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NULL HYPERSURFACE NORMALIZED BY THE CURVATURE VECTOR FIELD IN A LORENTZIAN MANIFOLD OF QUASI-CONSTANT CURVATURE

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Abstract. We introduce a class of null hypersurfaces M of Lorentzian manifolds of quasi-constant curvature $\overline{M}$, namely, ζ-rigging null hypersurfaces, whose curvature vector field ζ is a rigging for M. We extend some well-known results for Lorentzian manifolds of constant curvature and prove several classification theorems for such a null hypersurface. Next, we establish sufficient conditions to guarantee that such a null hypersurface must be totally geodesic. As a consequence, we prove that the ambient manifold is flat along M.

1. Introduction

A submanifold M of a semi-Riemannian manifold is null if the induced metric tensor is degenerated on M. Null hypersurfaces are specifically essential because of their applications in physics and mainly in general relativity. The principal differences between null and non-degenerate hypersurfaces stand up because of the absence of natural projections on the former. This prevents the usual geometric objects from being induced on null hypersurfaces. From the mathematical point of view, several methods have been developed to study these objects. A useful one is that of the rigging approach introduced in [**7**] and has proved to be a powerful tool for the analysis of a null hypersurface. Briefly, the primary concept is to choose a vector field E, called rigging, such that E_p is punctually transversal to M for all $p \in M$. From this precise arbitrary choice, we derive all the geometric objects needed to deal with a null hypersurface.

Among the most studied null hypersurfaces are those with an integrable screen distribution. They include the well-known totally umbilic screen distribution, screen conformal, screen quasi-conformal, respectively. The latter two represent natural classes to explore when it comes to classifying null hypersurfaces satisfying relevant geometric conditions, since in this context, the geometry of the null hypersurfaces is related to that of its screen distribution.

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In [**9**, Theorem 2.4], the author proved a characterization theorem for null hypersurfaces with totally umbilic screen distribution. He showed that the local second fundamental forms B and C of such a null hypersurface and its screen distribution $S(TM)$, respectively, satisfy $B = 0$ or $C = 0$. However, the classification of Einstein null hypersurfaces M with quasi-conformal screen distribution in semi-Riemannian manifolds $\overline{M}(c)$ of constant curvature c was studied in [**12**]. Authors proved that a screen quasi-conformal Einstein null hypersurface is locally a triple product $L \times M_1 \times M_2$, where L is a null geodesic and M_1, M_2 are two Riemannian manifolds (see [**12**, Theorem 5.8] for more details).

Several authors have studied the geometry of null hypersurfaces M of Lorentzian manifolds of quasi-constant curvature $\overline{M}$ (see [**8, 10**] and references therein). However, they failed to address the situation where the curvature vector field ζ of $\overline{M}$ is not tangent to M; instead, they have only examined the case in which ζ is tangent to M. In this work, we consider the case where the null hypersurface is transversal to the curvature vector field ζ. The latter is then a rigging E for M, and M is called ζ-rigging null hypersurface.

Our objective is to extend the above characterization theorems for ζ-rigging null hypersurface. Therefore, the organization of this paper is the following. Section 2 contains all preliminaries needed. In Sections 3, we prove a non-existence result of totally geodesic null hypersurfaces (Theorem 3.7), and we prove several classification theorems (Theorem 3.6, Theorem 3.11 and Theorem 3.13, Theorem 3.14, Theorem 3.15 and Theorem 3.22). Next, we establish sufficient conditions to guarantee that a ζ-rigging null hypersurface with totally umbilic screen distribution is totally geodesic (Theorem 3.26).

2. Preliminaries

In this section, we provide a brief review of Lorentzian manifolds of quasi-constant curvature and the rigging technique for null hypersurfaces.

2.1. Lorentzian manifold of quasi-constant curvature

Chen and Yano [**4**] introduced the notion of a Riemannian manifold of quasi-constant curvature as a Riemannian manifolds $(\overline{M}, \overline{g})$ endowed with the curvature tensor $\overline{R}$ satisfying the following equation:

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)Z,T) = {}& \overline{\alpha}\big\{\overline{g}(Y,Z)\overline{g}(X,T) - \overline{g}(X,Z)\overline{g}(Y,T)\big\} \\ & + \overline{\beta}\big\{\overline{g}(X,T)\theta(Y)\theta(Z) - \overline{g}(X,Z)\theta(Y)\theta(T) \\ & + \overline{g}(Y,Z)\theta(X)\theta(T) - \overline{g}(Y,T)\theta(X)\theta(Z)\big\} \end{aligned} \tag{1}$$

for any vector fields $X, Y, Z, T \in \Gamma(T\overline{M})$, where $\overline{\alpha}$ and $\overline{\beta}$ are smooth functions, and

$$\theta(X) = \overline{g}(X, \zeta)$$

is $\overline{g}$-dual to a non-vanishing smooth vector field ζ, called the curvature vector field of $\overline{M}$. It is well known that if the curvature tensor $\overline{R}$ is of the form (1), then $\overline{M}$

is conformally flat. If $\overline{\beta} = 0$, then $\overline{M}$ is a space of constant curvature. A non-flat Riemannian manifold of dimension $n(> 2)$ is called a quasi-Einstein manifold [**1**] if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies the condition

$$\overline{\mathrm{Ric}}(X, Y) = r\overline{g}(X, Y) + s\mu(X)\mu(Y),$$

where r and s are smooth functions such that $s \neq 0$ and μ is a non-vanishing 1-form such that $\overline{g}(X, U) = \mu(X)$ for any vector field X, where U is a unit vector field. If $s = 0$, then $\overline{M}$ is an Einstein manifold. It is easily to see that every Riemannian manifold of quasi-constant curvature is quasi-Einstein.

2.2. Rigging technique for null hypersurface

Let $(\overline{M}^{n+2}, \overline{g})$ be a Lorentzian manifold, and let (M, g) be a null hypersurface of $(\overline{M}, \overline{g})$. Following Gutiérrez and Olea (see [**7**]), a *rigging* for M is a vector field E defined on some open set of $\overline{M}$ containing M such that for each $p \in M$, $E_p \notin T_pM$. Given a rigging E for M, we set $\overline{\omega} = \overline{g}(E, \cdot)$, $\omega = i^*\overline{\omega}$, $\breve{g} = \overline{g} + \overline{\omega} \otimes \overline{\omega}$ and $\widetilde{g} = i^*\breve{g}$, where $i\colon M \hookrightarrow \overline{M}$ is the canonical inclusion map. It is well known that $\widetilde{g}$ is a Riemannian metric on M. The *rigged vector field* on M is the unique null vector field ξ given by $\widetilde{g}(\xi, .) = \omega$ and it satisfies $\overline{g}(E, \xi) = 1$. A rigging E defines a screen distribution $\mathscr{S}(E)$ given by $\mathscr{S}(E) = TM \cap E^{\perp} = \ker \omega$. The null transversal vector field on M is

$$N = E - \frac{1}{2}\overline{g}(E, E)\xi, \tag{2}$$

which is the unique null vector field such that $\overline{g}(N, \xi) = 1$. Moreover, it is worth noting that $T\overline{M}$ admits the following splitting:

$$T\overline{M}|_M = TM \oplus \mathrm{span}(N) = \{\mathscr{S}(E) \oplus \mathrm{span}(\xi)\} \oplus \mathrm{span}(N). \tag{3}$$

A null hypersurface M equipped with a rigging E is said to be *normalized* and is denoted (M, E).

According to the decomposition (3), the Gauss and Weingarten equations of M and $\mathscr{S}(E)$ are the following (see [**5**, p. 82–85]):

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad \nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{4}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \qquad \nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi, \ \tau(X) = \overline{g}(\overline{\nabla}_X N, \xi), \tag{5}$$

for all X, Y tangent to M. Here, ∇ and $\overset{\star}{\nabla}$ are induced linear connections on TM and $\mathscr{S}(TM)$, respectively, B is the null second fundamental form of M, and C is the second fundamental form on $\mathscr{S}(TM)$. Moreover, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM and $\mathscr{S}(TM)$, respectively, connected with the second fundamental forms by $B(X, Y) = g(\overset{\star}{A}_\xi X, Y)$ and $C(X, PY) = g(A_N X, PY)$, and τ is a 1-form on TM. The induced linear connection ∇ is not a metric connection. In fact, using the fact that $\overline{\nabla}\overline{g} = 0$, we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\omega(Z) + B(X, Z)\omega(Y) \quad \text{for all } X, Y, Z \in \Gamma(TM). \tag{6}$$

In addition, C is not symmetric since

(7) $C(X,Y)-C(Y,X)=g(\nabla_X Y-\nabla_Y X,N)=\eta([X,Y])$ for all $X,Y\in\mathscr{S}(\zeta)$.

Let us denote by $\overline{R}$ and R the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively. Using (4) and (5), we get the so called Gauss-Codazzi equations [**5**]

$$\begin{aligned}\langle\overline{R}(X,Y)Z,\xi\rangle &= (\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z)\\ &\quad+\tau(X)B(Y,Z)-\tau(Y)B(X,Z),\end{aligned}\tag{8}$$

$$\begin{aligned}\langle\overline{R}(X,Y)Z,PW\rangle &= \langle R(X,Y)Z,PW\rangle+B(X,Z)C(Y,PW)\\ &\quad-B(Y,Z)C(X,PW),\end{aligned}\tag{9}$$

$$\begin{aligned}\langle\overline{R}(X,Y)\xi,N\rangle &= \langle R(X,Y)\xi,N\rangle\\ &= C(Y,\overset{\star}{A}_\xi X)-C(\overset{\star}{A}_\xi Y,X)\\ &\quad-2d\tau(X,Y)\quad\text{for all } X,Y,Z,W\in\Gamma(TM|_{\mathscr{U}}).\end{aligned}\tag{10}$$

$$\begin{aligned}\langle\overline{R}(X,Y)PZ,N\rangle &= \langle(\nabla_X A_N)Y,PZ\rangle-\langle(\nabla_Y A_N)X,PZ\rangle\\ &\quad+\tau(Y)\langle A_N X,PZ\rangle-\tau(X)\langle A_N Y,PZ\rangle,\end{aligned}\tag{11}$$

for every X,Y and Z in $\Gamma(TM)$.

We define the *null mean curvature* $\mathbf{H}_\xi$ and the *screen mean curvature* $\mathbf{H}_N$ as

$$\mathbf{H}_\xi=\frac{1}{n}\sum_{i=1}^{n}B(\overset{\star}{E}_i,\overset{\star}{E}_i),\qquad \mathbf{H}_N\frac{1}{n}\sum_{i=1}^{n}C(\overset{\star}{E}_i,\overset{\star}{E}_i),$$

where $\{\overset{\star}{E}_1,\dots,\overset{\star}{E}_n\}$ is an orthonormal basis of $\mathscr{S}(\zeta)$.

3. ζ-rigging null hypersurface

The majority of authors who have studied null submanifolds of Lorentzian manifolds of quasi-constant curvature focused exclusively on the case in which the curvature vector field is tangent to M ([**8, 10**]). Here, we suppose that the curvature vector field ζ never belongs to the tangent space of the null hypersurface M. In this case, ζ can be taken as a rigging for M. This leads to the following definition.

Definition 3.1.

1. A null hypersurface M of Lorentzian manifolds of quasi-constant curvature such that the curvature vector field ζ is a rigging for M, is said to be *ζ-rigging null hypersurface.*
2. We say that the rigging ζ has a quasi-conformal screen distribution if the shape operators A_N and $\overset{\star}{A}_\xi$ of M and $\mathscr{S}(\zeta)$ satisfy

$$A_N X=\phi\overset{\star}{A}_\xi X+\sigma PX,\tag{12}$$

for any $X\in\Gamma(TM)$ and some functions ϕ and σ.

Equivalently,

$$C(X,PY) = \phi B(X,PY) + \sigma g(X,PY)$$

for any $X, Y \in \Gamma(TM)$. For $\sigma = 0$, we simply say that the rigging ζ has a conformal screen distribution.

3. We say that the quasi-conformal pair (ϕ, σ) is adapted if $\tau(X) = 0$, for all $X \in \mathscr{S}(\zeta)$, ϕ and σ are constant along $\mathscr{S}(\zeta)$.

Example 3.2. Let $\overline{M}$ be a Robertson-Walker space-time $(I \times_\psi F, -dt^2 + \psi^2(t)g_F(c))$, where $(F, g_F(c))$ is a $(n+1)$-dimensional Riemannian manifold of constant sectional curvature c. For any X, Y, Z, T on $\overline{M}$, we have

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,T) = {} & \frac{f'^2 + c}{f^2}\big\{\overline{g}(Y,Z)\overline{g}(X,T) - \overline{g}(X,Z)\overline{g}(Y,T)\big\} \\ & + \frac{ff'' - f(f'^2 + c)}{f^2}\big\{\overline{g}(X,T)\theta(Y)\theta(Z) - \overline{g}(X,Z)\theta(Y)\theta(T) \\ & \qquad + \overline{g}(Y,Z)\theta(X)\theta(T) - \overline{g}(Y,T)\theta(X)\theta(Z)\big\},\end{aligned}$$

where $\theta = \overline{g}(., -\partial t)$ (see [**11**, Proposition 2.3]). Thus $\overline{M}$ is a Lorentzian manifold of quasi-constant curvature with curvature vector field $\zeta = -\partial t$. Now, let M be an $(n+1)$-dimensional null hypersurface of $\overline{M}$. Since $\zeta = -\partial t$ is timelike we can use it as a rigging for M. Here, the screen distribution is given by $\ker\theta$. Let ξ denote the corresponding rigged vector field. The associated null transversal vector field is

$$N = -\partial t - \frac{1}{2}\overline{g}(\partial t, \partial t)\xi = \frac{1}{2}\xi - \partial t. \tag{13}$$

It is known that the vector field $K = \psi\partial t$ is a timelike closed conformal vector field with closed conformal factor ψ'. Then, multiplying equation (13) by ψ and differentiating in the direction of $X \in \Gamma(TM)$, we get

$$\begin{aligned}X(\psi)N + \psi(-A_N X + \tau(X)N) = {} & \frac{1}{2}\big((X\psi)\xi + \psi\big(-\overset{\star}{A}_\xi X - \tau(X)\xi\big)\big) \\ & - \psi'\big(PX + \eta(X)\xi\big).\end{aligned}$$

Matching the tangential, radical and transversal parts of the expressions above, we have

$$A_N X = \frac{1}{2}\overset{\star}{A}_\xi X + \ln(\psi)' PX, \qquad \tau(X) = -\ln(\psi)'\eta(X), \tag{14}$$

which means that M is a ζ-rigging null hypersurface with quasi-conformal screen distributions.

Proposition 3.3. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature, and let M be a ζ-rigging null hypersurface of $\overline{M}$. Then, for all $X, Y \in \Gamma(TM)$, we have*

$$\overline{\mathrm{Ric}}(X,Y) = \big((n+1)\overline{\alpha}_{|M} + \overline{g}(\zeta,\zeta)\overline{\beta}_{|M}\big)\overline{g}(X,Y) + n\overline{\beta}_{|M}\{\eta(X)\eta(Y)\}, \tag{15}$$

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \Big(n\overline{\alpha}_{|M} + \frac{1}{2}g(\overline{g}(\zeta,\zeta)\overline{\beta}_{|M}\Big)\overline{g}(X,Y) + 2\overline{\beta}_{|M}\{(n-1)\eta(X)\eta(Y)\} \\ &\quad + B(X,Y)\mathbf{H}_N - \overline{g}(A_N X, \overset{\star}{A}_\xi Y).\end{aligned} \tag{16}$$

$$\begin{aligned}\overset{\star}{\mathrm{Ric}}(X,Y) &= (n-1)\overline{\alpha}|_M g(X,Y) + n\big(B(X,Y)\mathbf{H}_N + g(A_N X,Y)\mathbf{H}_\xi\big) \\ &\quad - g(\overset{\star}{A}_\xi Y, A_N X) - g(\overset{\star}{A}_\xi X, A_N Y) \quad \textit{for all } X,Y \in \mathscr{S}(\zeta).\end{aligned} \tag{17}$$

Proof. Take $\zeta = E$ to be a rigging vector field. Since $\overline{M}$ is of quasi-constant curvature then $\theta(\xi) = 1$, $\theta(N) = \eta(N) = \frac{1}{2}\overline{g}(\zeta,\zeta)$, and $\theta(X) = \eta(X)$ for all $X \in \Gamma(TM)$. From this, we have from equation (1)

$$\begin{aligned}\overline{g}(\overline{R}(\xi,Y)X,N) &= \overline{\alpha}\big\{\overline{g}(Y,X)\overline{g}(\xi,N) - \overline{g}(\xi,X)\overline{g}(Y,N)\big\} \\ &\quad + \overline{\beta}\big\{\overline{g}(\xi,N)\theta(Y)\theta(X) - \overline{g}(\xi,X)\theta(Y)\theta(N) \\ &\quad + \overline{g}(Y,X)\theta(\xi)\theta(N) - \overline{g}(Y,N)\theta(\xi)\theta(X)\big\} \\ &= \overline{g}(Y,X)(\overline{\alpha} + \overline{\beta}\eta(N)) + \overline{\beta}(\eta(Y)\eta(X) - \eta(Y)\eta(X)) \\ &= \overline{g}(Y,X)(\overline{\alpha} + \overline{\beta}|_M \frac{1}{2}\overline{g}(\zeta,\zeta)).\end{aligned} \tag{18}$$

Also,

$$\overline{g}(\overline{R}(N,Y)X,\xi) = \overline{g}(Y,X)(\overline{\alpha}|_M + \overline{\beta}|_M \frac{1}{2}\overline{g}(\zeta,\zeta)). \tag{19}$$

Now, consider a quasi-orthonormal frame field $\{\xi; E_i\}$ such that $\mathscr{S}(\zeta) = \mathrm{Span}\{E_i\}$, and let $B = \{\xi, E_i, N\}$ be the corresponding frame field on M. Using this, we have

$$\overline{\mathrm{Ric}}(X,Y) = \sum_{i=1}^{n} \overline{g}(\overline{R}(E_i,X)Y,E_i) + \overline{g}(\overline{R}(\xi,X)Y,N) + \overline{g}(\overline{R}(N,X)Y,\xi). \tag{20}$$

But,

$$\begin{aligned}\sum_{i=1}^{n}\overline{g}(\overline{R}(E_i,X)Y,E_i) &= \sum_{i=1}^{n}\Big(\overline{\alpha}\big\{\overline{g}(Y,Z)\overline{g}(E_i,E_i) - \overline{g}(E_i,Y)\overline{g}(X,E_i)\big\} \\ &\quad + \overline{\beta}\big\{\overline{g}(E_i,E_i)\theta(X)\theta(Y) - \overline{g}(E_i,Y)\theta(X)\theta(E_i) \\ &\quad + \overline{g}(X,Y)\theta(E_i)\theta(E_i) - \overline{g}(X,E_i)\theta(E_i)\theta(Y)\big\}\Big) \\ &= \Big((n-1)\overline{\alpha} + \overline{\beta}\big(n\theta(X)\theta(Y) \\ &\quad + \theta(\zeta) - 2\theta(\xi)\theta(N)\big)\Big)\overline{g}(X,Y) \\ &\quad + \overline{\beta}\big(\eta(Y)\theta(X)\theta(\xi) + \eta(X)\theta(Y)\theta(\xi) - 2\theta(X)\theta(Y)\big) \\ &= \Big((n-1)\overline{\alpha} + n\overline{\beta}\big(\eta(X)\eta(Y)\big)\Big)\overline{g}(X,Y).\end{aligned} \tag{21}$$

Substituting equations (18), (19) and (21) in equation (20), we get equation (15).

Next, for all $X,Y \in \Gamma(TM)$, the Ricci tensors of M and $\overline{M}$ are related by ([**5**])

$$\mathrm{Ric}(X,Y) = \overline{\mathrm{R}}\mathrm{ic}(X,Y) + nB(X,Y)\mathbf{H}_N - \overline{g}(A_N X, \overset{\star}{A}_\xi Y) - \overline{g}(R(\xi,Y)X,N). \tag{22}$$

Substituting also equations (15) and (18) in equation (22), we have equation (16).

Finally, let $\overset{\star}{R}$ be the Riemannian curvature on $\mathscr{S}(\zeta)$. By straightforward computation, we have

$$\begin{aligned} R(X,Y)Z = {} & \overset{\star}{R}(X,Y)Z - C(Y,Z)\overset{\star}{A}_\xi X + C(X,Z)\overset{\star}{A}_\xi Y \\ & + \Big[C(X,\overset{\star}{\nabla}_Y Z) + X\cdot C(Y,Z) - C(Y,\overset{\star}{\nabla}_X Z) \\ & - Y\cdot C(X,Z) - C([X,Y],Z)\Big]\xi \quad \text{for all } X,Y,Z\in\mathscr{S}(\zeta). \end{aligned} \tag{23}$$

Thus, from equations (9)–(23), we have

$$\begin{aligned} \overline{g}(\overset{\star}{R}(X,Y)Z,W) = {} & \overline{g}(\overline{R}(X,Y)Z,W) + B(Y,Z)C(X,W) - B(X,Z)C(Y,W) \\ & + C(Y,Z)B(X,W) - C(X,Z)B(Y,W). \end{aligned} \tag{24}$$

Now, from equations (21) and (24), the Ricci curvature $\overset{\star}{\mathrm{Ric}}$ on $\mathscr{S}(\zeta)$ is given by

$$\begin{aligned} \overset{\star}{\mathrm{Ric}}(X,Y) = {} & (n-1)\overline{\alpha}|_M g(X,Y) + n\big(B(X,Y)\mathbf{H}_N + g(A_N X,Y)\mathbf{H}_\xi\big) \\ & - g(\overset{\star}{A}_\xi Y, A_N X) - g(\overset{\star}{A}_\xi X, A_N Y). \end{aligned}$$

This completes the proof. □

Observe that the induced Ricci tensor Ric is not symmetric (see (16)), so it has no geometric meaning. Nevertheless, null hypersurfaces with a quasi-conformal screen distribution in semi-Riemannian space forms do admit an induced symmetric Ricci tensor (see [**12**, Proposition 5.2.]). Using (16), we extend the above result in the context of Lorentzian manifolds of quasi-constant curvature by showing the following proposition.

Proposition 3.4. *Let* $(\overline{M},\overline{g})$ *be a Lorentzian manifold of quasi-constant curvature, and let* M *be a* ζ*-rigging null hypersurface with a quasi-conformal screen distribution. Then the tensor field* Ric *is an induced symmetric Ricci tensor of* M.

Proof. Since ζ has a quasi-conformal screen distribution, the claim follows using equation (12) in equation (16). □

Remark 3.5.

1. Suppose that M is a ζ-rigging null hypersurface of $\overline{M}$. Let $x\in M$ and $X\in\mathscr{S}(\zeta)$ be a unitary vector field, i.e., $\overline{g}(X,X)=1$. Let also $\Sigma=\mathrm{Span}(X,\xi)$ be a null plane contained in T_xM. The null sectional curvature with respect to ξ of Σ is given by [**7**]:

$$\begin{aligned} K_\xi(\Sigma) = \overline{g}(\overline{R}(\xi,X)X,\xi) \overset{(1)}{=} {} & \overline{\alpha}|_M\{\overline{g}(X,X)\overline{g}(\xi,\xi) - \overline{g}(\xi,X)\overline{g}(X,\xi)\} \\ & + \overline{\beta}|_M\{\overline{g}(\xi,\xi)\theta(X)\theta(X) - \overline{g}(\xi,X)\theta(X)\theta(\xi) \\ & + \overline{g}(X,X)\theta(\xi)\theta(\xi) - \overline{g}(X,\xi)\theta(\xi)\theta(X)\} \\ = {} & \overline{\beta}|_M g(X,X) \quad \text{for all } X\in T_xM. \end{aligned}$$

Also, from (15), it is worth noting that $\overline{\mathrm{Ric}}(\xi,\xi) = n\overline{\beta}_{|_M}$. Then $\overline{M}$ is of constant curvature $\overline{\alpha}|_M$ along M if and only if either $K_\xi(\Sigma)=0$ or $\overline{\mathrm{Ric}}(\xi,\xi)=0$.

2. Observe that if M is Einstein, then $\overline{M}$ is of constant curvature $\overline{\alpha}|_M$ along M. In fact, being M Einstein, there exists a smooth function ρ in M which is constant along $\mathscr{S}(\zeta)$ and satisfies $\mathrm{Ric}(X,Y)=\rho g(X,Y)$ for all $X,Y\in\Gamma(TM)$ (see [**12**]). Therefore, $0=\mathrm{Ric}(\xi,\xi)\overset{(16)}{=}\overline{\beta}|_M$.

Screen quasi-conformal or screen conformal null hypersurfaces satisfying classical geometric restrictions (totally geodesic, totally umbilical, Einstein) have been studied, and numerous classification results exist for such hypersurfaces when the ambient manifold has constant curvature (see [**6, 12**] and references therein). From item (2) of Remark 3.5 and by the characterization theorem in [**12**, Theorem 5.8], we have the following result.

Theorem 3.6. *Let $\overline{M}$ be Lorentzian manifold of quasi-constant curvature, and let M be a null Einstein ζ-rigging null hypersurface with quasi-conformal screen distribution and adapted quasi-conformal pair (ϕ,σ). If M has exactly two distinct screen principal curvatures, then M is locally diffeomorphic to a product $M=L\times M_1\times M_2$, where L is a null geodesic, and M_1, M_2 are two Riemannian manifolds. Moreover, if $\overline{\alpha}|_M=0$, then one of the screen principal curvatures vanishes.*

Proof. Since M is Einstein, item (2) of Remark 3.5 implies that $\overline{M}$ is of constant curvature $\overline{\alpha}|_M$ along M. Therefore, proceeding as in the proof of Theorem 5.8 in [**12**], the claim follows. Indeed, the proof of Theorem 5.8 is still true if the curvature vector field of $\overline{M}$ is constant just along the null hypersurface. □

The following result is an obstruction to the existence of totally geodesic null hypersurfaces. Recall that M is said to be totally umbilical (resp. totally geodesic) in $\overline{M}$ if there exists a smooth function k on M such that at each $p\in M$ and for all $u,v\in T_pM$,

$$B(u,v)=kg(u,v) \tag{25}$$

(respectively, B vanishes identically on M).

Theorem 3.7. *Let $(\overline{M},\overline{g})$ be a Lorentzian manifold of quasi-constant curvature, and let M be a ζ-rigging null hypersurface of $\overline{M}$. If M is totally umbilic with umbilic factor k, then k satisfies the partial differential equation*

$$\xi(k)-k^2+k\tau(\xi)-\overline{\beta}_{|M}=0. \tag{26}$$

$$PX(k)+k\tau(PX)=0. \tag{27}$$

Moreover, if $\overline{\beta}_{|M}\neq 0$, then M can not be totally geodesic.

Proof. Take $\zeta=E$ to be a rigging vector field, then $\theta(\xi)=1$. Replacing T by ξ in (1), we have

$$\overline{g}(\overline{R}(X,Y)Z,\xi)=\overline{\beta}_{|M}\big\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\big\}. \tag{28}$$

From equation (25), it is worth noting that

$$(\nabla_X B)(Y,Z)=XB(Y,Z)-B(\nabla_X Y,Z)-B(\nabla_X Z,Y)$$

$$= X(k)g(Y,Z) + kXg(Y,Z) - kg(\nabla_X Y, Z) - kg(\nabla_X Z, Y)$$
$$= X(k)g(Y,Z) + (\nabla_X g)(Y,Z)$$
$$= X(k)g(Y,Z) + k^2\big\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\big\}.$$

This leads to

(29)
$$\begin{aligned}&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\\ &= \big\{X(k) - k^2\eta(X) + k\tau(X)\big\}g(Y,Z) - \big\{Y(k) - k^2\eta(Y) + k\eta(Y)\big\}g(X,Z).\end{aligned}$$

From equation (29) together with equations (8) and (28), we have

$$\begin{aligned}&\overline{\beta}_{|M}\big\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\big\}\\ &= X(k)g(Y,Z) + k^2\big\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\big\} - Y(k)g(X,Z)\\ &\quad - k^2\big\{g(Y,X)\eta(Z) + g(Y,Z)\eta(X)\big\} - kg(X,Z)\tau(Y) + kg(Y,Z)\tau(X).\end{aligned} \tag{30}$$

Replacing X by ξ in equation (30), we have

$$\overline{\beta}_{|M}\big\{g(Y,Z)\big\} = \xi(k)g(Y,Z) - k^2\big\{g(Y,Z)\big\} + kg(Y,Z)\tau(\xi),$$

which gives equation (26).

Next, setting $X = PX$, $Y = PY$ and $Z = PZ$ in (30), we have

$$g(PX(k)PY - PY(k)PX - kgPX\tau(PY) + kPY\tau(PX), PZ) = 0.$$

As $\mathscr{S}(\zeta)$ is non-degenerate, this leads to $(PX(k) + k\tau(PX))PY = (PY(k) + k\tau(PY))PX$. Taking PX and PY to be linearly independent (rank($\mathscr{S}(\zeta)) > 1$) yields (27). The last claim follows setting $k = 0$ in equation (26). □

Definition 3.8. A null hypersurface M immersed in a Lorentzian manifold $(\overline{M}, \overline{g})$ is said to be isoparametric if the screen principal curvatures, with respect to $\overset{\star}{A}_\xi$, are constant along $\mathscr{S}(\zeta)$.

Theorem 3.9. *Any ζ-rigging totally umbilic null hypersurface of a Lorentzian manifold of quasi-constant curvature $(\overline{M}, \overline{g})$ such that the one form τ vanishes on $\mathscr{S}(\zeta)$, is isoparametric.*

Proof. The result follows from the differential equation (27) by considering the fact that τ vanishes on $\mathscr{S}(\zeta)$. □

The reader may compare the following with [**5**, Proposition 2.5.4, p. 77].

Proposition 3.10. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature, and M be a ζ- rigging null hypersurface of $\overline{M}$. If ζ is screen quasi-conformal with $A_N X = \phi\overset{\star}{A}_\xi X + \sigma PX$, then for all $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned}&B(X,PZ)(2\phi\tau(\xi) + \sigma - \xi(\phi))\\ &+ g(X,PZ)\Big[\sigma\tau(\xi) + \overline{\alpha}|_M + \frac{1}{2}\overline{g}(\zeta,\zeta)\overline{\beta}|_M - \xi(\sigma) - \phi\overline{\beta}|_M\Big] = 0,\end{aligned} \tag{31}$$

$$\begin{aligned}&\Big[PX(\phi)-2\phi\tau(PX)\Big]\overset{\star}{A}_\xi PY+\Big[2\phi\tau(Y)-PY(\phi)\Big]\overset{\star}{A}_\xi PX\\&+\Big[PX(\sigma)-\sigma\tau(PX)\Big]PY+\Big[\sigma\tau(PY)-PY(\sigma)\Big]PX=0.\end{aligned}\tag{32}$$

Proof. From equation (12), we have

$$C(Y,PZ)=\phi B(Y,PZ)+\sigma\overline{g}(Y,PZ)\quad\text{for all } Y,Z\in\Gamma(TM).\tag{33}$$

Differentiating equation (33) gives

$$\begin{aligned}(\nabla_X C)(Y,PZ)&=\phi(\nabla_X B)(Y,PZ)+(X\cdot\phi)B(Y,PZ)\\&\quad+(X\cdot\sigma)\overline{g}(Y,PZ)+\sigma B(X,PZ)\eta(Y).\end{aligned}$$

Interchanging X and Y in equations (34) and subtracting (34) from the new relation, we get

$$\begin{aligned}&(\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ)\\&=\phi\Big((\nabla_X B)(Y,PZ)-(\nabla_Y B)(X,PZ)\Big)+B(Y,PZ)(X\cdot\phi-\sigma\eta(X))\\&\quad+B(X,PZ)(\sigma\eta(X)-Y\cdot\phi)+(X\cdot\sigma)\overline{g}(Y,PZ)-(Y\cdot\sigma)\overline{g}(X,PZ)\\&\overset{(8)}{=}\phi\overline{g}(\overline{R}(X,Y)PZ,\xi)+B(Y,PZ)(X\cdot\phi-\sigma\eta(X)-\phi\tau(X))\\&\quad+B(X,PZ)(\sigma\eta(X)-Y\cdot\phi+\phi\tau(Y))+(X\cdot\sigma)\overline{g}(Y,PZ)-(Y\cdot\sigma)\overline{g}(X,PZ)\\&\overset{(1)}{=}B(Y,PZ)(X\cdot\phi-\sigma\eta(X)-\phi\tau(X))+B(X,PZ)(\sigma\eta(X)-Y\cdot\phi+\phi\tau(Y))\\&\quad+\overline{g}(Y,PZ)\big(X\cdot\sigma+\phi\overline{\beta}|_M\theta(X)\theta(\xi)\big)-\overline{g}(X,PZ)\big(Y\cdot\sigma+\phi\overline{\beta}|_M\theta(Y)\theta(\xi)\big).\end{aligned}\tag{34}$$

But,

$$\begin{aligned}&(\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ)\\&\overset{(11)}{=}\overline{g}(\overline{R}(X,Y)Z,N)+\tau(X)\phi B(Y,PZ)-\phi\tau(Y)B(X,PZ)\\&\quad+\sigma\tau(X)\overline{g}(Y,PZ)-\sigma\tau(Y)\overline{g}(X,PZ)\\&\overset{(1)}{=}\tau(X)\phi B(Y,PZ)-\phi\tau(Y)B(X,PZ)\\&\quad+\overline{g}(Y,PZ)\big(\sigma\tau(X)+\overline{\alpha}|_M\eta(X)+\overline{\beta}|_M\theta(X)\theta(N)\big)\\&\quad-\overline{g}(X,PZ)\big(\sigma\tau(Y)+\overline{\alpha}|_M\eta(Y)+\overline{\beta}|_M\theta(Y)\theta(N)\big)\\&\quad+\overline{\beta}_M\theta(PZ)\big(\eta(X)\theta(Y)-\eta(Y)\theta(X)\big).\end{aligned}\tag{35}$$

Substituting equation (35) in equation (34), we get

$$\begin{aligned}&B(Y,PZ)\Big[X(\phi)-2\phi\tau(X)-\sigma\eta(X)\Big]+B(X,PZ)\Big[2\phi\tau(Y)+\sigma\eta(Y)-Y(\phi)\Big]\\&=-g(Y,PZ)\Big[X(\sigma)-\sigma\tau(X)-\overline{\alpha}|_M\eta(X)-\overline{\beta}|_M\theta(X)\theta(N)+\phi\overline{\beta}|_M\theta(X)\theta(\xi)\Big]\\&\quad-g(X,PZ)\Big[\sigma\tau(Y)+\overline{\alpha}|_M\eta(Y)+\overline{\beta}|_M\theta(Y)\theta(N)-Y(\sigma)-\phi\overline{\beta}|_M\theta(Y)\theta(\xi)\Big]\\&\quad+\overline{\beta}_M\theta(PZ)(\eta(X)\theta(Y)-\eta(Y)\theta(X)).\end{aligned}\tag{36}$$

As $\zeta = E$, then $\theta(\xi) = 1$ and $\theta(PZ) = 0$, replacing Y by ξ in equation (36), we get equation (31).

Finally, setting $X = PX$, $Y = PY$ in (36) together with the fact $\mathscr{S}(\zeta)$ is non-degenerate, we obtain equation (32). □

The following result is a transversal version of [**10**, Theorem 3.3 and Theorem 3.4], where it was assumed that the structural vector field is tangent to the null hypersurface and $\mathscr{S}(\zeta)$ totally umbilic in M. We recall also that the screen distribution $\mathscr{S}(\zeta)$ is totally umbilical (resp. totally geodesic) in M if there is a smooth function λ such that

$$C(X, PY) = \lambda g(X, Y) \tag{37}$$

for all $X, Y \in \Gamma(TM)$ (respectively, C vanishes identically). (See [**3, 5**].)

Theorem 3.11. *Let $(\overline{M}^{n+2}, \overline{g})$ $(n > 2)$ be a $(n+2)$-dimensional Lorentzian manifold of quasi-constant curvature, and let M be a ζ-rigging null hypersurface of $\overline{M}$. We suppose that $\mathscr{S}(\zeta)$ is totally umbilic with umbilic factor λ. Then M is locally a product manifold $L \times \overset{\star}{M}$, where L is a null curve, and $\overset{\star}{M}$ is an Einstein manifold of constant curvature.*

Proof. As $\mathscr{S}(\zeta)$ is totally umbilic with umbilic factor λ, then it is integrable. From [**5**], M is locally a product manifold $L \times \overset{\star}{M}$, where L is a null curve, and $\overset{\star}{M}$ is a leaf of $S(\zeta)$ as a codimension two spacelike submanifold of $\overline{M}$. Setting $\phi = 0$ and $\sigma = \lambda$ in equation (31), we have

$$\lambda B(Y, Z) = g(Y, Z)\Big(- \overline{\alpha}|_M - \frac{1}{2}\overline{g}(\zeta, \zeta)\overline{\beta}|_M - \lambda\tau(\xi) + \xi(\lambda)\Big). \tag{38}$$

We distinguish two cases:

(1) If $\lambda = 0$, then from equations (38) and in (17), we have $\overline{\alpha}|_M = -\frac{1}{2}\overline{g}(\zeta, \zeta)\overline{\beta}|_M$ and $\overset{\star}{\mathrm{Ric}}(X, Y)Z = (n-1)(\overline{\alpha}|_M)$. Thus $\overset{\star}{M}$ is an Einstein manifold of constant curvature $(\overline{\alpha}|_M)$ since $n > 2$.

(2) Assume that $\lambda \neq 0$. From equation (38), we have

$$B(Y, Z) = kg(X, Y) \quad \text{for all } X, Y \in \Gamma(TM), \tag{39}$$

where

$$k = \lambda^{-1}\Big(- \overline{\alpha}_{|_M} - \frac{1}{2}\overline{g}(\zeta, \zeta)\overline{\beta}_{|_M} - \lambda\tau(\xi) + \xi(\lambda)\Big). \tag{40}$$

This means that M is totally umbilic with umbilic factor k.

Substituting equations (37) and (39) in equation (17) of Proposition 3.3, we have

$$\overset{\star}{\mathrm{Ric}}(X, Y)Z = (n-1)(\overline{\alpha}|_M + 2k\lambda).$$

This implies that $\overset{\star}{M}$ is an Einstein manifold of constant curvature $(\overline{\alpha}|_M + 2k\lambda)$ since $n > 2$. This completes the proof. □

We recall the following from [**9**].

Theorem 3.12. *Let (M, ζ) be a lightlike hypersurface of a semi-Riemannian space form $(\overline{M}(\overline{\alpha}), \overline{g})$ such that $\mathscr{S}(\zeta)$ is totally umbilic. Then $C = 0$ or $B = 0$. Moreover,*

(1) *$C = 0$ implies $\mathscr{S}(\zeta)$ is totally geodesic and $\overline{\alpha} = 0$;*
(2) *$B = 0$ implies that M is totally geodesic immersed in $\overline{M}(\overline{\alpha})$ and the induced connection ∇ is metric.*

From equations (26) and (40), we have

$$\xi(k) \overset{(26)}{=} k^2 - k\tau(\xi) + \overline{\beta}_{|_M}$$

and

$$\xi(\lambda) \overset{(40)}{=} k\lambda + \overline{\alpha}_{|_M} + \frac{1}{2}\overline{g}(\zeta,\zeta)\overline{\beta}_{|_M} + \lambda\tau(\xi).$$

As $\xi(\overline{\alpha}|_M + 2k\lambda)$ is constant, together with the above equations, we have

$$0 = \xi(\overline{\alpha}|_M + 2k\lambda) = \xi(\overline{\alpha}|_M) + 2(\xi(k)\lambda) + k\xi(\lambda)) \tag{41}$$

$$\begin{aligned} &\xi(\overline{\alpha}|_M) + 2k(\overline{\alpha}|_M + 2k\lambda) + 2\overline{\beta}|_M\Big(\lambda + \frac{1}{2}\overline{g}(\zeta,\zeta)k\Big) \\ &= \xi(\overline{\alpha}|_M) + 2\lambda(2k^2 + \overline{\beta}|_M) + 2k\Big(\overline{\alpha}|_M + \frac{1}{2}\overline{g}(\zeta,\zeta)\overline{\beta}|_M\Big). \end{aligned} \tag{42}$$

Using this, we present an extension of Theorem 3.12, where it was supposed $\overline{M}$ of constant curvature.

Theorem 3.13. *Let $(\overline{M}^{n+2}, \overline{g})$ $(n > 2)$ be a $(n+2)$-dimensional Lorentzian manifold of quasi-constant curvature, and let M be a ζ-rigging null hypersurface of $\overline{M}$ such that $\mathscr{S}(\zeta)$ is totally umbilic with umbilic factor λ. If any one of the following conditions holds:*

(i) *$\overline{\beta}|_M = 0$,*
(ii) *$\overline{\alpha}|_M$ is constant along the integral curves of ξ and $\lambda + \frac{1}{2}\overline{g}(\zeta,\zeta)k = 0$,*

then $C = 0$ or $B = 0$.

Proof. Suppose that $C \neq 0$. Under the hypothesis (i) or (ii), equation (41) of Theorem 3.11 leads to $k(\overline{\alpha}|_M + 2k\lambda) = 0$. Therefore, $k = 0$ or $(\overline{\alpha}|_M + 2k\lambda) = 0$. From this, by following closely the argument of Theorem 2.4 in [**9**], we have the result. Indeed, if $(\overline{\alpha}|_M + 2k\lambda) = 0$, then $\overset{\star}{M}$ is a semi-Euclidean space and the second fundamental form C of $\overset{\star}{M}$ satisfies $C = 0$. It is a disagreement to $C \neq 0$. Thus we have $k = 0$. Consequently, we get $B = 0$ by equation (39). Thus M is totally geodesic in $\overline{M}$. Also, it is worth noting that $(\nabla_X g)(X, Z) = 0$ for all $X, Y, Z \in \Gamma(TM)$, that is, the induced connection ∇ on M is a metric one. □

Theorem 3.14. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature, and let M be a ζ-rigging null hypersurface of $\overline{M}$ such that $\mathscr{S}(\zeta)$ is totally umbilic with umbilic factor λ. If λ and $\overline{\alpha}|_M$ are constant along the integral curves of ξ, and $\overline{\beta}|_M$ is positive, then $C = 0$ or ($B = 0$ and $\overline{M}$ is flat along M).*

Proof. Suppose that $C \neq 0$. Under the hypothesis, equation (42) of Theorem 3.11 leads to $2\lambda k(2k^2 + \overline{\beta}|_M) = 0$. Therefore, $2k^2 = -\overline{\beta}|_M$ as $\lambda \neq 0 (C \neq 0)$. Since $\overline{\beta}$ is positive, we have $k = 0$ and $\overline{\beta}|_M = 0$. As λ is constant along the integral curves of ξ, the screen mean curvature $H_N = \lambda \neq \{0\}$ is constant. Using this, equation (50) leads to

$$\overline{\alpha}_{|_M} + \frac{1}{2}\overline{g}(\zeta,\zeta)\overline{\beta}_{|_M} = \overline{\alpha}_{|_M} = 0.$$

This completes the proof. □

From equation (31), for all $X, Y \in \Gamma(TM)$, we have

$$B(X,PZ)\Lambda + g(X,PZ)\Big[\sigma\tau(\xi) + \overline{\alpha}|_M + \overline{\beta}|_M\Big(\frac{1}{2}\overline{g}(\zeta,\zeta) - \phi\Big) - \xi(\sigma)\Big] = 0, \tag{43}$$

where

$$\Lambda = 2\phi\tau(\xi) + \sigma - \xi(\phi). \tag{44}$$

From this, we have the following theorem.

Theorem 3.15. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature, and let M be a ζ rigging-null hypersurface with a quasi-conformal screen distribution.*

(i) *If $\Lambda = 0$, then $\overline{\alpha}|_M$ and $\overline{\beta}|_M$ satisfy the following equation:*

$$\sigma\tau(\xi) + \overline{\alpha}|_M + \overline{\beta}|_M\Big(\frac{1}{2}\overline{g}(\zeta,\zeta) - \phi\Big) - \xi(\sigma) = 0.$$

(ii) *If $\Lambda \neq 0$, both $\mathscr{S}(\zeta)$ and M are totally umbilic immersed in M and $\overline{M}$.*

Proof. Setting $\Lambda = 0$ in equation (43), we have item (i).
Now we assume that $\Lambda \neq 0$. From equations (33) and (31), we have

$$B(X,Y) = \frac{\xi(\sigma) - \sigma\tau(\xi) - \overline{\alpha}|_M - \overline{\beta}|_M\Big(\frac{1}{2}\overline{g}(\zeta,\zeta) - \phi\Big)}{\Lambda} g(X,Y),$$

$$C(X,Y) = \Big[\frac{\phi}{\Lambda}\Big(\xi(\sigma) - \sigma\tau(\xi) - \overline{\alpha}|_M - \overline{\beta}|_M\Big(\frac{1}{2}\overline{g}(\zeta,\zeta) - \phi\Big)\Big) + \sigma\Big]g(X,Y),$$

which means that both $\mathscr{S}(\zeta)$ and M are totally umbilical. □

Remark 3.16. If $\zeta = \frac{\overline{g}(\zeta,\zeta)}{2}\xi + N$ is closed conformal with conformal factor Ψ, then M is screen integrable. From [**1**], it follows that

$$\begin{cases} \frac{1}{2}\overline{g}(\zeta,\zeta)\overset{\star}{A}_\xi X + A_N X + \Psi PX = 0, \\ X \cdot \Big(\frac{1}{2}\overline{g}(\zeta,\zeta)\Big) = \Psi\eta(X), \\ \tau(X) = 0. \end{cases}$$

Thus, ζ has a quasi-conformal screen distribution with conformal factor $(\phi, \sigma) = (-\frac{1}{2}\overline{g}(\zeta,\zeta), -\Psi)$. Using this in (44), we have

$$\Lambda = 2\phi\tau(\xi) + \sigma - \xi(\phi) = -\Psi + \xi \cdot \Big(\frac{1}{2}\overline{g}(\zeta,\zeta)\Big) = -\Psi + \Psi = 0.$$

This leads to the following corollary.

Corollary 3.17. *Let* $(\overline{M}, \overline{g})$ *be a Lorentzian manifold of quasi-constant curvature, and let* M *be a* ζ *rigging-null hypersurface such that* ζ *is closed conformal with conformal factor* Ψ*. Then* $\overline{\alpha}|_M$ *and* $\overline{\beta}|_M$ *satisfy the following equation:*

$$\overline{\alpha}|_M + \overline{g}(\zeta, \zeta)\overline{\beta}|_M + \xi(\Psi) = 0. \tag{45}$$

Moreover, if ζ *is closed homothetic, then* $\overline{M}$ *is flat along* M *if either* $\overline{\alpha}|_M = 0$ *or* $\overline{\beta}|_M = 0$.

Proof. Being ζ closed conformal, we have from Remark 3.16 that ζ has a quasi-conformal screen distribution with quasi-conformal factor $(\phi, \sigma) = (-\frac{1}{2}\overline{g}(\zeta, \zeta), -\Psi)$ and $\Lambda = 0$. Using this in item (i) of Theorem 3.15, we have equation (45). Now, if ζ is closed homothetic, i.e, if Ψ is constant, then (45) becomes $\overline{\alpha}|_M + \overline{g}(\zeta, \zeta)\overline{\beta}|_M = 0$. From this, the last claim follows. □

If M is conformal screen, then equation (43) becomes

$$B(X, PZ)\Big[\Lambda_1\Big] + g(X, PZ)\Big[\overline{\alpha}|_M + \overline{\beta}|_M\Big(\frac{1}{2}\overline{g}(\zeta, \zeta) - \phi\Big)\Big] = 0 \tag{46}$$

for all $X, Y \in \Gamma(TM)$, where $\Lambda_1 = 2\phi\tau(\xi) - \xi(\phi)$. From this, we have the following theorem.

Theorem 3.18. *Let* $(\overline{M}, \overline{g})$ *be a Lorentzian manifold of quasi-constant curvature, and let* M *be a* ζ*-rigging null hypersurface with a conformal screen distribution. We suppose that* $\mathrm{Ric}(\xi, \xi) = 0$.

(1) *If* $\Lambda_1 = 0$*, then* $\overline{M}$ *is flat along* M.
(2) *If* $\Lambda_1 \neq 0$*, then both* $\mathscr{S}(\zeta)$ *and* M *are totally geodesic immersed in* M *and* $\overline{M}$*. Moreover,* $\overline{M}$ *is flat along* M.

Proof. Since $\mathrm{Ric}(\xi, \xi) = 0$, from Remark 3.16, we have $\overline{\beta}|_M = 0$. Now, if $\Lambda_1 = 0$, then it is worth nothing from (46) that $\overline{\alpha}|_M = -\overline{\beta}|_M\Big(\frac{1}{2}\overline{g}(\zeta, \zeta) - \phi\Big) = 0$. Which gives item (1).

Now we assume that $\Lambda_1 \neq 0$. From equation (46) together with the fact that $\overline{\beta}|_M = 0$, we have that

$$\begin{aligned} B(X, Y) &= -\frac{\overline{\alpha}|_M}{\Lambda_1} g(X, Y), \\ C(X, PY) &= -\frac{\phi\overline{\alpha}|_M}{\Lambda_1} g(X, PY), \end{aligned} \tag{47}$$

Since $\overline{M}$ is of constant curvature $\overline{\alpha}_M$ along M, by applying the Theoren 3.12, we get $B = 0$ or $C = 0$, which is equivalent to $\overline{\alpha}|_M = 0$ or $\phi = 0$. Notice that $\phi = 0$ implies that $\Lambda_1 = 0$, which is a contradiction to the assumption. Thus we have $\overline{\alpha}|_M = 0$ and $B = C = 0$. □

Corollary 3.19. *Let* $(\overline{M}, \overline{g})$ *be a Lorentzian manifold of quasi-constant curvature, and let* M *be a* ζ*-rigging null hypersurface with a conformal screen distribution. Then* $\mathrm{Ric}(\xi, \xi) = 0$ *if and only if* $\overline{M}$ *is flat along* M.

Remark 3.20. Take $K = \zeta$ to be a rigging vector field. Replacing T by N in (1) together with the fact that $\theta(N) = \frac{1}{2}\overline{g}(\zeta,\zeta)$, we have that

$$
\begin{aligned}
\overline{g}(\overline{R}(X,Y)Z,N) &= \overline{\alpha}_{|M}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \\
&\quad + \overline{\beta}_{|M}\Big\{\eta(Y)\eta(Z)\eta(X) - \overline{g}(X,Z)\eta(Y)\Big(\frac{1}{2}\overline{g}(\zeta,\zeta)\Big) \\
&\qquad + \overline{g}(Y,Z)\eta(X)\Big(\frac{1}{2}\overline{g}(\zeta,\zeta)\Big) - \eta(Y)\eta(X)\eta(Z)\Big\} \\
&= \overline{\alpha}_{|M}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \\
&\quad + \frac{1}{2}\overline{g}(\zeta,\zeta)\overline{\beta}_{|M}\{\overline{g}(Y,Z)\eta(X) - \overline{g}(X,Z)\eta(Y)\} \\
&= \Big(\overline{\alpha}_{|M} + \overline{\beta}_{|M}\frac{1}{2}\overline{g}(\zeta,\zeta)\Big)\big(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big).
\end{aligned}
\tag{48}
$$

From (10), we have

$$
\overline{g}(\overline{R}(X,Y)\xi,N) = C(Y,\overset{\star}{A}_\xi X) - C(X,\overset{\star}{A}_\xi Y) - 2d\tau(X,Y) \tag{49}
$$

for all $X, Y \in \Gamma(TM)$. Due to (48), the left hand side of (49) vanishes. Moreover if $\mathscr{S}(\zeta)$ is totally umbilic or ζ has a quasi-conformal screen distribution, then

$$
C(Y,\overset{\star}{A}_\xi X) - C(X,\overset{\star}{A}_\xi Y) = \overline{g}(A_N Y,\overset{\star}{A}_\xi X) - \overline{g}(A_N X,\overset{\star}{A}_\xi Y) \overset{(37)}{=} 0.
$$

That is $d\tau(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$. Therefore, there exists a smooth function $f\colon M \to \mathbb{R}$ such that $\tau = df$ whenever M is simply connected. If we take $\overline{\zeta} = \gamma\zeta$, then the corresponding rigged is given by $\overline{\xi} = \gamma^{-1}\xi$. It follows that $\tau(X) = \overline{\tau}(X) + X(Ln(\gamma^{-1}))$. Setting $\gamma = \exp(f)^{-1}$ in this equation, we get that $\overline{\tau}(X) = 0$ for any $X \in \Gamma(TM)$.

Lemma 3.21. *Let $(\overline{M},\overline{g})$ be a Lorentzian manifold of quasi-constant curvature, and let (M,ζ) be a ζ-rigging null hypersurface of $\overline{M}$. Then*

$$
\operatorname{div}^{\overset{\star}{\nabla}}(A_N\xi) = \xi(\cdot n\mathbf{H}_N) + n(\overline{\alpha}_{|M} + \overline{\beta}_{|M}(\frac{1}{2}\overline{g}(\zeta,\zeta))). \tag{50}
$$

Proof.

$$
\begin{aligned}
\operatorname{div}^{\overset{\star}{\nabla}}(A_N\xi) &= \sum_{i=1}^{n}\overline{g}\big(\overset{\star}{\nabla}_{\overset{\star}{E}_i}(A_N\xi),\overset{\star}{E}_i\big) = \sum_{i=1}^{n}\overline{g}\big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N)\xi,\overset{\star}{E}_i\big) \\
&= \sum_{i=1}^{n}\overline{g}\big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N)\xi + C(\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N,\xi)\xi,\overset{\star}{E}_i\big) = \sum_{i=1}^{n}\overline{g}\big((\nabla_{\overset{\star}{E}_i}A_N)\xi,\overset{\star}{E}_i\big).
\end{aligned}
$$

Gauss-Codazzi equation leads to

$$
\begin{aligned}
\operatorname{div}^{\overset{\star}{\nabla}}(A_N\xi) &= \sum_{i=1}^{n}\overline{g}\big((\nabla_\xi A_N)\overset{\star}{E}_i,\overset{\star}{E}_i\big) + \sum_{i=1}^{n}\overline{g}\big(\overline{R}(\overset{\star}{E}_i,\xi)\overset{\star}{E}_i,N\big) \\
&= \sum_{i=1}^{n}\overline{g}\big((\nabla_\xi A_N)\overset{\star}{E}_i,\overset{\star}{E}_i\big) + n\big(\overline{\alpha}_{|M} + \overline{\beta}_{|M}(\frac{1}{2}\overline{g}(\zeta,\zeta)\big) \\
&= \xi(\cdot n\mathbf{H}_N) - n\big(\overline{\alpha}_{|M} + \overline{\beta}_{|M}(\frac{1}{2}\overline{g}(\zeta,\zeta)\big).
\end{aligned}
$$

□

Theorem 3.14 still holds if $\overline{\beta}$ is not positive but $\overline{M}$ is simply connected as shown in the following theorem.

Theorem 3.22. *Let $(\overline{M}^{n+2}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature, and let M be a ζ-rigging null hypersurface of $\overline{M}$. If $\mathscr{S}(\zeta)$ is totally umbilic with umbilic factor λ constant along the integral curves of ξ and $\overline{M}$ simply connected, then $C = 0$ or ($B = 0$ and $\overline{M}$ is flat along M).*

Proof. Suppose that $C \neq 0$. As λ is constant along the integral curves of ξ, the screen mean curvature $H_N = \lambda \neq \{0\}$ is constant. Using this, (50) leads to

$$\overline{\alpha}_{|_M} + \frac{1}{2}\overline{g}(\zeta,\zeta)\overline{\beta}_{|_M} = 0. \tag{51}$$

Substituting this in (38), we have that $B(X,Y) = -\tau(\xi)g(X,Y)$, which implies that M is either totally umbilic or totally geodesic. Replacing k by $-\tau(\xi)$ in (26), we get

$$\xi(\tau(\xi)) + 2\tau(\xi)^2 + \overline{\beta}_{|_M} = 0. \tag{52}$$

Since $\overline{M}$ is simply connected, from Remark 15, we have $\tau(\xi) = 0$, which implies froom (52) that $\overline{\beta}|_M = 0$. Using this in (51), we get $\overline{\alpha}|_M = 0$. This completes the proof. □

In the following, we establish sufficient conditions to guarantee that a ζ-rigging null hypersurface is totally geodesic and that $\overline{M}$ is flat along the null hypersurface.

Theorem 3.23. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature, and let M be a ζ-rigging null hypersurface of $\overline{M}$. We suppose that the screen shape operator $\overset{\star}{A}_\xi$ is parallel with respect to ∇. Then M is totally geodesic if and only if $\overline{\beta}|_M = 0$.*

Proof. Let $X \in \mathcal{S}(\zeta)$. We have

$$\begin{aligned}\overline{g}(R(X,\xi)\xi, X) &= \overline{g}(\nabla_X\nabla_\xi\xi, X) - \overline{g}(\nabla_\xi\nabla_X\xi, X) - \overline{g}(\nabla_{[X,\xi]}\xi, X)\\ &= \tau(\xi)\overline{g}(\overset{\star}{A}_\xi(X), X) - \overline{g}(\nabla_\xi(-\tau(X)\xi - \overset{\star}{A}_\xi(X)), X) + \overline{g}(\overset{\star}{A}_\xi([X,\xi]), X)\\ &= \overline{g}(\nabla_\xi\overset{\star}{A}_\xi(X), X) + \overline{g}(\overset{\star}{A}_\xi(\nabla_X\xi) - \overline{g}(\overset{\star}{A}_\xi(\nabla_\xi X), X)\\ &= \overline{g}((\nabla_\xi\overset{\star}{A}_\xi)(X), X) - \overline{g}(\overset{\star}{A}_\xi(X), \overset{\star}{A}_\xi(X)).\end{aligned} \tag{53}$$

But,

$$\overline{g}(R(X,\xi)\xi, X) \overset{(9)}{=} \overline{g}(\bar{R}(X,\xi)\xi, X) \overset{(1)}{=} \overline{\beta}|_M\overline{g}(X,X). \tag{54}$$

If M is $\overset{\star}{A}_\xi$ parallel then the above relations imply that

$$g(\overset{\star}{A}_\xi(X), \overset{\star}{A}_\xi(X)) = -\overline{\beta}_M,$$

which means that M is totally geodesic if and only if $\overline{\beta}|_M = 0$ as the screen distribution is positive definite. □

Theorem 3.24. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature with $\overline{\beta}_{|_M}$ non-negative, and let M be a ζ-rigging simply connected compact null hypersurface of $\overline{M}$. If ζ has a quasi-conformal distribution with quasi-conformal pair (ϕ, σ), then M is totally geodesic in $\overline{M}$ and $\mathscr{S}(\zeta)$ is totally umbilic in $\overline{M}$ with umbilic factor σ. Moreover, if σ is constant along the integral curves of ξ, then $\overline{M}$ is flat along M.*

Proof. Since M is simply connected with a quasi-conformal screen distribution, then from Remark 3.20, there exists a normalization ζ such that the corresponding rigged ξ satisfies $\tau(\xi) = 0$. From [**2**, Remark 3], we have

$$\overline{\mathrm{Ric}}(\xi) = \xi(\mathbf{H}_\xi) + \tau(\xi)\mathbf{H}_\xi - |\overset{\star}{A}_\xi|^2.$$

But $\overline{\mathrm{Ric}}(\xi) = n\overline{\beta}_{|M}$ and $\tau(\xi) = 0$, it follows that $\xi(\mathbf{H}_\xi) - |\overset{\star}{A}_\xi|^2 - n\overline{\beta}_{|_M} = 0$. Using the inequality $|\overset{\star}{A}_\xi|^2 \geq \frac{1}{n}\mathbf{H}_\xi^2$, we obtain $\xi(\mathbf{H}_\xi) - \frac{1}{n}\mathbf{H}_\xi^2 - n\overline{\beta}_{|M} \geq 0$, and since ξ is complete (M being compact), we get that $\mathbf{H}_\xi = 0$.

From the relation $\xi(\mathbf{H}_\xi) - |\overset{\star}{A}_\xi|^2 - n\overline{\beta}_{|_M} = 0$, it follows that $|\overset{\star}{A}_\xi|^2 = -n\overline{\beta}_{|_M}$, which leads to $\overset{\star}{A}_\xi = 0$ and $\overline{\beta}_{|_M} = 0$ on M since $\overline{\beta}_{|_M}$ is non negative. Being M totally geodesic together with the fact that ζ has a quasi-conformal screen distribution, we have from equation (12) that $\mathscr{S}$ is totally umbilic with umbilic factor σ. Now, if σ is constant along the integral curves of ξ, then from equation (38) together with the fact that M is totally geodesic and $\overline{\beta}|_M = 0$, the last claim follows. □

Corollary 3.25. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature with $\overline{\beta}_{|_M}$ non-negative, and let M be a ζ-rigging simply connected compact null hypersurface of $\overline{M}$. If ζ has a conformal screen distribution with conformal factor ϕ, then both $\mathscr{S}(\zeta)$ and M are totally geodesic immersed in M and $\overline{M}$. Moreover, $\overline{M}$ is flat along M.*

In the Theorem 3.26, the hypothesis quasi-conformal screen distribution and simply connected imply that $\tau(X) = 0$ for all $X \in TM$. Similar conclusion holds if ζ is closed conformal (see Remark 3.16). This leads to the following theorem.

Theorem 3.26. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold of quasi-constant curvature with $\overline{\beta}_{|_M}$ non-negative, and let M be a ζ-rigging compact null hypersurface of $\overline{M}$. If ζ is closed conformal with conformal factor Ψ, then M is totally geodesic in $\overline{M}$, $\mathscr{S}(\zeta)$ is totally umbilic in $\overline{M}$ with umbilic factor Ψ. Moreover, if Ψ is constant along the integral curves of ξ, then $\overline{M}$ is flat along M.*

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References

1. Atindogbé C. and Kemajou Mbiakop T., *Normalized null hypersurfaces in Lorentzian manifolds admitting a conformal vector field*, Differ. Geom. Dyn. Syst. **23** (2021), 1–17.
2. Atindogbé C. and Fosting H. T., *Newton transformations on null hypersurfaces*, Commun. Math. **23** (2015), 57–83.
3. Atindogbé C. and Kemajou Mbiakop T., *Null hypersurfaces in Brinkmann spacetimes*, Bull. Math. Anal. Appl. **12**(3) (2020), 60–75.
4. Chen B-Y. and Yano K., *Hypersurfaces of a conformally flat space*, Tensor (NS) **26** (1972), 318–322.
5. Duggal K. L. and Bejancu A., *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic, Dordrecht, 1996.
6. Duggal K. L. and Jin D-H., *A classification of Einstein lightlike hypersurfaces of a Lorentzian space form*, J. Geom. Phys. **60** (2010), 1881–1889.
7. Gutiérrez M. and Olea B., *Induced Riemannian structures on null hypersurfaces*, Math. Nachr. **289** (2016), 1219–1236.
8. Jin D-H., *A characterization theorem for lightlike hypersurfaces of semi-Riemannian manifolds of quasi-constant curvatures*, East Asian Math. J. **30** (2014), 15–22.
9. Jin D-H., *Lightlike hypersurfaces with totally umbilic screen distributions*, J. Chungcheong Math. Soc. **22** (2009), 409–415.
10. Jin D-H., *Lightlike hypersurfaces of a semi-Riemannian manifold of quasi-constant curvature*, Commun. Korean Math. Soc. **27** (2012), 763–770.
11. Kang T.-H., *On lightlike hypersurfaces of a GRW space-time*, Bull. Korean Math. Soc. **49** (2012), 863–874.
12. Navarro M., Palmas O. and Solis D. A., *Null screen quasi-conformal hypersurfaces in semi-Riemannian manifolds and applications*, Math. Nachr. **293** (2020), 1534–1553.

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