D_2(e)) + e\cdot D(e) - D(e) = -(1-e)\cdot D(e).\end{aligned}$$

Setting $\eta = -D_2(e)$, we deduce that $\|\eta\| \le 2\|D\|$. □

Lemma 2.3. *Let $\mathcal{A}$ be a unital (approximately) left φ-amenable Banach algebra with the identity e for some $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$. If $\mathcal{X}$ is a Banach $(\varphi, \mathcal{A})$-bimodule and $D\colon \mathcal{A} \to \mathcal{X}^*$ is a derivation, then there is an element $x^* \in e\cdot\mathcal{X}^*$ (resp., a net $(x_\alpha^*) \subseteq e\cdot\mathcal{X}^*$) and $\eta \in \mathcal{X}^*$ such that*

$$\|\eta\| \le 2\|D\| \qquad \textit{and} \qquad D = \delta_{x^*} + \delta_\eta \qquad (\textit{resp.,} \quad D(a) = \lim_\alpha \delta_{x_\alpha^*}(a) + \delta_\eta(a)).$$

Proof. Lemma 2.2 makes D decompose into D_1 and δ_η. On the other hand, from the left φ-amenability of D_1 and $e\cdot\mathcal{X}^* = (\mathcal{X}\cdot e)^*$, it follows that $D_1 = \delta_{x^*}$ for some $x^* \in e\cdot\mathcal{X}^*$. For the case of approximately left φ-amenability, the argument is similar. □

Let $\mathcal{A}$ be a Banach algebra without identity and let $\mathcal{A}^\# = \mathcal{A} \oplus \mathbb{C}e$ denote the unitization of $\mathcal{A}$ by adjoining an identity e. Then every $\varphi \in \sigma(\mathcal{A})$ extends uniquely to an element $\widetilde{\varphi} \in \sigma(\mathcal{A}^\#)$ by $\widetilde{\varphi}(a + \lambda e) = \varphi(a) + \lambda$ $(a \in \mathcal{A},\ \lambda \in \mathbb{C})$.

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra without a unit and $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$. Then $\mathcal{A}$ is [approximately] left (right) φ-amenable if and only if $\mathcal{A}^\#$ is [approximately] left (right) $\widetilde{\varphi}$-amenable.*

Proof. We obtain the result for the left case. Suppose $\mathcal{X}$ is a Banach $(\widetilde{\varphi}, \mathcal{A}^{\#})$-bimodule and $D\colon \mathcal{A}^{\#} \to \mathcal{X}^*$ is a derivation. By Lemma 2.2, $D = D_1 + \delta_\eta$, where $D_1\colon \mathcal{A}^{\#} \to e \cdot \mathcal{X}^*$ and $\eta \in \mathcal{X}^*$. Since $D_1(e) = 0$ by the assumption, $D_1|_{\mathcal{A}}$ is [approximately] inner, so is D.

Conversely, assume that $\mathcal{X}$ is a $(\varphi, \mathcal{A})$-bimodule and $D\colon \mathcal{A} \to \mathcal{X}^*$ is a derivation. Consider the mapping

$$\widetilde{D}\colon \mathcal{A}^{\#} \to \mathcal{X}^*, \qquad \widetilde{D}(a + \lambda e) = D(a) \qquad (a \in \mathcal{A},\ \lambda \in C).$$

If we define $(a + \lambda e) \cdot x = (\varphi(a) + \lambda)x$ and $e \cdot x = x$, then $\mathcal{X}$ is a $(\widetilde{\varphi}, \mathcal{A}^{\#})$-bimodule and $\widetilde{D}$ is a derivation so there exists a net $(x^*_\alpha) \subseteq \mathcal{X}^*$ such that

$$D(a) = \widetilde{D}(a + \lambda e) = \lim_\alpha (a + \lambda e) \cdot x^*_\alpha - (\varphi(a) + \lambda)x^*_\alpha = \lim_\alpha a \cdot x^*_\alpha - \varphi(a)x^*_\alpha.$$

This shows that D is approximately inner. $\square$

Note that the above proposition was proved in [**14**, Lemma 3.7] with a different method. In fact, they obtained the result by using the concept of a mean while we have used derivations.

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras. Then $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately left (right) character amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are approximately left (right) character amenable.*

Proof. We obtain the result for the left case. The other case is similar. Suppose that $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately left character amenable. By [**14**, Proposition 3.1], $\mathcal{A}$ and $\mathcal{B}$ are approximately left character amenable since the natural projections on $\mathcal{A}$ and $\mathcal{B}$ are continuous epimorphisms.

Conversely, assume that $\mathcal{A}$ and $\mathcal{B}$ are both approximately left character amenable, $\varphi \in \sigma(\mathcal{A}\oplus_1\mathcal{B}) \cup \{0\}$, $\mathcal{X}$ is a Banach $(\varphi, \mathcal{A}\oplus_1\mathcal{B})$-bimodule and $D\colon \mathcal{A}\oplus_1\mathcal{B} \to \mathcal{X}^*$ is a derivation. Set $Y_1 = e_{\mathcal{A}} \cdot \mathcal{X}^*$, $Y_2 = e_{\mathcal{B}} \cdot \mathcal{X}^*$ and $Y_3 = (1 - e_{\mathcal{A}})(1 - e_{\mathcal{B}}) \cdot \mathcal{X}^*$, where $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ are the identities of $\mathcal{A}$ and $\mathcal{B}$, respectively. Thus we have decomposition $\mathcal{X}^* = Y_1 + Y_2 + Y_3$ and all of the mappings $D_i\colon \mathcal{A} \oplus_1 \mathcal{B} \to Y_i$ (i=1,2,3) are derivations. We have two cases as follows: if $\varphi|_{\mathcal{B}} \neq 0$, then $\varphi(e_{\mathcal{B}}) = 1$, and hence $\varphi(e_{\mathcal{A}}) = 0$. For each $a \in \mathcal{A}$, we have

$$D_1(a \oplus 0) = D_1(a \oplus 0) \cdot (e_{\mathcal{A}} \oplus 0) + (a \oplus 0) \cdot D_1(e_{\mathcal{A}} \oplus 0).$$

Since $D_1(e_{\mathcal{A}} \oplus 0) \in Y_1$, there exists an element $x^*_1 \in \mathcal{X}^*$ such that $e_{\mathcal{A}} \cdot x^*_1 = D_1(e_{\mathcal{A}} \oplus 0) \in Y_1$. So,

$$D_1(a \oplus 0) = (a \oplus 0) \cdot e_{\mathcal{A}} \cdot x^*_1 = (a \oplus b)e_{\mathcal{A}} \cdot x^*_1.$$

It is similar to the x^*_2 below. For each $b \in \mathcal{B}$, we get

$$\begin{aligned} D_1(0 \oplus b) &= D_1(0 \oplus e_{\mathcal{B}}).(0 \oplus b) + (0 \oplus e_{\mathcal{B}}) \cdot D_1(0 \oplus b) \\ &= D_1(0 \oplus e_{\mathcal{B}}) \cdot (0 \oplus b) = \varphi(a \oplus b)D_1(0 \oplus e_{\mathcal{B}}) \\ &= D_1(0 \oplus e_{\mathcal{B}}) \cdot (a \oplus b) = -e_{\mathcal{A}} \cdot x^*_2 \cdot (a \oplus b). \end{aligned}$$

Therefore, $D_1(a \oplus b) = (a \oplus b) \cdot e_{\mathcal{A}} \cdot x^*_1 - e_{\mathcal{A}} \cdot x^*_2 \cdot (a \oplus b)$. On the other hand, $0 = D_1(e_{\mathcal{A}} \oplus e_{\mathcal{B}}) = e_{\mathcal{A}} \cdot x^*_1 - e_{\mathcal{A}} \cdot x^*_2$. It follows that $D_1(a\oplus b) = (a\oplus b) \cdot x^* - x^* \cdot (a\oplus b)$, where $x^* = e_{\mathcal{A}} \cdot x_1^* = e_{\mathcal{A}} \cdot x_2^* \in Y_1$. Moreover, for the derivation D_2, we have $D_2(a \oplus 0) = 0$. By assumption, for the derivation $D_2 \colon 0 \oplus \mathcal{B} \to e_{\mathcal{B}} \cdot \mathcal{X}^*$, there exists a net $(x_\alpha^*) \subseteq e_{\mathcal{B}} \cdot \mathcal{X}^*$ such that

$$D_2(0 \oplus b) = \lim_\alpha (0 \oplus b) \cdot x_\alpha^* - x_\alpha^*(0 \oplus b) = \lim_\alpha (a \oplus b) \cdot x_\alpha^* - x_\alpha^* \cdot (a \oplus b)$$

for all $b \in \mathcal{A}$. Similarly, one can prove that the derivation D_3 is approximately inner, and so D is approximately inner. Now, if $\varphi|_{\mathcal{A}} \neq 0$, then $\varphi(e_{\mathcal{A}}) = 1$, and hence $\varphi(e_{\mathcal{B}}) = 0$. The above method can be repeated to show that $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately left character amenable. □

Consider $\mathcal{A}^{**}$ as a Banach algebra with the first Arens product. Then every $\varphi \in \sigma(\mathcal{A})$ extends uniquely to some element $\varphi^{**} \in \mathcal{A}^{**}$. The part (ii) of the next theorem was proved in [**14**, Proposition 3.9] but our proof is different.

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$.*

(i) *The following conditions are equivalent:*

  (a) *$\mathcal{A}$ is approximately left (right) φ-amenable;*

  (b) *For each Banach $(\varphi, \mathcal{A})$-bimodule ($(\mathcal{A}, \varphi)$-bimodule) $\mathcal{X}$, every derivation $D \colon \mathcal{A} \to \mathcal{X}^{**}$ is approximately inner.*

(ii) *If $\mathcal{A}^{**}$ is approximately left (right) φ^{**}-amenable, then $\mathcal{A}$ is approximately left (right) φ-amenable.*

Proof. (i): (a) $\Rightarrow$ (b). It is clear.

(b) $\Rightarrow$ (a). By [**14**, Lemma 5.1], it suffices to show that every derivation $D \colon \mathcal{A} \to \mathcal{X}$ is approximately inner for each Banach $(\mathcal{A}, \varphi)$-bimodule $\mathcal{X}$. The mapping $\iota \circ D \colon \mathcal{A} \to \mathcal{X}^{**}$ is a derivation, where $\iota \colon \mathcal{X} \to \mathcal{X}^{**}$ is the canonical embeding. By hypothesis, there is a net $(x_\alpha^{**}) \subseteq \mathcal{X}^{**}$ such that $\iota \circ D(a) = \lim_\alpha a \cdot x_\alpha^{**} - \varphi(a) x_\alpha^{**} \quad (a \in \mathcal{A})$. Now, take $\epsilon > 0$ and nonempty finite sets $E \subseteq \mathcal{A}$, and $F \subseteq \mathcal{X}^*$. Then, there is α such that

$$|\langle x^*, \iota \circ D(a) - (a \cdot x_\alpha^{**} - \varphi(a) x_\alpha^{**}) \rangle| < \epsilon \quad (x^* \in F, a \in E).$$

By Goldstine's theorem [**3**, Theorem A.3.29], there exists an element $x_\alpha \in \mathcal{X}$ such that $|\langle x^*, \iota \circ D(a) - (a \cdot x_\alpha - \varphi(a) x_\alpha) \rangle| < \epsilon$. Hence, there is a net $(x_\alpha) \subseteq \mathcal{X}$ such that $D(a) = w - \lim_\alpha (a \cdot x_\alpha - \varphi(a) x_\alpha)$ for all $a \in \mathcal{A}$, where $w - \lim_\alpha$ denotes the weak topology on X. Finally, for each finite subset $E = \{a_1, ..., a_n\} \subseteq \mathcal{A}$, we get

$$(a_1 \cdot x_\alpha - \varphi(a_1) x_\alpha, \ldots, a_n \cdot x_\alpha - \varphi(a_n) x_\alpha) \text{ converges to } (D(a_1), \ldots, D(a_n))$$

weakly in $\mathcal{X}^n$. By Mazur's theorem, we have

$$(D(a_1), \ldots, D(a_n)) \in \overline{\mathrm{co}}^{\|\cdot\|} \{(a_1 \cdot x_\alpha - \varphi(a_1) x_\alpha, \ldots, a_n \cdot x_\alpha - \varphi(a_n) x_\alpha)\}.$$

Therefore, there exists a net $(x_{(\epsilon,F)}) \subseteq \mathrm{co}\{x_\alpha\}$ such that $D(a) = \lim a \cdot x_{(\epsilon,F)} - \varphi(a) x_{(\epsilon,F)} \quad (a \in E)$. This completes the proof of (i).

(ii) Suppose that $D \colon \mathcal{A} \to \mathcal{X}^*$ is a derivation and $\mathcal{X}$ is a Banach $(\varphi, \mathcal{A})$-bimodule. Then $D^{**} \colon \mathcal{A}^{**} \to \mathcal{X}^{***}$ is a derivation by [**4**, Proposition 1.7], where

$\mathcal{X}^{***}$ is considered as a Banach $(\mathcal{A}^{**}, \varphi^{**})$-bimodule. Since $\mathcal{A}^{**}$ is approximately left φ^{**}-amenable, by [**14**, Lemma 5.1], there exists a net $(x_\alpha^{***}) \subseteq \mathcal{X}^{***}$ such that

$$D^{**}(a^{**}) = \lim_\alpha a^{**} \cdot x_\alpha^{***} - \varphi^{**}(a^{**})x_\alpha^{***} \qquad (a^{**} \in \mathcal{A}^{**}).$$

If $P \colon \mathcal{X}^{***} \to \mathcal{X}^*$ is the natural projection, then

$$D(a) = \lim_\alpha a \cdot P(x_\alpha^{***}) - \varphi(a)P(x_\alpha^{***}) \qquad (a \in \mathcal{A}).$$

It implies that $\mathcal{A}$ is approximately left φ-amenable. The proof of the right φ-amenability is similar. $\square$

It was proved in [**14**, Proposition 3.11] that if $\mathcal{A}, \mathcal{B}$ are two Banach algebras, $\varphi \in \sigma(\mathcal{A})$ and $\psi \in \sigma(\mathcal{B})$, then the approximate $\varphi \otimes \phi$-amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ implies the approximate φ-amenability of $\mathcal{A}$ and the approximate ϕ-amenability of $\mathcal{B}$. This result has not been proved for approximate character amenability so far. Indeed, this consequence holds if $\mathcal{A}$ and $\mathcal{B}$ are commutative. In the upcoming proposition, we prove this result in the unital case.

Proposition 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with identities $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. If $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is approximately left character amenable, then both $\mathcal{A}$ and $\mathcal{B}$ are approximately left character amenable.*

Proof. Since both $\mathcal{A}$ and $\mathcal{B}$ have the identity, so has $\mathcal{A}\widehat{\otimes}\mathcal{B}$. By [**14**, Proposition 2.8], it follow that $\mathcal{A}, \mathcal{B}$ and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ are approximately left 0-amenable. Now, if $\psi \in \sigma(\mathcal{A}\widehat{\otimes}\mathcal{B})$, then $\psi = \varphi \otimes \phi$, where $\varphi \in \sigma(\mathcal{A})$ and $\phi \in \sigma(\mathcal{B})$. Obviously, both mappings $\mathcal{A}\widehat{\otimes}\mathcal{B} \to \mathcal{A}$; $a \otimes b \mapsto \phi(b)a$ and $\mathcal{A}\widehat{\otimes}\mathcal{B} \to \mathcal{B}$; $a \otimes b \mapsto \varphi(a)b$ are continuous epimorphisms, so the desired result is obtained from [**14**, Proposition 3.1]. $\square$

Remark 2.8. If $\mathcal{A}$ is commutative, it is obvious that $\mathcal{A}$ is left character amenable if and only if it is approximately left character amenable. Thus the converse of Proposition 2.7 can be hold in case that $\mathcal{A}$ or $\mathcal{B}$ is commutative, and the proof is similar to the proof of [**10**, Theorem 3.3].

Example 2.9. (i) For a locally compact group G, it can be seen that $L^1(G)$ is approximately (left) right character amenable if and only if G is amenable [**14**, Theorem 7.1].
(ii) It was proved in [**14**, Lemma 2.3] that if $\varphi \in \sigma(\mathcal{A})$ and the ideal $\ker(\varphi)$ has a right approximate identity, then $\mathcal{A}$ is approximately φ-amenable. Since every closed ideal of a C^*-algebra $\mathcal{A}$ has a bounded approximate identity, it is approximately φ-amenable for all $\varphi \in \sigma(\mathcal{A})$.

The following result is a direct consequence of [**8**, Proposition 2.13].

Proposition 2.10. *Suppose that $\mathcal{A}$ is approximately charater amenable and let $\mathcal{J}$ be a weakly complemented left ideal of $\mathcal{A}$. Then $\mathcal{J}$ has a right approximate identity. In particular, $\mathcal{J}^2$ is norm dense in $\mathcal{J}$.*

Let G be a locally compact group, $C_0(G)$ be the space of all bounded continuous complex-valued functions vanishing at infinity G, and $LUC(G)$ be the space of left uniformly continuous complex-valued functions on G. Then $LUC(G)^* = M(G) \oplus$

$C_0(G)^{\perp}$. In the following theorem, we prove [**14**, Theorem 3.4] by using a directed method.

Theorem 2.11. *For a locally compact group G, $L^1(G)^{**}$ is approximately character amenable if and only if G is finite.*

Proof. Assume that G is finite. Then $L^1(G)^{**}$ is amenable by [**7**, Theorem 1.3], and thus it is approximately character amenable.

For the converse, we follow the standard argument in [**5**, Theorem 3.3]. Suppose that $L^1(G)^{**}$ is approximately character amenable with the right identity $\mathcal{E}$. Then the map $T\colon L^1(G)^{**} \to \mathcal{E}L^1(G)^{**} \cong LUC(G)^*; F \mapsto \mathcal{E}F$ is an epimorphism in which $C_0(G)^{\perp}$ is a closed two-sided ideal. By Proposition 2.7, $M(G)$ is approximately character amenable and G is discrete by [**8**, Example 2.14]. Let $\mathfrak{M}$ be the set of left invariant means on $l^1(G)^{**}$. For each $n \in \mathfrak{M}$ and $m \in l^{\infty}(G)^*$, we have $mn = \langle m, 1\rangle n$. Note that n is idempotent. Consider the right ideal $\mathcal{J} = nL^1(G)^{**}$ in $L^1(G)^{**}$. Now, it follows from the last equality that $\mathcal{J}$ is also a left ideal. The composition $L^1(G)^{**} = \mathcal{J} \oplus (1-n)L^1(G)^{**}$ shows that $\mathcal{J}$ is a weakly complemented ideal. Hence, by Proposition 2.10, $\mathcal{J}$ contains a right approximate identity (m_α). Now, the proof of [**5**, Theorem 3.3] shows that G is finite. □

It is shown in [**10**, Theorem 1.4] that the left character amenability is equivalent to the existence of a certain bounded net which is not unique (see also [**12**, Theorem 2.3]). Set the left character amenability constant $C(\mathcal{A})$ of $\mathcal{A}$ to be infimum of the norms of such nets. A Banach algebra $\mathcal{A}$ is said to be K-left character amenable if its left character amenability constant is at most K.

Example 2.12. Let $(\mathcal{A}_n)$ be a sequence of left character amenable Banach algebras such that $C(\mathcal{A}_n) \to \infty$. Then $\mathcal{B} = c_0(\mathcal{A}_n^{\#})$ is approximately left character amenable, non-character amenable Banach algebra. For this, let $\mathcal{B}$ be character amenable and (u_α) be the net in $\mathcal{B}$ obtained by [**10**, Theorem 1.4]. Restricting (u_α) to the n^{th} coordinate of this net of bound K yields a net in $\mathcal{A}_n$ with bound at most K. This is a contradiction, and thus $\mathcal{B}$ can not be character amenable. Now, define

$$\mathcal{B}_K = \{(x_n) \in c_0(\mathcal{A}_n^{\#}) : x_n = 0 \quad \text{for} \quad n > k\}.$$

Set $E_n = (e_1, e_2, \dots, e_n, 0, 0, \dots)$. Thus (E_n) is a central approximate identity for $\mathcal{B}$ bounded by 1. Suppose that $\varphi \in \sigma(\mathcal{B}) \cup \{0\}$ and $D\colon \mathcal{B} \to \mathcal{X}^*$ is a derivation in which $\mathcal{X}$ is a Banach $(\varphi, \mathcal{B})$-bimodule. Then, by restricting D to B_n, we have a derivation $D_n\colon B_n \to \mathcal{X}^*$. Since B_n is unital, according to Lemma 2.3, $D_n = \delta_{\varepsilon_n} + \delta_{\eta_n}$, where $\varepsilon_n \in E_n \cdot \mathcal{X}^*$ and $\|\eta_n\| \leq 2\|D\|$. Note that for each $a \in \mathcal{B}$, $\delta_{\eta_n}(E_n a - a) \to 0$ because (E_n) is an approximate identity and (η_n) is bounded. Moreover, we have equalities $aE_n \cdot \varepsilon_n = a.\varepsilon_n$ and $\varepsilon_n \cdot aE_n = \varphi(aE_n)\varepsilon_n = \varphi(a)\varepsilon_n$ for $a \in \mathcal{B}$, $x \in \mathcal{X}$. So

$$\begin{aligned} D(a) = \lim_n((E_n a) &= \lim_n((aE_n) \cdot \varepsilon_n) - \varepsilon_n \cdot (aE_n) + ad_{\eta_n}(E_n a)) \\ &= \lim_n(a \cdot \varepsilon_n - \varphi(a)\varepsilon_n + a \cdot \eta_n - \varphi(a)\eta_n). \end{aligned}$$

Hence D is approximately inner.

In the following example we will present a Banach algebra that is approximately left character amenable, but it is neither left character amenable nor approximately amenable.

Example 2.13. Let $\mathcal{A}$ be a Banach algebra. It is easy to see that $\mathcal{A}$ is approximately amenable if and only if $M_n(\mathcal{A})$ is approximately amenable for $n \in \mathbb{N}$. If $\mathcal{A}$ is not approximately amenable, then $M_n(\mathcal{A})$ is not approximately amenable. It is well-known that $\sigma(M_n) = \emptyset$ for $n \geq 2$. So $M_n(\mathcal{A})$ is always approximately left character amenable. If $c_0(\mathcal{A}_n^{\#})$ is the Banach algebra as in Example 2.12, then by Theorem 2.5, $\mathcal{B} = c_0(\mathcal{A}_n^{\#}) \oplus M_n(\mathcal{A})$ is approximately left character amenable. On the other hand, in view of the projections $P_1 \colon \mathcal{B} \to c_0(\mathcal{A}_n^{\#})$, $P_2 \colon \mathcal{B} \to M_n(\mathcal{A})$ and also [**12**, Theorem 2.6] as well as [**5**, Proposition 2.2], we find that $\mathcal{B}$ is neither left character amenable nor approximately amenable.

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B. Shojaee, Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran, *e-mail* : `shoujaei@kiau.ac.ir`

A. Bodaghi, Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran, *e-mail* : `abasalt.bodaghi@gmail.com`