Acta Math. Univ. Comenianae
Vol. XCIV, 3 (2025), pp. 209–224

A FINITE CALCULUS APPROACH TO THE PARTIAL SUM AND BOUNDS OF THE HARMONIC SERIES

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Abstract. This study develops a new general partial summation formula for harmonic series, utilizing finite calculus techniques. The formula provides highly accurate upper and lower bounds without a correction term within which the exact partial sum lies. Additionally, an improved approximation formula for the summation of the harmonic series is introduced. The proposed general formula offers a straightforward and precise method for summing harmonic series, but has an unsolved term. The derived bounds are the closest to the exact partial sum, marking a significant advancement in the field. Comparisons with Euler's formula, based on general and RMS errors, demonstrate that the proposed approximation formula achieves superior accuracy. These findings bring a novel approach to harmonic series analysis, with potential applications in numerical analysis and theoretical research.

1. Introduction

The harmonic series $H_n(a,d)$ is defined as the sum of a harmonic progression, which is formed using the reciprocals of successive terms of an arithmetic progression:

$$H_n(a,d) = \frac{1}{a+d} + \frac{1}{a+2d} + \frac{1}{a+3d} + \cdots + \frac{1}{a+nd}\,, \tag{1}$$

where a, d and n indicate the first term, common difference and number of terms, respectively.

The most basic case of $H_n(a,d)$ occurs when $a = 0$ and $d = 1$. This case is the most extensively studied and frequently used for approximations:

$$H_n(a=0,d=1) = \sum_{x=1}^{n} \frac{1}{x} = H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\,, \tag{2}$$

Received December 14, 2024; revised July 1, 2025.
2020 *Mathematics Subject Classification.* Primary 65B10.
Key words and phrases. Finite calculus; harmonic series; partial sum; discrete derivative; discrete anti-derivative; Euler's formula.
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where n is a positive integer. The symbol H_n represents the harmonic sum of the first n terms in the harmonic series, also known as the nth partial sum of the harmonic series [**8**]. It is well established that the harmonic series is divergent [**8**]. Despite its divergence, the harmonic series appears in various mathematical problems and applications, such as the Leaning Tower of Pisa, Gabriel's Wedding Cake, Euler's proof that the harmonic series of prime numbers diverges [**8**], and more. It also arises in the analysis of algorithms, particularly in the study of series convergence, and has applications in physics.

The Euler-Maclaurin formula is a powerful method that provides a relationship between an integral and the corresponding sum of a function. It can be used to approximate an integral by a sum and vice versa [**2**].

Although there is no closed-form summation formula for the harmonic series, various approximation methods have been developed, such as Euler's approximation formula [**7**], Knuth's approximation [**9**], and approximations using hyperbolic functions [**1**]. Additionally, several closest bounds have been proposed [**4**]. The majority of these approaches are based on the Euler-Maclaurin formula and involve the use of the Euler-Mascheroni constant. However, if an exact partial sum of the harmonic series were to exist, the need for a correction term such as the Euler-Mascheroni constant would be eliminated. So, it is crucial to identify closest bounds for the harmonic series that exclude correction terms such as the Euler-Mascheroni constant, as this may lead to determining the exact partial sum. Resolving this would not only address a key issue but also open new directions for research. Therefore, exploring alternative methods is necessary to make progress in this field.

The primary objective of this paper is to develop a new general formula using finite difference calculus to find the exact partial summation of the harmonic series (1). Using this general formula, the paper presents the closest lower and upper bounds without correction term within which the exact partial sum of the harmonic series (1) lies. Additionally, the paper introduces an approximation formula for the summation of the harmonic series (2), which offers improved accuracy compared to the well-known Euler approximation formula.

2. Methodology

2.1. Finite calculus

In mathematics, the concept of function is essential. Functions can be categorized into two main types: continuous functions, which are defined for all values within a given interval, and discontinuous functions, which are defined only for specific discrete values within a set. The Calculus of Finite Differences primarily deals with discontinuous functions, although it can be applied to both continuous and discontinuous cases.

Given a function $f(x)$ defined at the points $x = x_1, x_2, x_3, \ldots$, finite differences are typically considered when the values of the variable x are equidistant. That is,

$$x_{i+1} - x_i = h\,,$$

where h is independent of i [**7**]. For the purposes of this paper, we will assume $h = 1$. According to the definition of derivatives, we know

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = g(x). \tag{3}$$

The anti-derivative of $g(x)$ is given by

$$\int g(x)\,\mathrm{d}x = f(x) + c.$$

The definite integral represents the area under the curve of the function $g(x)$ between points a and b on the x-axis, dividing the area into infinitely small rectangles with width $\mathrm{d}x$ as $\mathrm{d}x \to 0$:

$$\int_a^b g(x)\,\mathrm{d}x = f(b) - f(a). \tag{4}$$

We can explore equation (4) according to Figure 1 further as follows:

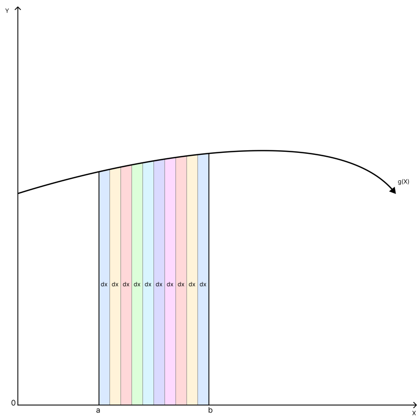

Figure 1. Geometrical representation of definite integral

$$\int_a^b g(x)\,\mathrm{d}x = g(a)\,\mathrm{d}x + g(a + \mathrm{d}x)\,\mathrm{d}x + g(a + 2\mathrm{d}x)\,\mathrm{d}x + \cdots + g(b)\,\mathrm{d}x. \tag{5}$$

Now, referring back to equation (3), in differential calculus, we usually consider $h \to 0$, but in finite calculus, we consider $h \to 1$.

Let us rewrite equation (3) using a difference operator Δ as follows:

$$\Delta f(x) = \lim_{h \to 1} \frac{f(x+h) - f(x)}{h} = H(x).$$

Thus, we have

$$\Delta f(x) = f(x+1) - f(x) = H(x). \tag{6}$$

Similar to equation (5), we can expand equation (6) as follows:

$$X = H(a) \cdot 1 + H(a+1) \cdot 1 + H(a+2) \cdot 1 + \cdots + H(b) \cdot 1. \tag{7}$$

Equation (7) represents a sequence of finite sums of the function $H(x)$ over $[a, b]$. We can now write equation (7) as

$$X = \sum_{a}^{b} H(x). \tag{8}$$

From the theory of definite integrals, we know that

$$\int_a^b g(x)\,\mathrm{d}x = f(b) - f(a).$$

In finite calculus, we aim to prove that

$$\sum_{a}^{b} H(x) = f(b+1) - f(a).$$

According to [**5**], we have the following definitions.

Definition (Discrete Derivative)**.** The definition of the discrete derivative of $f(x)$ is

$$\Delta f(x) = f(x+1) - f(x). \tag{9}$$

Definition (Discrete Anti-Derivative)**.** A function $f(x)$ is called the discrete anti-derivative of $g(x)$ if it has the characteristic of

$$\Delta f(x) = g(x).$$

We represent the indefinite sum of $g(x)$ as the class of functions that satisfy this property:

$$\sum g(x)\,\Delta x = f(x) + C.$$

As we consider $\Delta x \to 1$ for finite difference calculus, we can rewrite as

$$\sum g(x) \ = f(x) + C, \tag{10}$$

where C is an arbitrary constant.

The fundamental theorem of finite difference calculus states the following:

Theorem 1. *Let $f(x)$ be a discrete anti-derivative of $g(x)$, then*

$$\sum_{a}^{b} g(x) = f(b+1) - f(a).$$

Proof.

$$\sum_{a}^{b} g(x) = \sum_{a}^{b} \Delta f(x) = \sum_{a}^{b} (f(x+1) - f(x))$$
$$= \sum_{a}^{b} f(x+1) - \sum_{a}^{b} f(x) = f(b+1) - f(a)$$

Thus, for a given function $g(x)$, if we can find a function $f(x)$ such that $\Delta f(x) = g(x)$, then $f(x)$ will serve as a closed-form solution for the finite sum of $g(x)$. □

2.2. Derivation of the general partial sum of harmonic series

Consider the series

$$H_n(a,d) = \frac{1}{a+d} + \frac{1}{a+2d} + \frac{1}{a+3d} + \cdots + \frac{1}{a+nd}.$$

As we know from the Maclaurin series,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots . \tag{11}$$

From equation (11), we obtain

$$\ln\left(1+\frac{d}{a+xd}\right) = \frac{d}{a+xd} - \frac{d^2}{2(a+xd)^2} + \frac{d^3}{3(a+xd)^3} - \frac{d^4}{4(a+xd)^4} + \cdots . \tag{12}$$

Taking the anti-discrete derivative in equation (12), we get

$$\begin{aligned} \sum \ln\left(1+\frac{d}{a+xd}\right) &= \sum \frac{d}{a+xd} + \sum\left(-\frac{d^2}{2(a+xd)^2} - \frac{d^4}{4(a+xd)^4} - \cdots\right) \\ &\quad + \sum\left(\frac{d^3}{3(a+xd)^3} + \frac{d^5}{5(a+xd)^5} + \cdots\right). \end{aligned} \tag{13}$$

Let

$$-\frac{d^2}{2(a+xd)^2} - \frac{d^4}{4(a+xd)^4} - \cdots = A(x),$$

$$\frac{d^3}{3(a+xd)^3} + \frac{d^5}{5(a+xd)^5} + \cdots = B(x).$$

Then,

$$\sum \ln\left(1+\frac{d}{a+xd}\right) = \sum \frac{d}{a+xd} + \sum A(x) + \sum B(x). \tag{14}$$

Similarly, from the Maclaurin series, we know

$$\ln\left(1-\frac{d}{a+xd}\right) = -\frac{d}{a+xd} - \frac{d^2}{2(a+xd)^2} - \frac{d^3}{3(a+xd)^3} - \frac{d^4}{4(a+xd)^4} - \cdots . \tag{15}$$

Thus, we can write

$$\sum \ln\left(1-\frac{d}{a+xd}\right) = -\sum \frac{d}{a+xd} + \sum A(x) - \sum B(x). \tag{16}$$

Adding equations (14) and (16), we obtain

$$\sum A(x) = \frac{1}{2}\left(\sum \ln\left(1+\frac{d}{a+xd}\right) + \sum \ln\left(1-\frac{d}{a+xd}\right)\right). \tag{17}$$

Using the discrete anti-derivative, we get

$$\sum \ln\left(1+\frac{d}{a+xd}\right) = \ln(a+xd) + c_1$$

$$\text{and } \sum \ln\left(1-\frac{d}{a+xd}\right) = -\ln(a+xd-d) + c_2.$$

For further details on the proof of the formula, see appendix A. Thus, from equation (17), we get

$$\sum A(x) = \frac{1}{2}\ln\left(\frac{a+xd}{a+xd-d}\right) + c_3. \tag{18}$$

Now, combining equations (14) and (18), we find

$$\sum B(x) + \sum \frac{d}{a+xd} = \sum \ln\left(1+\frac{d}{a+xd}\right) - \frac{1}{2}\ln\left(\frac{a+xd}{a+xd-d}\right) - c_3$$

$$\text{or } \sum B(x) + \sum \frac{d}{a+xd} = \ln(a+xd) - \frac{1}{2}\ln\left(\frac{a+xd}{a+xd-d}\right) + c_4$$

$$\text{or } \sum B(x) + \sum \frac{d}{a+xd} = \ln\left(\sqrt{(a+xd)(a+xd-d)}\right) + c_4. \tag{19}$$

Using the fundamental theorem of finite calculus, from equation (19), we get

$$\sum_{x=2}^{n} \frac{d}{a+xd} = \ln\left(\sqrt{(a+nd)(a+nd+d)}\right) - \ln\left(\sqrt{(a+2d)(a+2d-d)}\right) - \sum_{x=2}^{n} B(x).$$

We know that

$$\sum_{x=1}^{1} \frac{d}{a+xd} = \frac{d}{a+d}.$$

Then,

$$\sum_{x=1}^{n} \frac{d}{a+xd} = \ln\left(\sqrt{\frac{(a+nd)(a+nd+d)}{(a+2d)(a+2d-d)}}\right) + \frac{d}{a+d} - \sum_{x=2}^{n} B(x). \tag{20}$$

We now determine the possible values of the parameters a, d, and n, as well as the variable x, that satisfy the equations (20) and (13). As x and n represent term indices, they must be positive integers. The validity of equation (13) requires $\left|\frac{d}{a+xd}\right| < 1$ and $a + xd \neq 0$.

For $a = 0$ and $d = 1$, we find the classical harmonic series (2). In this case, the condition $\left|\frac{d}{a+xd}\right| < 1$ simplifies to

$$\left|\frac{1}{x}\right| < 1 \implies x \geq 2.$$

Thus, the range of x is $2 \leq x \leq n$, implying $n \geq 2$.

To extend the formula to include $x = 1$, we explicitly add the following term in equation (20). Consequently, equation (20) can compute $\sum_{x=1}^{n} \frac{d}{a+xd}$ for $n \geq 2$.

$$\sum_{x=1}^{1} \frac{d}{a+xd} = \frac{d}{a+d}.$$

Therefore, equation (20) holds for any a and d satisfying $\left|\frac{d}{a+xd}\right| < 1$ and $a+xd \neq 0$ for $2 \leq x \leq n$, with $n \geq 2$.

2.3. Derivation of the approximation of harmonic series

By setting $a = 0$ and $d = 1$, the series

$$H_n(a,d) = \frac{1}{a+d} + \frac{1}{a+2d} + \frac{1}{a+3d} + \cdots + \frac{1}{a+nd}$$

becomes

$$\sum_{x=1}^{n} \frac{1}{x} = H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n},$$

where n is a positive integer. So, from equation (20)

$$\sum_{x=1}^{n} \frac{1}{x} = \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + 1 - \sum_{x=2}^{n} B(x). \tag{21}$$

Now, considering the terms $1 - \sum_{x=2}^{n} B(x)$, where $B(x) = \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots$, using python programming in Thonny IDE, it is found that for $n = \infty$,

$$1 - \sum_{x=2}^{\infty} B(x) \approx 0.9237892551\ldots.$$

So, the approximation partial summation formula for the harmonic series is

$$\sum_{x=1}^{n} \frac{1}{x} \approx \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + 0.9237892551\ldots. \tag{22}$$

2.4. Derivation of the upper and lower bound functions

Now, we will derive a lower bound and an upper bound functions within which the exact partial summation of the harmonic series exists.

By removing the discrete anti-derivative from equation (17), we get

$$A(x) = \frac{1}{2}\left(\ln\left(1 + \frac{d}{a+xd}\right) + \ln\left(1 - \frac{d}{a+xd}\right)\right). \tag{23}$$

Now, taking the derivative of both sides,

$$\frac{\mathrm{d}}{\mathrm{d}x}A(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1 + \frac{d}{a+xd}\right) + \ln\left(1 - \frac{d}{a+xd}\right)\right) \tag{24}$$

$$\frac{d^3}{(a+xd)^3} + \frac{d^5}{(a+xd)^5} + \cdots = -\frac{1}{2}\left[\frac{d^2}{(a+xd+d)(a+xd)}\right] + \frac{1}{2}\left[\frac{d^2}{(a+xd-d)(a+xd)}\right].$$

Taking the discrete anti-derivative of both sides, we get

$$\sum\left(\frac{d^3}{(a+xd)^3} + \frac{d^5}{(a+xd)^5} + \cdots\right) = -\frac{d^2}{2(a+xd-d)(a+xd)} + C. \tag{25}$$

For further details on the proofs of the formula, see appendix A.

Let

$$\frac{d^3}{(a+xd)^3} + \frac{d^5}{(a+xd)^5} + \cdots = P(x).$$

Using the fundamental theorem of finite calculus,

$$\sum_{x=2}^{n} P(x) = \frac{d^2}{2(a+2d)(a+d)} - \frac{d^2}{2(a+nd+d)(a+nd)}. \tag{26}$$

It is clearly visible that

$$\sum_{x=2}^{n}\left(\frac{d^3}{(a+xd)^3} + \frac{d^5}{(a+xd)^5} + \cdots\right) \geq \sum_{x=2}^{n}\left(\frac{d^3}{3(a+xd)^3} + \frac{d^5}{5(a+xd)^5} + \cdots\right).$$

Therefore,

$$\sum_{x=2}^{n} P(x) \geq \sum_{x=2}^{n} B(x)$$

Using equation (20), we can conclude that

$$\begin{aligned}\sum_{x=1}^{n}\frac{d}{a+xd} &= \ln\left(\sqrt{\frac{(a+nd)(a+nd+d)}{(a+2d)(a+2d-d)}}\right) + \frac{d}{a+d} - \sum_{x=2}^{n} B(x)\\ &\geq \ln\left(\sqrt{\frac{(a+nd)(a+nd+d)}{(a+2d)(a+2d-d)}}\right) + \frac{d}{a+d} - \sum_{x=2}^{n} P(x).\end{aligned} \tag{27}$$

Again, from equation (20), we can conclude that

$$\sum_{x=1}^{n}\frac{d}{a+xd} \leq \ln\left(\sqrt{\frac{(a+nd)(a+nd+d)}{(a+2d)(a+2d-d)}}\right) + \frac{d}{a+d}.$$

Thus, the final result is

$$\begin{aligned}&\ln\left(\sqrt{\frac{(a+nd)(a+nd+d)}{(a+2d)(a+2d-d)}}\right) + \frac{d}{a+d}\\ &\quad + \frac{d^2}{2(a+nd+d)(a+nd)} - \frac{d^2}{2(a+2d)(a+d)}\\ &\leq \sum_{x=1}^{n}\frac{d}{a+xd} \leq \ln\left(\sqrt{\frac{(a+nd)(a+nd+d)}{(a+2d)(a+2d-d)}}\right) + \frac{d}{a+d}.\end{aligned} \tag{28}$$

This represents a novel lower bound and upper bound functions within which the exact partial sum of the harmonic series lies. This formulation is one of the key contributions of this paper. Furthermore, by using (28), we can establish lower and upper bound functions for various harmonic series by selecting different values of a and d.

3. Result

3.1. The general partial sum of harmonic series

From equation (20), using finite calculus, the partial sum of the harmonic series is

$$\sum_{x=1}^{n} \frac{d}{a+xd} = \ln\left(\sqrt{\frac{(a+nd)(a+nd+d)}{(a+2d)(a+d)}}\right) + \frac{d}{a+d} - \sum_{x=2}^{n} B(x),$$

where

$$B(x) = \frac{d^3}{3(a+xd)^3} + \frac{d^5}{5(a+xd)^5} + \cdots .$$

This formula is based on the harmonic series

$$H_n(a,d) = \frac{1}{a+d} + \frac{1}{a+2d} + \frac{1}{a+3d} + \cdots + \frac{1}{a+nd}. \tag{29}$$

Now, using $a = 3$ and $d = 2$, the series (29) becomes

$$H_n(a=3, d=2) = \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots + \frac{1}{3+2n}.$$

Then the partial sum formula for this series is

$$\sum_{x=1}^{n} \frac{2}{3+2x} = \ln\left(\sqrt{\frac{(3+2n)(5+2n)}{35}}\right) + \frac{2}{5} - \sum_{x=2}^{n} B(x),$$

where

$$B(x) = \frac{8}{3(3+2x)^3} + \frac{32}{5(3+2x)^5} + \cdots .$$

Next, using $a = 3$ and $d = 3$, the series (29) becomes

$$H_n(a=3, d=3) = \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \cdots + \frac{1}{3+3n}.$$

The partial sum of this series is

$$\sum_{x=1}^{n} \frac{3}{3+3x} = \ln\left(\sqrt{\frac{(3+3n)(6+3n)}{54}}\right) + \frac{3}{6} - \sum_{x=2}^{n} B(x),$$

where

$$B(x) = \frac{3^3}{3(3+3x)^3} + \frac{3^5}{5(3+3x)^5} + \cdots .$$

In this way, we can generate many harmonic series and find their corresponding partial sum formulas.

3.2. The approximation of the harmonic series

Now, this paper presents an approximation of the harmonic series, where $a = 0$ and $d = 1$ in series (1). Then the series

$$\sum_{x=1}^{n} \frac{1}{x} = H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is the most vital and well-known harmonic series with numerous applications in mathematics, algorithm analysis, etc.

From equation (22), the approximation formula derived in this paper is

$$\sum_{x=1}^{n} \frac{1}{x} \approx \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + 0.9237892551\ldots.$$

Now, let us compare this formula with the famous Euler formula for approximating the harmonic series [**6**]:

$$H_n \approx \ln(n) + \gamma,$$

where γ is the Euler-Mascheroni constant, with $\gamma = 0.5772156649\ldots$.

Table 1. Comparison of exact values and Euler's formula values with accuracy and error percentage

Size of n	**Exact value of H_n (calculated manually)**	**Value of H_n by Euler's formula**	**Accuracy**	**Error**
n = 5	2.283333333333333	2.186653577334100	95.76%	4.23%
n = 20	3.597739657143682	3.572947938453991	99.31%	0.68%
n = 50	4.499205338329423	4.489238670328146	99.77%	0.22%
n = 100	5.187377517639621	5.182385850888092	99.90%	0.09%
n = 1000	7.485470860550343	7.484970943882137	99.993%	0.006%

Table 2. Comparison of exact values with proposed formula values with accuracy and error percentage

Size of n	**Exact value of H_n (calculated manually)**	**Value of H_n by proposed formula**	**Accuracy**	**Error**
n = 5	2.283333333333333	2.277814355732005	99.75%	0.24%
n = 20	3.597739657143682	3.597343020539634	99.98%	0.01%
n = 50	4.499205338329423	4.499139983977163	99.99%	0.001%
n = 100	5.187377517639621	5.187361016315602	99.999%	0.0003%
n = 1000	7.485470860550343	7.485470694049606	99.99999%	0.000002%

From Tables 1 and 2, we can conclude that the proposed approximation formula produces better results compared to Euler's formula. As the value of n increases, the proposed formula produces more accurate results.

3.3. The upper and lower bound functions

Another contribution of this paper is demonstrating that the lower and upper bound functions within the partial sum of the harmonic series exist. Now we will find upper and lower bound function for most vital harmonic series

$$\sum_{x=1}^{n} \frac{1}{x} = H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

From (28), using $a = 0$ and $d = 1$,

$$\ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + \frac{1}{2n(n+1)} + \frac{3}{4} \leq \sum_{x=1}^{n} \frac{1}{x} \leq \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + 1.$$

Let

$$L(n) = \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + \frac{1}{2n(n+1)} + \frac{3}{4}$$

and

$$U(n) = \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + 1.$$

Now, let us find the maximum and minimum difference between the lower and upper bound functions:

$$\begin{aligned} D(n) = U(n) - L(n) = \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + 1 \\ - \left(\ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + \frac{1}{2n(n+1)} + \frac{3}{4}\right), \end{aligned} \tag{30}$$

where n is a positive integer.

The difference function is then

$$D(n) = \frac{1}{4} - \frac{1}{2n(n+1)}. \tag{31}$$

Now, setting $n = 2$ and $n = \infty$ in equation (31),

$$D(2) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

and

$$D(\infty) = \frac{1}{4} - 0 = \frac{1}{4}.$$

Thus, the function $D(n)$ yields a minimum difference of $\frac{1}{6}$ and a maximum difference of $\frac{1}{4}$. Therefore, we establish the following inequality:

$$\frac{1}{6} \leq D(n) \leq \frac{1}{4}.$$

This inequality demonstrates that the upper and lower bounds are remarkably close, confirming the tightness of the approximation.

4. Discussion

4.1. Comparison and validation of equation (22)

The Root Mean Square (RMS) error quantifies the average magnitude of errors between the exact harmonic sum and the approximations provided by different formulas. The RMS error is calculated using the following formula [**3**]:

$$\text{RMS Error} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{x}_i - x_i)^2},$$

where $\hat{x}_i$ represents the predicted values (approximations) and x_i represents the actual values of the harmonic series, n represents number of observations.

Table 3 presents a comparative analysis of the RMS errors for the proposed formula and the Euler formula across different values of n. The results unequivocally demonstrate the superior accuracy and effectiveness of the proposed approximation in comparison to the traditional Euler method.

Table 3. RMS Error Comparison between Proposed Formula and Euler's Formula

Range of n	**RMS Error (proposed formula)**	**RMS Error (Euler's formula)**
1 to 100	0.008282	0.056700
100 to 200	8.97e-06	0.003536
200 to 300	2.85e-06	0.002041
300 to 400	1.40e-06	0.001443
400 to 500	8.39e-07	0.001118
500 to 600	5.58e-07	0.000913
600 to 700	3.98e-07	0.000771
700 to 800	2.98e-07	0.000668

Figure 2 visually underscores the substantial superiority of the proposed formula over Euler's method in approximating the harmonic series. The plot clearly illustrates the consistent lower RMS errors achieved by the proposed formula across all tested ranges of n. For instance, within the range of $n = 1$ to $n = 100$, the RMS error of the proposed formula is approximately 0.0083, while Euler's formula yields a significantly higher error of 0.0567. As n increases, both formulas exhibit decreasing errors, but the error reduction of the proposed formula is notably more pronounced. In the range $n = 700$ to $n = 800$, the RMS error of the proposed formula is as low as 2.98×10^{-7}, while Euler's formula maintains a considerably higher error of approximately 0.000668. This substantial disparity unequivocally highlights the superior precision and effectiveness of the proposed formula for approximating the harmonic series.

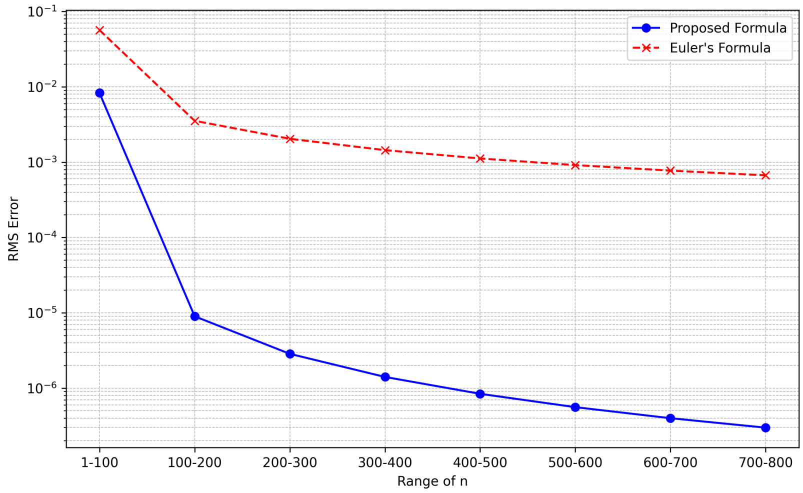

Figure 2. Comparison of RMS errors between the proposed formula and Euler's formula across different n ranges.

4.2. Analysis of upper and lower bounds

In this study, we have successfully demonstrated the existence of precise upper and lower bound functions for the partial sum of the harmonic series:

$$\sum_{x=1}^{n} \frac{1}{x} = H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

Our derived bounds are

$$L(n) = \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + \frac{1}{2n(n+1)} + \frac{3}{4}$$

and

$$U(n) = \ln\left(\sqrt{\frac{n(n+1)}{2}}\right) + 1.$$

The difference between these bounds is expressed as

$$D(n) = \frac{1}{4} - \frac{1}{2n(n+1)}.$$

Although the bounds do not involve correction terms, they reveal that the bounds are exceptionally close to each other. Specifically, as n ranges from 2 to ∞, $D(n)$ ranges from $\frac{1}{6}$ to $\frac{1}{4}$, indicating a tight approximation. For $n = 2$, $D(n = 2) = \frac{1}{6}$, and as n approaches infinity, $D(n = \infty)$ approaches $\frac{1}{4}$.

The established bounds for the partial sum of the harmonic series, with the difference between the upper and lower functions ranging from $\frac{1}{6}$ to $\frac{1}{4}$, provide an exceptionally tight approximation. This result is significant as it offers a highly accurate range within which the exact harmonic sum exists, facilitating precise calculations and enhancing the reliability of approximations. By laying a foundation for future research, this work not only aids in finding the exact sum but also has practical implications for fields relying on harmonic series computations, such as number theory and algorithm analysis.

5. Limitation of proposed formula

While our developed general formula provides significant results, including the closest bounds and the most accurate approximation formula, there is still room for improvement. Our formula is given by:

$$\sum_{x=1}^{n} \frac{d}{a+xd} = \ln\left(\sqrt{\frac{(a+nd)(a+nd+d)}{(a+2d)(a+2d-d)}}\right) + \frac{d}{a+d} - \sum_{x=2}^{n} B(x).$$

In this formula, the term $\sum_{x=2}^{n} B(x)$ remains unsolved. Solving this term could lead to finding the exact closed partial sum of the harmonic series, which would be a significant breakthrough. Further research on this term could also improve our proposed approximation formula and the upper and lower bounds even more.

6. Conclusion

This study introduces a new general method for calculating partial sums of harmonic series using finite difference calculus. Although the general formula does not provide an exact closed-form solution due to the presence of one unsolved term, it allows us to derive a highly accurate approximation formula. Additionally, it establishes straightforward and closest upper and lower bounds without correction term for the exact sums, setting a new standard for precision in this field.

By comparing our approximation formula with Euler's formula, we demonstrate that our method outperforms in terms of both general and RMS errors. Furthermore, we have proven that the upper and lower bound functions are remarkably close. This improvement is significant for various applications, including numerical analysis and theoretical research. Overall, our study makes a valuable contribution to the field of harmonic series, offering a practical tool for future research and applications.

Acknowledgement. The authors express their sincere gratitude to the anonymous referee for insightful comments and constructive suggestions that have significantly improved the quality of this manuscript. The authors sincerely thank Md. Mojammel Haque for his valuable guidance and mentorship during this research.

Appendix A. Proof of Formula

In this appendix, we provide proofs for some formulas used in this paper. We start with the formula:

$$\sum \ln\left(1+\frac{d}{a+xd}\right) = \ln(a+xd) + c_1.$$

Let

$$y = \ln(a+xd).$$

Taking the discrete derivative on both sides, we obtain

$$\Delta y = \ln(a+xd+d) - \ln(a+xd)$$

$$\text{or} \quad \Delta y = \ln\left(\frac{a+xd+d}{a+xd}\right)$$

$$\text{or} \quad \Delta y = \ln\left(1+\frac{d}{a+xd}\right).$$

Now, taking the discrete anti-derivative on both sides, we have

$$\sum \ln\left(1+\frac{d}{a+xd}\right) = y + c_1,$$

$$\sum \ln\left(1+\frac{d}{a+xd}\right) = \ln(a+xd) + c_1.$$

Similarly, we can prove

$$\sum \ln\left(1-\frac{d}{a+xd}\right) = -\ln(a+xd-d) + c_2,$$

and

$$\sum\left(-\frac{1}{2}\left[\frac{d^2}{(a+xd+d)(a+xd)}\right] + \frac{1}{2}\left[\frac{d^2}{(a+xd-d)(a+xd)}\right]\right)$$
$$= -\frac{d^2}{2(a+xd-d)(a+xd)} + C.$$

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