Acta Math. Univ. Comenianae
Vol. XCIV, 2 (2025), pp. 47–58

STUDY OF NEW GENERAL INTEGRAL RESULTS UNDER ORIGINAL PRIMITIVE-LIKE INEQUALITY ASSUMPTIONS

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Abstract. This article proposes new general inequalities based on the integral of two adaptive functions. The main originality of these results lies in two key aspects: (i) the assumptions considered, which combine monotonicity and primitive-like inequalities, and (ii) the expression of the bounds, in which the power and logarithmic transformations play a significant role. Several complementary integral inequalities are established under additional assumptions, including convexity assumptions. Applications and future directions are also discussed.

1. Introduction

Integrals are undoubtedly one of the most important tools in mathematics. They are used to define various measures of areas, volumes and quantities in many different contexts. These include physics, engineering, economics, statistics and probability. However, despite the wide range of available techniques, integral evaluations often do not have exact values. In such cases, bounds are essential for estimation purposes. These bounds are a consequence of sharp integral inequalities, which have been of interest for many years. See [**1, 2, 4, 6, 15, 16**] for an overview of this topic.

Integrals defined using two adaptive functions are of particular interest. They can represent an inner product or measure functional relations. For the purposes of this article, we will highlight three well-known integral inequalities that fit this description. These are the monotonic, Steffensen and Chebyshev integral inequalities. Let $a, b \in \mathbb{R}$ with $a < b$, and let f and g be positive functions defined on $[a, b]$. Then, under additional assumptions on f and g, the results below are well established.

Monotonic integral inequality. We assume that g is continuous and monotonic. Then we have

$$\int_a^b f(t)g(t)\mathrm{d}t \leq \max[g(b), g(a)] \int_a^b f(t)\mathrm{d}t.$$

Received January 5, 2025; revised May 16, 2025.
2020 *Mathematics Subject Classification.* Primary 26D15, 33E20.
Key words and phrases. Integral inequalities; primitive-like inequality assumptions; convexity; Young product inequality; power transformation.

Steffensen integral inequality. We assume that f is non-increasing, and $0 \leq g(t) \leq 1$ for any $t \in [a, b]$. Then we have

$$\int_a^b f(t)g(t)\mathrm{d}t \leq \int_a^{a+\int_a^b g(t)\mathrm{d}t} f(t)\mathrm{d}t.$$

Chebyshev integral inequality. We assume that f is non-increasing and g is non-decreasing. Then we have

$$\int_a^b f(t)g(t)\mathrm{d}t \leq \frac{1}{b-a}\left[\int_a^b f(t)\mathrm{d}t\right]\left[\int_a^b g(t)\mathrm{d}t\right].$$

Taken together, these results provide various insights into integral relations. Recent related studies in different mathematical contexts include those in [**3, 5, 8, 7, 9, 10, 11, 12, 13, 14, 17**].

In this article, we take a novel approach to the topic by introducing new types of general integral inequality. These inequalities involve two adaptive functions, f and g, and are based on the integral $\int_a^b f(t)g(t)\mathrm{d}t$. In this way, we adhere to the principles of the well-known inequalities presented above. The main originality lies in the assumptions made about f and g. These are monotonicity and "primitive-like inequality assumptions", i.e., assumptions of the following form:

$$\text{“}\int_x^b g(t)\mathrm{d}t \leq f^{\alpha}(x)\text{”} \quad \text{or} \quad \text{“}\int_a^x g(t)\mathrm{d}t \leq f^{\alpha}(x)\text{”},$$

where α is an additional power parameter, positive or negative, that plays a role in the obtained bounds. We categorize our integral results into two types: those with the primitive-like inequality assumptions based on $\int_x^b g(t)\mathrm{d}t$, called "of the first type", and those with the primitive-like inequality assumptions based on $\int_a^x g(t)\mathrm{d}t$, called "of the second type". As an illustrative example, under certain assumptions, including those of the first type, the following inequality is proved:

$$\text{“}\int_a^b f(t)g(t)\mathrm{d}t \leq \frac{1}{\alpha+1}f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1}f^{\alpha+1}(a).\text{”}$$

In addition, we use the main results to prove other secondary integral results for each type, some of which require additional assumptions on f, such as convexity. For example, under certain assumptions, including those of the first type, the following lower bound is established:

$$\begin{aligned}&\text{“}\frac{1}{\alpha+1}f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1}f^{\alpha+1}(a)\\ &\geq \max\left[\int_a^b f(t)g(t)\mathrm{d}t, f^{\alpha+1}\left(\frac{b+\alpha a}{\alpha+1}\right), f(b)f^{\alpha}(a)\right].\text{”}\end{aligned}$$

To the best of our knowledge, this type of inequality is new to the literature and opens up new perspectives for applications.

The rest of the article is divided into three sections: Section 2 presents the results of the first type. Section 3 does the same for the second type. Section 4 provides a conclusion.

2. New integral results of the first type

2.1. Main result

The theorem below is our main result of the first type. It shows that, under certain monotonicity and primitive-like inequality assumptions, new upper bounds for $\int_a^b f(t)g(t)\mathrm{d}t$ can be established.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$, f be a positive differentiable non-decreasing function defined on $[a, b]$ and g be an integrable function defined on $[a, b]$.*

1. *We assume that there exists $\alpha \in \mathbb{R} \setminus \{-1\}$ such that, for any $x \in [a, b]$, we have*

$$\int_x^b g(t)\mathrm{d}t \leq f^{\alpha}(x). \tag{1}$$

Then we have

$$\int_a^b f(t)g(t)\mathrm{d}t \leq \frac{1}{\alpha+1} f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1} f^{\alpha+1}(a).$$

2. *We assume that, for any $x \in [a, b]$, we have*

$$\int_x^b g(t)\mathrm{d}t \leq \frac{1}{f(x)}. \tag{2}$$

Then we have

$$\int_a^b f(t)g(t)\mathrm{d}t \leq 1 + \ln\left[\frac{f(b)}{f(a)}\right].$$

Proof. The two points have the same mathematical basis. Since f is differentiable, for any $t \in [a, b]$, we can write

$$f(t) = [f(t) - f(a)] + f(a) = \int_a^t f'(u)\mathrm{d}u + f(a).$$

This, combined with a change in the order of integration made possible by the Fubini integral theorem, gives

$$\begin{aligned} \int_a^b f(t)g(t)\mathrm{d}t &= \int_a^b \left[\int_a^t f'(u)\mathrm{d}u + f(a)\right] g(t)\mathrm{d}t \\ &= \int_a^b \int_a^t f'(u)g(t)\mathrm{d}u\mathrm{d}t + f(a)\int_a^b g(t)\mathrm{d}t \\ &= \int_a^b \int_u^b f'(u)g(t)\mathrm{d}t\mathrm{d}u + f(a)\int_a^b g(t)\mathrm{d}t \\ &= \int_a^b f'(u)\left[\int_u^b g(t)\mathrm{d}t\right]\mathrm{d}u + f(a)\int_a^b g(t)\mathrm{d}t. \end{aligned} \tag{3}$$

Let us now make a distinction between the assumptions made in points 1 and 2.

1. Using equation (1) twice and the fact f is positive, differentiable and non-decreasing, which means that $f'(u) \geq 0$ for any $u \in [a, b]$, we get

$$\begin{aligned}
&\int_a^b f'(u)\left[\int_u^b g(t)\mathrm{d}t\right]\mathrm{d}u + f(a)\int_a^b g(t)\mathrm{d}t \\
&\leq \int_a^b f'(u)f^{\alpha}(u)\mathrm{d}u + f(a)f^{\alpha}(a) \\
&= \left[\frac{1}{\alpha+1}f^{\alpha+1}(u)\right]_{u=a}^{u=b} + f^{\alpha+1}(a) \\
&= \frac{1}{\alpha+1}f^{\alpha+1}(b) - \frac{1}{\alpha+1}f^{\alpha+1}(a) + f^{\alpha+1}(a) \\
&= \frac{1}{\alpha+1}f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1}f^{\alpha+1}(a).
\end{aligned} \tag{4}$$

It follows from equations (3) and (4) that

$$\int_a^b f(t)g(t)\mathrm{d}t \leq \frac{1}{\alpha+1}f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1}f^{\alpha+1}(a).$$

The given inequality is determined.

2. Using equation (2) twice and the fact f is positive, differentiable and non-decreasing, which means that $f'(u) \geq 0$ for any $u \in [a, b]$, we get

$$\begin{aligned}
&\int_a^b f'(u)\left[\int_u^b g(t)\mathrm{d}t\right]\mathrm{d}u + f(a)\int_a^b g(t)\mathrm{d}t \\
&\leq \int_a^b f'(u)\frac{1}{f(u)}\mathrm{d}u + f(a)\frac{1}{f(a)} \\
&= \left\{\ln[f(u)]\right\}_{u=a}^{u=b} + 1 \\
&= \left\{\ln[f(b)] - \ln[f(a)]\right\} + 1 \\
&= 1 + \ln\left[\frac{f(b)}{f(a)}\right].
\end{aligned} \tag{5}$$

It follows from equations (3) and (5) that

$$\int_a^b f(t)g(t)\mathrm{d}t \leq 1 + \ln\left[\frac{f(b)}{f(a)}\right].$$

This is the desired inequality. This ends the proof of Theorem 2.1. □

To the best of our knowledge, the results in Theorem 2.1 are new to the literature. They open up new perspectives for applications.

It is worth noting that α can be negative in equation (1). In addition, the point 2 is in fact the case $\alpha = -1$, which needs a particular treatment. Note that, by the Chasles integral relation, the assumption in equation (1) is equivalent to

$$\int_a^x g(t)\mathrm{d}t \geq \int_a^b g(t)\mathrm{d}t - f^\alpha(x),$$

which gives another look to the primitive-like inequality assumption of the first type.

Another remark is that, in the context of Theorem 2.1, the following assumption is not valuable to replace that in equation (1): there exists $\alpha \in \mathbb{R}$ such that, for any $x \in [a, b]$, we have

$$\int_x^b g(t)\mathrm{d}t \geq f^\alpha(x).$$

In fact, if we take $x = b$, we find that $0 = \int_b^b g(t)\mathrm{d}t \geq f^\alpha(b) \geq 0$, so $f(b) = 0$. Since f is supposed to be non-decreasing on $[a, b]$, this is only possible if f is the zero function, which makes an integral inequality based on it uninteresting.

2.2. Secondary results

In this subsection, we derive several results related to Theorem 2.1, focusing on the case $\alpha > 0$.

The proposition below shows a simple product lower bound depending on $f(a)$, $f(b)$ and α for a central term.

Proposition 2.2. *Let $a, b \in \mathbb{R}$ with $a < b$, f be a positive continuous function defined on $[a, b]$ and $\alpha > 0$. Then we have*

$$\frac{1}{\alpha+1} f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1} f^{\alpha+1}(a) \geq f(b)f^\alpha(a).$$

Proof. The Young product inequality states that, for any $p, q > 1$ such that $1/p + 1/q = 1$ and $x, y \geq 0$, we have

$$\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy. \tag{6}$$

Applying this inequality with $p = \alpha + 1$ and $q = (\alpha + 1)/\alpha$, which satisfy $p > 1$ and $q > 1$ since $\alpha > 0$, $x = f(b)$ and $y = f^\alpha(a)$, we get

$$\frac{1}{\alpha+1} f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1} f^{\alpha+1}(a) \geq f(b)f^\alpha(a).$$

The desired inequality is obtained, ending the proof of Proposition 2.2. □

In the context of Theorem 2.1 under the assumption in equation (1), it follows from Proposition 2.2 that

$$\frac{1}{\alpha+1} f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1} f^{\alpha+1}(a) \geq \max\left[\int_a^b f(t)g(t)\mathrm{d}t, f(b)f^\alpha(a)\right]. \tag{7}$$

Below, a variant of Proposition 2.2 is proposed under an additional convexity assumption on f.

Proposition 2.3. *Let $a, b \in \mathbb{R}$ with $a < b$, f be a positive twice differentiable convex function defined on $[a, b]$ and $\alpha > 0$. Then we have*

$$\frac{1}{\alpha+1} f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1} f^{\alpha+1}(a) \geq f^{\alpha+1}\left(\frac{b+\alpha a}{\alpha+1}\right).$$

Proof. Since $\alpha > 0$, f is positive, and twice differentiable and convex, meaning that $f''(x) \geq 0$ for any $x \in [a, b]$, we have

$$[f^{\alpha+1}(x)]'' = (\alpha+1) f^{\alpha-1}(x) \left[\alpha [f'(x)]^2 + f(x) f''(x)\right] \geq 0.$$

Therefore $f^{\alpha+1}$ is convex. This means, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f^{\alpha+1}[\lambda x + (1-\lambda) y] \leq \lambda f^{\alpha+1}(x) + (1-\lambda) f^{\alpha+1}(y).$$

This applied with $x = a$, $y = b$ and $\lambda = \alpha/(\alpha+1) \in [0, 1]$, gives

$$\begin{aligned} \frac{1}{\alpha+1} f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1} f^{\alpha+1}(a) &\geq f^{\alpha+1}\left(\frac{1}{\alpha+1} b + \frac{\alpha}{\alpha+1} a\right) \\ &= f^{\alpha+1}\left(\frac{b+\alpha a}{\alpha+1}\right). \end{aligned} \tag{8}$$

The stated lower bound is determined, completing the proof of Proposition 2.3. □

Therefore, in the context of Theorem 2.1 under the assumption in equation (1), and under the additional assumption of convexity on f, it follows from Proposition 2.3 that

$$\frac{1}{\alpha+1} f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1} f^{\alpha+1}(a) \geq \max\left[\int_a^b f(t) g(t) \mathrm{d}t, f^{\alpha+1}\left(\frac{b+\alpha a}{\alpha+1}\right)\right].$$

If we take into account equation (7), we also have

$$\frac{1}{\alpha+1} f^{\alpha+1}(b) + \frac{\alpha}{\alpha+1} f^{\alpha+1}(a) \geq \max\left[\int_a^b f(t) g(t) \mathrm{d}t, f^{\alpha+1}\left(\frac{b+\alpha a}{\alpha+1}\right), f(b) f^{\alpha}(a)\right],$$

as sketched in the introduction section.

2.3. Complements

The theorem below generalizes Theorem 2.1 with the use of an additional function h.

Theorem 2.4. *Let $a, b \in \mathbb{R}$ with $a < b$, f be a positive differentiable non-decreasing function defined on $[a, b]$, g be an integrable function defined on $[a, b]$ and h be a differentiable function defined on $[0, +\infty)$. We assume that, for any $x \in [a, b]$, we have*

$$\int_x^b g(t) \mathrm{d}t \leq h'[f(x)]. \tag{9}$$

Then we have

$$\int_a^b f(t) g(t) \mathrm{d}t \leq h[f(b)] - h[f(a)] + f(a) h'[f(a)].$$

If the inequality in equation (9) is reversed, then the final inequality is reversed.

Proof. The proof is similar to that of Theorem 2.1, so we refer to it for the details. For the main line, starting with equation (3) and using equation (9) twice, we get

$$\begin{aligned}\int_a^b f(t)g(t)\mathrm{d}t &= \int_a^b f'(u)\left[\int_u^b g(t)\mathrm{d}t\right]\mathrm{d}u + f(a)\int_a^b g(t)\mathrm{d}t \\ &\leq \int_a^b f'(u)h'[f(u)]\mathrm{d}u + f(a)h'[f(a)] \\ &= \left\{h[f(u)]\right\}_{u=a}^{u=b} + f(a)h'[f(a)] \\ &= h[f(b)] - h[f(a)] + f(a)h'[f(a)].\end{aligned}$$

The desired bound is determined. Clearly, if the inequality in equation (9) is reversed, then the final inequality is also reversed. The details are omitted. □

Theorem 2.4 thus gains in generality but loses in interpretation compared to that of Theorem 2.1, which has a clearer line with simple functions. However, it can be used with h adapted to a particular context and thus help to solve specific problems.

The rest of the article is devoted to new integral results of the second type.

3. New integral results of the second type

3.1. Main result

The theorem below is our main result of the second type. It shows that, under certain monotonicity and primitive-like inequality assumptions, new integral inequalities can be established for $\int_a^b f(t)g(t)\mathrm{d}t$. These inequalities are modified versions of those obtained in Theorem 2.1.

Theorem 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, f be a positive differentiable non-increasing function defined on $[a, b]$ and g be an integrable function defined on $[a, b]$.*

1. *We assume that there exists $\alpha \in \mathbb{R}\backslash\{-1\}$ such that, for any $x \in [a, b]$, we have*

$$\int_a^x g(t)\mathrm{d}t \leq f^{\alpha}(x). \tag{10}$$

Then we have

$$\int_a^b f(t)g(t)\mathrm{d}t \leq \frac{\alpha}{\alpha+1} f^{\alpha+1}(b) + \frac{1}{\alpha+1} f^{\alpha+1}(a).$$

2. *We assume that, for any $x \in [a, b]$, we have*

$$\int_a^x g(t)\mathrm{d}t \leq \frac{1}{f(x)}. \tag{11}$$

Then we have

$$\int_a^b f(t)g(t)\mathrm{d}t \leq 1 + \ln\left[\frac{f(a)}{f(b)}\right].$$

Proof. The two points have the same mathematical basis. Since f is differentiable, for any $t \in [a, b]$, we can write

$$\begin{aligned} f(t) &= f(b) - [f(b) - f(t)] \\ &= f(b) - \int_t^b f'(u)\mathrm{d}u. \end{aligned}$$

With this and a change in the order of integration, which is possible thanks to the Fubini integral theorem, we have

$$\begin{aligned} \int_a^b f(t)g(t)\mathrm{d}t &= \int_a^b \left[f(b) - \int_t^b f'(u)\mathrm{d}u\right] g(t)\mathrm{d}t \\ &= f(b)\int_a^b g(t)\mathrm{d}t - \int_a^b \int_t^b f'(u)g(t)\mathrm{d}u\mathrm{d}t \\ &= f(b)\int_a^b g(t)\mathrm{d}t - \int_a^b \int_a^u f'(u)g(t)\mathrm{d}t\mathrm{d}u \\ &= f(b)\int_a^b g(t)\mathrm{d}t + \int_a^b [-f'(u)]\left[\int_a^u g(t)\mathrm{d}t\right]\mathrm{d}u. \end{aligned} \tag{12}$$

Let us now make a distinction between the assumptions made in points 1 and 2.

1. Using equation (10) twice and the fact that f is positive, differentiable and non-increasing, which means that $f'(u) \leq 0$, so $-f'(u) \geq 0$, for any $u \in [a, b]$, we get

$$\begin{aligned} & f(b)\int_a^b g(t)\mathrm{d}t + \int_a^b [-f'(u)]\left[\int_a^u g(t)\mathrm{d}t\right]\mathrm{d}u \\ &\leq f(b)f^{\alpha}(b) + \int_a^b [-f'(u)]f^{\alpha}(u)\mathrm{d}u \\ &= f^{\alpha+1}(b) - \left[\frac{1}{\alpha+1}f^{\alpha+1}(u)\right]_{u=a}^{u=b} \\ &= f^{\alpha+1}(b) - \frac{1}{\alpha+1}f^{\alpha+1}(b) + \frac{1}{\alpha+1}f^{\alpha+1}(a) \\ &= \frac{\alpha}{\alpha+1}f^{\alpha+1}(b) + \frac{1}{\alpha+1}f^{\alpha+1}(a). \end{aligned} \tag{13}$$

It follows from equations (12) and (13) that

$$\int_a^b f(t)g(t)\mathrm{d}t \leq \frac{\alpha}{\alpha+1}f^{\alpha+1}(b) + \frac{1}{\alpha+1}f^{\alpha+1}(a).$$

This is the claimed inequality.

2. Using equation (11) twice and the fact that f is positive, differentiable and non-decreasing, which means that $f'(u) \leq 0$, so $-f'(u) \geq 0$, for any $u \in [a, b]$, we get

$$\begin{aligned}
&f(b)\int_a^b g(t)\mathrm{d}t + \int_a^b [-f'(u)]\left[\int_a^u g(t)\mathrm{d}t\right]\mathrm{d}u \\
&\leq f(b)\frac{1}{f(b)} + \int_a^b [-f'(u)]\frac{1}{f(u)}\mathrm{d}u \\
&= 1 - \{\ln[f(u)]\}_{u=a}^{u=b} \\
&= 1 - \{\ln[f(b)] - \ln[f(a)]\} \\
&= 1 + \ln\left[\frac{f(a)}{f(b)}\right].
\end{aligned} \tag{14}$$

It follows from equations (12) and (14) that

$$\int_a^b f(t)g(t)\mathrm{d}t \leq 1 + \ln\left[\frac{f(a)}{f(b)}\right].$$

We get the expected upper bound. This ends the proof of Theorem 3.1. □

To the best of our knowledge, the integral inequalities obtained in Theorem 3.1 are new to the literature. The main novelty is a consequence of the primitive inequality assumptions that lead to the original bounds. Of particular interest is the logarithmic bound obtained in point 2. While the results are similar to those in Theorem 2.1, there is a notable difference in the expressions.

Note that, by the Chasles integral relation, the assumption in equation (10) is equivalent to

$$\int_x^b g(t)\mathrm{d}t \geq \int_a^b g(t)\mathrm{d}t - f^{\alpha}(x),$$

which offers another view to the primitive-like inequality assumption.

Also, it is worth mentioning that, in the context of Theorem 3.1, the following assumption is not valuable to replace that in equation (10): there exists $\alpha \in \mathbb{R}$ such that, for any $x \in [a, b]$, we have

$$\int_a^x g(t)\mathrm{d}t \geq f^{\alpha}(x).$$

Just, if we take $x = a$, we find that $0 = \int_a^a g(t)\mathrm{d}t \geq f^{\alpha}(a) \geq 0$, so $f(a) = 0$. Since f is supposed to be non-increasing on $[a, b]$, this is only possible if f is the zero function, which makes an integral inequality based on it uninteresting.

3.2. Secondary results

In this subsection, we derive several results related to Theorem 3.1, focusing on the case $\alpha > 0$.

The proposition below determines a simple product lower bound depending on $f(a)$, $f(b)$ and α for a central term.

Proposition 3.2. *Let $a, b \in \mathbb{R}$ with $a < b$, f be a positive continuous function defined on $[a, b]$ and $\alpha > 0$. Then we have*

$$\frac{\alpha}{\alpha+1} f^{\alpha+1}(b) + \frac{1}{\alpha+1} f^{\alpha+1}(a) \geq f^{\alpha}(b) f a).$$

Proof. The proof is similar to that of Proposition 2.2. The Young product inequality, recalled in equation (6), applied with $p = (\alpha+1)/\alpha$ and $q = \alpha+1$, which satisfy $p > 1$ and $q > 1$ since $\alpha > 0$, $x = f^{\alpha}(b)$ and $y = f(a)$, gives

$$\frac{\alpha}{\alpha+1} f^{\alpha+1}(b) + \frac{1}{\alpha+1} f^{\alpha+1}(a) \geq f^{\alpha}(b) f(a).$$

The expected lower bound is obtained, ending the proof of Proposition 3.2. □

In the context of Theorem 3.1 under the assumption in equation (10), it follows from Proposition 3.2 that

$$\frac{\alpha}{\alpha+1} f^{\alpha+1}(b) + \frac{1}{\alpha+1} f^{\alpha+1}(a) \geq \max\left[\int_a^b f(t) g(t) \mathrm{d}t, f^{\alpha}(b) f(a)\right]. \tag{15}$$

A variant of Proposition 3.2 is presented below, under a convexity assumption on f.

Proposition 3.3. *Let $a, b \in \mathbb{R}$ with $a < b$, f be a positive twice differentiable convex function defined on $[a, b]$ and $\alpha > 0$. Then we have*

$$\frac{\alpha}{\alpha+1} f^{\alpha+1}(b) + \frac{1}{\alpha+1} f^{\alpha+1}(a) \geq f^{\alpha+1}\left(\frac{\alpha b + a}{\alpha+1}\right).$$

Proof. The proof is similar to that of Proposition 2.3. Since $\alpha > 0$, f is positive, and twice differentiable and convex, we have $f''(x) \geq 0$ for any $x \in [a, b]$, so that

$$[f^{\alpha+1}(x)]'' = (\alpha+1) f^{\alpha-1}(x) \left[\alpha [f'(x)]^2 + f(x) f''(x)\right] \geq 0.$$

Therefore $f^{\alpha+1}$ is convex. This means, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f^{\alpha+1}[\lambda x + (1-\lambda) y] \leq \lambda f^{\alpha+1}(x) + (1-\lambda) f^{\alpha+1}(y).$$

This applied with $x = a$ $y = b$ and $\lambda = 1/(\alpha+1) \in [0, 1]$ gives

$$\begin{aligned} \frac{\alpha}{\alpha+1} f^{\alpha+1}(b) + \frac{1}{\alpha+1} f^{\alpha+1}(a) &\geq f^{\alpha+1}\left(\frac{\alpha}{\alpha+1} b + \frac{1}{\alpha+1} a\right) \\ &= f^{\alpha+1}\left(\frac{\alpha b + a}{\alpha+1}\right). \end{aligned}$$

The desired inequality is established, completing the proof of Proposition 3.3. □

Therefore, in the context of Theorem 3.1 under the assumption in equation (10), and under the additional assumption of convexity on f, it follows from Proposition 3.3 that

$$\frac{\alpha}{\alpha+1} f^{\alpha+1}(b) + \frac{1}{\alpha+1} f^{\alpha+1}(a) \geq \max\left[\int_a^b f(t) g(t) \mathrm{d}t, f^{\alpha+1}\left(\frac{\alpha b + a}{\alpha+1}\right)\right].$$

If we take into account equation (15), we also have

$$\frac{\alpha}{\alpha+1} f^{\alpha+1}(b)+\frac{1}{\alpha+1} f^{\alpha+1}(a) \geq \max \left[\int_a^b f(t) g(t) \mathrm{d} t, f^{\alpha+1}\left(\frac{\alpha b+a}{\alpha+1}\right), f^\alpha(b) f(a)\right].$$

Again, we highlight the novelty of these inequalities.

3.3. Complements

A generalization of Theorem 3.1 is proposed below, with the use of an adaptive function h.

Theorem 3.4. *Let $a, b \in \mathbb{R}$ with $a<b$, f be a positive differentiable non-decreasing function defined on $[a, b]$, g be an integrable function defined on $[a, b]$ and h be a differentiable function defined on $[0,+\infty)$. We assume that, for any $x \in[a, b]$, we have*

$$\int_a^x g(t) \mathrm{d} t \leq h^{\prime}[f(x)]. \tag{16}$$

Then we have

$$\int_a^b f(t) g(t) \mathrm{d} t \leq f(b) h^{\prime}[f(b)]-h[f(b)]+h[f(a)].$$

If the inequality in equation (16) is reversed, then the final inequality is reversed.

Proof. The proof is similar to that of Theorem 3.1. Starting with equation (12) and using equation (16) twice, we get

$$\begin{aligned}
\int_a^b f(t) g(t) \mathrm{d} t & =f(b) \int_a^b g(t) \mathrm{d} t+\int_a^b\left[-f^{\prime}(u)\right]\left[\int_a^u g(t) \mathrm{d} t\right] \mathrm{d} u \\
& \leq f(b) h^{\prime}[f(b)]+\int_a^b\left[-f^{\prime}(u)\right] h^{\prime}[f(u)] \mathrm{d} u \\
& =f(b) h^{\prime}[f(b)]-\{h[f(u)]\}_{u=a}^{u=b} \\
& =f(b) h^{\prime}[f(b)]-h[f(b)]+h[f(a)] .
\end{aligned}$$

The bound we want is exhibited. Clearly, if the inequality in equation (16) is reversed, then the final inequality is also reversed. The details are omitted. □

The generality of Theorem 3.4 opens up some perspectives for applications that are beyond the scope of this article.

4. Conclusion

In conclusion, this article contributes to the development of general integral inequalities. The main originality lies in the use of non-standard primitive-like inequality assumptions that lead to new and original results. Other results are proved under additional assumptions, including convexity assumptions. Since the main integral term has the standard form of the product of functions, it can be used in theory and applications dealing with the certain inner product or measures of functions. Furthermore, these results provide a robust framework for analyzing functional relations and estimating integral values in applied mathematics, physics and engineering.

As a perspective, we are thinking of extending the results of this article to the multivariate case, dealing with the following main integral term:

$$\int_a^b \dots \int_a^b f(t_1, \dots, t_n) g_1(t_1) \dots g_n(t_n) \mathrm{d}t_1 \dots \mathrm{d}t_n.$$

In this case, the primitive-like inequality assumptions must be adjusted. The development of such extensions would open up new possibilities for the study of multivariate integral inequalities, which are particularly relevant in areas such as probability theory, optimization and partial differential equations. We leave this challenging work for the future, in the hope that it will inspire further research.

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