SOME INTEGRAL INEQUALITIES FOR CONVEX STOCHASTIC PROCESSES

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Abstract. In this paper, we extend the Hermite-Hadamard-type inequality and Jensen-type inequality for convex stochastic processes. The generalization of Hermite-Hadamard-type inequality for convex stochastic processes is also obtained and some special cases of this result are given.

1. Introduction

The classical Hermite-Hadamard inequality which was first published in [9] gives an estimate of the mean value of a convex function \( f: I \rightarrow \mathbb{R} \),

\[
    f\left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{2} \tag{1.1}
\]

History of this inequality can be found in [8]. Surveys on various generalizations and developments can be found in [1] and [7]. Recently in [3], the author established this inequality for twice differentiable functions. In case \( f \) is convex, there exists an estimation better than (1.1) and that is why the following question was posed.

In [4], Farissi gave the refinement of the inequality (1.1) as follows.

**Theorem 1.** Assume that \( f: I \rightarrow \mathbb{R} \) is a convex function on \( I \). Then for all \( \lambda \in [0, 1] \), we have

\[
f\left( \frac{a + b}{2} \right) \leq l(\lambda) \leq \frac{1}{b - a} \int_a^b f(x)\,dx \leq L(\lambda) \leq \frac{f(a) + f(b)}{2},
\]

where

\[
l(\lambda) := \lambda f\left( \frac{\lambda b + (2 - \lambda)a}{2} \right) + (1 - \lambda) f\left( \frac{(1 + \lambda)b + (1 - \lambda)a}{2} \right)
\]

and

\[
L(\lambda) := \frac{1}{2} (f(\lambda b + (1 - \lambda)a) + \lambda f(a) + (1 - \lambda)f(b)).
\]
In [5], Gao generalized the above inequalities for convex function using the Jensen inequality.


Let \((\Omega, \mathcal{A}, P)\) be an arbitrary probability space. A function \(X: \Omega \to \mathbb{R}\) is called a random variable if it is \(\mathcal{A}\)-measurable. A function \(X: I \times \Omega \to \mathbb{R}\), where \(I \subset \mathbb{R}\) is an interval, is called a stochastic process if for every \(t \in I\), the function \(X(t, \cdot)\) is a random variable.

Recall that the stochastic process \(X: I \times \Omega \to \mathbb{R}\) is called

(i) continuous in probability in interval \(I\) if for all \(t_0 \in I\), we have

\[
P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),
\]

where \(P - \lim\) denotes the limit in probability,

(ii) mean-square continuous in the interval \(I\) if for all \(t_0 \in I\).

\[
\lim_{t \to t_0} E[(X(t) - X(t_0))^2] = 0,
\]

where \(E[X(t)]\) denotes the expectation value of the random variable \(X(t, \cdot)\).

Obviously, mean-square continuity implies continuity in probability, but the converse implication is not true.

**Definition 1.** Suppose a sequence \(\{\Delta^m\}\) of partitions, \(\Delta^m = \{a_{m,0}, \ldots, a_{m,n_m}\}\) is given. We say that the sequence \(\{\Delta^m\}\) is a normal sequence of partitions if the length of the greatest interval in the \(n\)-th partition tends to zero, i.e.,

\[
\lim_{m \to \infty} \sup_{1 \leq i \leq n_m} |a_{m,i} - a_{m,i-1}| = 0.
\]

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties see [11].

Let \(X: I \times \Omega \to \mathbb{R}\) be a stochastic process with \(E[X(t)^2] < \infty\) for all \(t \in I\). Let \([a, b] \subset I\), \(a = t_0 < t_1 < t_2 < \cdots < t_n = b\) be a partition of \([a, b]\) and \(\Theta_k \in [t_{k-1}, t_k]\) for all \(k = 1, \ldots, n\). A random variable \(Y: \Omega \to \mathbb{R}\) is called the mean-square integral of the process \(X\) on \([a, b]\) if we have

\[
\lim_{n \to \infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0
\]

for all normal sequence of partitions of the interval \([a, b]\) and for all \(\Theta_k \in [t_{k-1}, t_k]\), \(k = 1, \ldots, n\). Then, we write

\[
Y(\cdot) = \int_{a}^{b} X(s, \cdot)ds \quad (a.e.).
\]

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process \(X\).
Throughout the paper, we frequently use the monotonicity of the mean-square integral. If \( X(t, \cdot) \leq Y(t, \cdot) \) (a.e.) in an interval \([a, b]\), then
\[
\int_a^b X(t, \cdot) dt \leq \int_a^b Y(t, \cdot) dt \quad \text{(a.e.)}
\]
Of course, this inequality is the immediate consequence of the definition of the mean-square integral.

We say that a stochastic process \( X: I \times \Omega \rightarrow \mathbb{R} \) is convex if for all \( \lambda \in [0, 1] \) and \( u, v \in I \), the inequality
\[
X(\lambda u + (1 - \lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) \quad \text{(a.e.)} \quad (1.2)
\]
is satisfied. If the above inequality is assumed only for \( \lambda = \frac{1}{2} \), then the process \( X \) is Jensen-convex or \( \frac{1}{2} \)-convex. A stochastic process \( X \) is concave if \( (-X) \) is convex.

Some interesting properties of convex and Jensen-convex processes are presented in \([2, 11]\).

Now, we present some results proved by Kotrys \([6]\) about Hermite-Hadamard inequality for convex stochastic processes.

**Lemma 1.** If \( X: I \times \Omega \rightarrow \mathbb{R} \) is a stochastic process of the form \( X(t, \cdot) = A(\cdot) t + B(\cdot) \), where \( A, B: \Omega \rightarrow \mathbb{R} \) are random variables, such that \( E[A^2] < \infty \), \( E[B^2] < \infty \) and \([a, b] \subset I\), then
\[
\int_a^b X(t, \cdot) dt = A(\cdot) \frac{b^2 - a^2}{2} + B(\cdot)(b - a) \quad \text{(a.e.)}
\]

**Proposition 1.** Let \( X: I \times \Omega \rightarrow \mathbb{R} \) be a convex stochastic process and \( t_0 \in \text{int} \, I \). Then there exists a random variable \( A: \Omega \rightarrow \mathbb{R} \) such that \( X \) is supported at \( t_0 \) by the process \( A(\cdot)(t - t_0) + X(t_0, \cdot) \). That is,
\[
X(t, \cdot) \geq A(\cdot)(t - t_0) + X(t_0, \cdot) \quad \text{(a.e.)}
\]
for all \( t \in I \).

**Theorem 2.** Let \( X: I \times \Omega \rightarrow \mathbb{R} \) be Jensen-convex stochastic process, mean-square continuous in the interval \( I \). Then for any \( u, v \in I \), we have
\[
X\left(\frac{u + v}{2}, \cdot\right) \leq \frac{1}{v - u} \int_u^v X(t, \cdot) dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad \text{(a.e.)} \quad (1.3)
\]

The aim of this paper is to extend the classical Hermite-Hadamard-type and Jensen inequalities to convex stochastic processes.

2. Main Results

**Theorem 3 (Hermite-Hadamard-type inequality).** Let \( X: I \times \Omega \rightarrow \mathbb{R} \) be Jensen-convex stochastic process, mean-square continuous in the interval \( I \). Then for any
\( u, v \in I \) and for all \( \lambda \in [0, 1] \), we have
\[
X \left( \frac{u + v}{2}, \cdot \right) \leq h(\lambda) \leq \frac{1}{v - u} \int_{u}^{v} X(t, \cdot) dt \leq H(\lambda) \leq \frac{X(u, \cdot) + X(v, \cdot)}{2},
\]
(2.1)

where
\[
h(\lambda) := \lambda X \left( \frac{\lambda v + (2 - \lambda)u}{2}, \cdot \right) + (1 - \lambda) X \left( \frac{(1 + \lambda)v + (1 - \lambda)u}{2}, \cdot \right)
\]
and
\[
H(\lambda) := \frac{1}{2} \left( X(\lambda v + (1 - \lambda)u, \cdot) + \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) \right).
\]

Proof. We give two different proofs for the theorem.

Firstly, let \( X \) be a Jensen-convex stochastic process, mean-square continuous in the interval \( I \). Applying (1.3) on the subinterval \([u, \lambda v + (1 - \lambda)u]\) with \( \lambda \neq 0 \), we get
\[
X \left( \frac{\lambda v + (2 - \lambda)u}{2}, \cdot \right) \leq \frac{1}{\lambda(v - u)} \int_{u}^{v} X(t, \cdot) dt \leq \frac{X(u, \cdot) + X(\lambda v + (1 - \lambda)u, \cdot)}{2}.
\]
(2.2)

Applying (1.3) on \([\lambda v + (1 - \lambda)u, v]\) with \( \lambda \neq 1 \) again, we get
\[
X \left( \frac{(1 + \lambda)v + (1 - \lambda)u}{2}, \cdot \right) \leq \frac{1}{(1 - \lambda)(v - u)} \int_{u}^{v} X(t, \cdot) dt \leq \frac{X(\lambda v + (1 - \lambda)u, \cdot) + X(v, \cdot)}{2}.
\]
(2.3)

Multiplying (2.2) by \( \lambda \), (2.3) by \( (1 - \lambda) \), and adding the resulting inequalities, we get
\[
h(\lambda) \leq \frac{1}{v - u} \int_{u}^{v} X(t, \cdot) dt \leq H(\lambda),
\]
(2.4)

where \( h(\lambda) \) and \( H(\lambda) \) are defined as in Theorem 3.

In [2], Nikodem proved that every Jensen-convex stochastic process and continuous in probability is convex. Using this fact, we obtain
\[
X \left( \frac{u + v}{2}, \cdot \right) = X \left( \frac{\lambda v + (2 - \lambda)u}{2}, \cdot \right) + (1 - \lambda) X \left( \frac{(1 + \lambda)v + (1 - \lambda)u}{2}, \cdot \right)
\]
\[
\leq \lambda X \left( \frac{\lambda v + (2 - \lambda)u}{2}, \cdot \right) + (1 - \lambda) X \left( \frac{(1 + \lambda)v + (1 - \lambda)u}{2}, \cdot \right)
\]
\[
\leq \frac{1}{2} \left( X(\lambda v + (2 - \lambda)u, \cdot) + \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) \right) \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}.
\]
(2.5)
Then by (2.4) and (2.5), we get (2.1).

Secondly, since the process $X$ is mean-square continuous, it is continuous in probability. Nikodem [2] proved that every Jensen-convex probability stochastic continuous in process is convex. Since $X$ is convex, then by Proposition 1, it has a supporting process at any point $t_0 \in \text{int} \ I$. Let us take a support at $t_0 = \frac{u + v}{2}$, then we have

$$X(t, \cdot) \geq A(\cdot) \left( t - \frac{u + v}{2} \right) + X \left( \frac{u + v}{2}, \cdot \right) \quad (a.e.).$$

Using Lemma 1 on the subinterval $[u, \lambda v + (1 - \lambda)u]$ with $\lambda \neq 0$, we can write

$$\int_u^x X(t, \cdot) dt \geq \int_u^{\lambda v + (1 - \lambda)u} \left[ A(\cdot) \left( t - \frac{u + \lambda v + (1 - \lambda)u}{2} \right) + X \left( \frac{u + \lambda v + (1 - \lambda)u}{2}, \cdot \right) \right] dt$$

$$= \lambda X \left( \frac{u + \lambda v + (1 - \lambda)u}{2}, \cdot \right) (v - u) \quad (a.e.),$$

and again on $[\lambda v + (1 - \lambda)u, v]$ with $\lambda \neq 1$, we get

$$\int_{\lambda v + (1 - \lambda)u}^v X(t, \cdot) dt \geq \int_{\lambda v + (1 - \lambda)u}^v \left[ A(\cdot) \left( t - \frac{v + \lambda v + (1 - \lambda)u}{2} \right) + X \left( \frac{v + \lambda v + (1 - \lambda)u}{2}, \cdot \right) \right] dt$$

$$= (1 - \lambda) X \left( \frac{v + \lambda v + (1 - \lambda)u}{2}, \cdot \right) (v - u) \quad (a.e.).$$

From (2.6) and (2.7), we obtain

$$\frac{1}{v - u} \int_u^x X(t, \cdot) dt \geq \lambda X \left( \frac{\lambda v + (2 - \lambda)u}{2}, \cdot \right) + (1 - \lambda) X \left( \frac{(1 + \lambda)v + (1 - \lambda)u}{2}, \cdot \right)$$

$$\geq X \left( \frac{u + v}{2}, \cdot \right).$$

It finishes the proof of first two inequalities in (2.1).

If we take $t = \lambda u + (1 - \lambda)v$ in inequality (1.2), then $\lambda = \frac{t - u}{u - v}$ and by the convexity of $X$, we have

$$X(t, \cdot) \leq \frac{t - v}{u - v} X(u, \cdot) + \frac{u - t}{u - v} X(v, \cdot)$$

$$= \frac{X(v, \cdot) - X(u, \cdot)}{v - u} t + \frac{X(v, \cdot)u - X(u, \cdot)v}{u - v} \quad (a.e.).$$
As before, using Lemma 1 on the subinterval \([u, \lambda v + (1 - \lambda)u]\) with \(\lambda \neq 0\), we can write

\[
\int_{u}^{\lambda v + (1 - \lambda)u} X(t, \cdot) \, dt \leq \int_{u}^{\lambda v + (1 - \lambda)u} \left[ \frac{X(\lambda v + (1 - \lambda)u, \cdot) - X(u, \cdot)}{\lambda(v - u)} \right] \, dt
\]

\[
+ \frac{X(\lambda v + (1 - \lambda)u, \cdot) u - X(u, \cdot) v}{\lambda(v - u)} \right] \, dt
\]

\[
= \left[ X(\lambda v + (1 - \lambda)u, \cdot) - X(u, \cdot) \right] \left( \frac{\lambda v + (2 - \lambda)u}{2} \right)
\]

\[
- X(\lambda v + (1 - \lambda)u, \cdot) u + X(u, \cdot) (\lambda v + (1 - \lambda)u)
\]

and again on \([\lambda v + (1 - \lambda)u, v]\) with \(\lambda \neq 1\), we get

\[
\int_{\lambda v + (1 - \lambda)u}^{v} X(t, \cdot) \, dt \leq \int_{\lambda v + (1 - \lambda)u}^{v} \left[ \frac{X(v, \cdot) - X(\lambda v + (1 - \lambda)u, \cdot)}{(1 - \lambda)(v - u)} \right] \, dt
\]

\[
+ \frac{X(\lambda v + (1 - \lambda)u, \cdot) v - X(v, \cdot) (\lambda v + (1 - \lambda)u)}{(1 - \lambda)(v - u)} \right] \, dt
\]

\[
= \left[ X(v, \cdot) - X(\lambda v + (1 - \lambda)u, \cdot) \right] \left( \frac{1 + \lambda} {2} \right) v + (1 - \lambda)u
\]

\[
+ X(\lambda v + (1 - \lambda)u, \cdot) v - X(v, \cdot) (\lambda v + (1 - \lambda)u).
\]

From (2.8) and (2.9), we obtain

\[
\frac{1}{v - u} \int_{u}^{v} X(t, \cdot) \, dt \leq \frac{1}{2} (X(\lambda v + (1 - \lambda)u, \cdot) + \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot))
\]

\[
\leq \frac{X(u, \cdot) + X(v, \cdot)}{2},
\]

which completes the proof. \(\square\)

Now, we start the following important inequality for convex stochastic processes

**Theorem 4** (Jensen-type inequality). If \(X: I \times \Omega \to \mathbb{R}\) is a convex stochastic process, then we have

\[
X\left( \frac{1}{b - a} \int_{a}^{b} \varphi(t, \cdot) \, dt, \cdot \right) \leq \frac{1}{b - a} \int_{a}^{b} X \circ \varphi(t, \cdot) \, dt
\]

for an arbitrary non-negative integrable stochastic process \(\varphi: I \times \Omega \to I \subset \mathbb{R}\).
Proof. From Proposition 1, we have

\[
\frac{1}{b-a} \int_a^b X \circ \varphi(t, \cdot)dt - X\left(\frac{1}{b-a} \int_a^b \varphi(t, \cdot)dt, \cdot\right) \\
= \frac{1}{b-a} \int_a^b \left[ X \circ \varphi(t, \cdot) - X\left(\frac{1}{b-a} \int_a^b \varphi(t, \cdot)dt, \cdot\right) \right] dt \\
\geq A(\cdot)\left\{ \frac{1}{b-a} \int_a^b \varphi(t, \cdot) dt - \frac{1}{b-a} \int_a^b \varphi(t, \cdot) dt \right\} \\
= A(\cdot)\left\{ \frac{1}{b-a} \int_a^b \varphi(t, \cdot) dt - \frac{1}{b-a} \int_a^b \varphi(t, \cdot) dt \right\} \\
= 0,
\]

which completes the proof. \(\Box\)

Theorem 5 (Generalized Hermite–Hadamard-type inequality). Let \(X: I \times \Omega \to \mathbb{R}\) be a convex stochastic process. If the non-negative integrable stochastic process \(\varphi: I \times \Omega \to I \subset \mathbb{R}\) such that \(X \circ \varphi(t, \cdot)\) is also convex stochastic process, then we have

\[
X\left(\frac{1}{b-a} \int_a^b \varphi(t, \cdot)dt, \cdot\right) \leq h(\lambda_1, \ldots, \lambda_n) \leq \frac{1}{b-a} \int_a^b X \circ \varphi(t, \cdot)dt \\
\leq H(\lambda_1, \ldots, \lambda_n) \leq X \circ \varphi(a, \cdot) + X \circ \varphi(b, \cdot)
\]

for \(n \in \mathbb{N}, \lambda_0 = 0, \lambda_{n+1} = 1\) and arbitrary \(0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq 1\), where

\[
h(\lambda_1, \ldots, \lambda_n) = \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) X\left(\frac{1}{(1-\lambda_{k+1})a+\lambda_{k+1}b} \int_{(1-\lambda_k)a+\lambda_kb}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} \varphi(t, \cdot)dt, \cdot\right),
\]

\[
H(\lambda_1, \ldots, \lambda_n) = \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) \left( \frac{X \circ \varphi((1-\lambda_k)a + \lambda_kb, \cdot)}{2} \right. \\
\left. + \frac{X \circ \varphi((1-\lambda_{k+1})a + \lambda_{k+1}b, \cdot)}{2} \right).
\]

Proof. From Theorem 4 and convexity of \(X \circ \varphi(t, \cdot)\), we have

\[
X\left(\frac{1}{b-a} \int_a^b \varphi(t, \cdot)dt, \cdot\right) \leq \frac{1}{b-a} \int_a^b X \circ \varphi(t, \cdot)dt \\
\leq \frac{X \circ \varphi(a, \cdot) + X \circ \varphi(b, \cdot)}{2}.
\]
Applying (2.10) to \([(1 - \lambda_k)a + \lambda_kb, (1 - \lambda_{k+1})a + \lambda_{k+1}b]\), \(k = 0, 1, \ldots, n\), we have

\[
X\left(\frac{1}{(\lambda_{k+1} - \lambda_k)(b - a)} \int_{(1 - \lambda_k)a + \lambda_kb}^{(1 - \lambda_{k+1})a + \lambda_{k+1}b} \varphi(t, \cdot)dt, \cdot \right)
\]

(2.11)

\[
\leq \frac{1}{(\lambda_{k+1} - \lambda_k)(b - a)} \int_{(1 - \lambda_k)a + \lambda_kb}^{(1 - \lambda_{k+1})a + \lambda_{k+1}b} X \circ \varphi(t, \cdot)dt
\]

\[
\leq \frac{X \circ \varphi((1 - \lambda_k)a + \lambda_kb, \cdot) + X \circ \varphi((1 - \lambda_{k+1})a + \lambda_{k+1}b, \cdot)}{2}.
\]

After multiplying each term in (2.11) by \((\lambda_{k+1} - \lambda_k)\) and later summing the result over \(k\) from 0 to \(n\), we have

\[
\sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) X\left(\frac{1}{(\lambda_{k+1} - \lambda_k)(b - a)} \int_{(1 - \lambda_k)a + \lambda_kb}^{(1 - \lambda_{k+1})a + \lambda_{k+1}b} \varphi(t, \cdot)dt, \cdot \right)
\]

\[
\leq \frac{1}{b - a} \sum_{k=0}^{n} \frac{(1 - \lambda_{k+1})a + \lambda_{k+1}b}{(1 - \lambda_k)a + \lambda_kb} \int_{(1 - \lambda_k)a + \lambda_kb}^{(1 - \lambda_{k+1})a + \lambda_{k+1}b} X \circ \varphi(t, \cdot)dt
\]

\[
\leq \frac{\sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) X \circ \varphi((1 - \lambda_k)a + \lambda_kb, \cdot) + X \circ \varphi((1 - \lambda_{k+1})a + \lambda_{k+1}b, \cdot)}{2}.
\]

That is,

\[
h(\lambda_1, \ldots, \lambda_n) \leq \frac{1}{b - a} \sum_{k=0}^{n} \frac{(1 - \lambda_{k+1})a + \lambda_{k+1}b}{(1 - \lambda_k)a + \lambda_kb} \int_{(1 - \lambda_k)a + \lambda_kb}^{(1 - \lambda_{k+1})a + \lambda_{k+1}b} X \circ \varphi(t, \cdot)dt \leq H(\lambda_1, \ldots, \lambda_n).
\]

Using the convexity of \(X(t, \cdot)\) and \(X \circ \varphi(t, \cdot)\) for \(\sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) = 1\), we get

\[
X\left(\frac{1}{b - a} \int_{a}^{b} \varphi(t, \cdot)dt, \cdot \right)
\]

\[
= X\left(\sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) \frac{1}{(\lambda_{k+1} - \lambda_k)(b - a)} \int_{(1 - \lambda_k)a + \lambda_kb}^{(1 - \lambda_{k+1})a + \lambda_{k+1}b} \varphi(t, \cdot)dt, \cdot \right)
\]

\[
\leq \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) X\left(\frac{1}{(\lambda_{k+1} - \lambda_k)(b - a)} \int_{(1 - \lambda_k)a + \lambda_kb}^{(1 - \lambda_{k+1})a + \lambda_{k+1}b} \varphi(t, \cdot)dt, \cdot \right)
\]
\begin{align*}
&\leq \sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k)X \circ \varphi((1 - \lambda_k)a + \lambda_k b, \cdot)\frac{X \circ \varphi((1 - \lambda_{k+1})a + \lambda_{k+1} b, \cdot)}{2} \\
&\leq \sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k)\frac{\lambda_k X \circ \varphi(a, \cdot) + \lambda_k X \circ \varphi(b, \cdot)}{2} \\
&\quad + \frac{(1 - \lambda_{k+1})X \circ \varphi(a, \cdot) + \lambda_{k+1} X \circ \varphi(b, \cdot)}{2} \\
&= \frac{1}{2} \sum_{k=0}^{n}((1 - \lambda_k) - (1 - \lambda_{k+1}))((1 - \lambda_k) + (1 - \lambda_{k+1}))X \circ \varphi(a, \cdot) \\
&\quad + \frac{1}{2} \sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k)(\lambda_{k+1} + \lambda_k)X \circ \varphi(b, \cdot) \\
&= \frac{1}{2} \sum_{k=0}^{n}((1 - \lambda_k)^2 - (1 - \lambda_{k+1})^2)X \circ \varphi(a, \cdot) + \frac{1}{2} \sum_{k=0}^{n}(\lambda_{k+1}^2 - \lambda_k^2)X \circ \varphi(b, \cdot) \\
&= \frac{X \circ \varphi(a, \cdot) + X \circ \varphi(b, \cdot)}{2}.
\end{align*}

This completes the proof. \hfill \Box

**Corollary 1.** Under assumption of Theorem 5 with $\lambda_k = \lambda$ for $k = 1, 2, \ldots, n$, we have the inequality

$$X\left(\frac{1}{b-a} \int_a^b \varphi(t, \cdot)dt, \cdot\right) \leq h_1(\lambda) \leq \frac{1}{b-a} \int_a^b X \circ \varphi(t, \cdot)dt \\
\leq H_1(\lambda) \leq \frac{X \circ \varphi(a, \cdot) + X \circ \varphi(b, \cdot)}{2},$$

where

$$h_1(\lambda) = \lambda X\left(\frac{1}{\lambda(b-a)} \int_a^b \varphi(t, \cdot)dt, \cdot\right) \\
+ (1 - \lambda)X\left(\frac{1}{(1 - \lambda)(b-a)} \int_a^b \varphi(t, \cdot)dt, \cdot\right)$$

$$H_1(\lambda) = \frac{1}{2} (X(\lambda a + (1 - \lambda)b, \cdot) + \lambda X(a, \cdot) + (1 - \lambda)X(b, \cdot)).$$

**Remark 1.** If we choose $\varphi(t, \cdot) = t$ in Corollary 1, then inequality (2.12) reduces inequality (2.1).
References


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