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SOME INTEGRAL INEQUALITIES FOR CONVEX STOCHASTIC PROCESSES

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Abstract. In this paper, we extend the Hermite-Hadamard-type inequality and Jensen-type inequality for convex stochastic proceses. The generalization of Hermite-Hadamard-type inequality for convex stochastic proceses is also obtained and some special cases of this result are given.

1. Introduction

The classical Hermite-Hadamard inequality which was first published in [**9**] gives an estimate of the mean value of a convex function $f\colon I \to \mathbb{R}$,

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

History of this inequality can be found in [**8**]. Surveys on various generalizations and developments can be found in [**1**] and [**7**]. Recently in [**3**], the author established this inequality for twice differentiable functions. In case f is convex, there exists an estimation better than (1.1) and that is why the following question was posed.

In [**4**], Farissi gave the refinement of the inequality (1.1) as follows.

Theorem 1. *Assume that $f\colon I \to \mathbb{R}$ is a convex function on I. Then for all $\lambda \in [0,1]$, we have*

$$f\Big(\frac{a+b}{2}\Big) \le l(\lambda) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le L(\lambda) \le \frac{f(a)+f(b)}{2},$$

where

$$l(\lambda) := \lambda f\Big(\frac{\lambda b + (2-\lambda)a}{2}\Big) + (1-\lambda) f\Big(\frac{(1+\lambda)b + (1-\lambda)a}{2}\Big)$$

and

$$L(\lambda) := \frac{1}{2}(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)).$$

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In [**5**], Gao generalized the above inequalities for convex function using the Jensen inequality.

In 1980, Nikodem [**2**] introduced convex stochastic processes and investigated their regularity properties. In 1992, Skwronski [**10**] obtained some further results on convex functions.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X\colon \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. A function $X\colon I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$, the function $X(t, \cdot)$ is a random variable.

Recall that the stochastic process $X\colon I \times \Omega \to \mathbb{R}$ is called

(i) continuous in probability in interval I if for all $t_0 \in I$, we have

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P - \lim$ denotes the limit in probability,

(ii) *mean-square continuous* in the interval I if for all $t_0 \in I$.

$$\lim_{t \to t_0} E\left[(X(t) - X(t_0))^2\right] = 0,$$

where $E[X(t)]$ denotes the expectation value of the random variable $X(t, \cdot)$.

Obviously, *mean-square* continuity implies continuity in probability, but the converse implication is not true.

Definition 1. Suppose a sequence $\{\Delta^m\}$ of partitions, $\Delta^m = \{a_{m,0}, \dots, a_{m,n_m}\}$ is given. We say that the sequence $\{\Delta^m\}$ is a normal sequence of partitions if the length of the greatest interval in the n-th partition tends to zero, i.e.,

$$\lim_{m \to \infty} \sup_{1 \le i \le n_m} |a_{m,i} - a_{m,i-1}| = 0.$$

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties see [**11**].

Let $X\colon I \times \Omega \to \mathbb{R}$ be a stochastic process with $E\left[X(t)^2\right] < \infty$ for all $t \in I$. Let $[a, b] \subset I$, $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, \dots, n$. A random variable $Y\colon \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a, b]$ if we have

$$\lim_{n \to \infty} E\Big[\big(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\big)^2\Big] = 0$$

for all normal sequence of partitions of the interval $[a, b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, \dots, n$. Then, we write

$$Y(\cdot) = \int_a^b X(s, \cdot)\mathrm{d}s \qquad \text{(a.e.)}.$$

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process X.

Throughout the paper, we frequently use the monotonicity of the mean-square integral. If $X(t,\cdot) \le Y(t,\cdot)$ (a.e.) in an interval $[a,b]$, then

$$\int_a^b X(t,\cdot)\mathrm{d}t \le \int_a^b Y(t,\cdot)\mathrm{d}t \qquad \text{(a.e.)}.$$

Of course, this inequality is the immediate consequence of the definition of the mean-square integral.

We say that a stochastic process $X\colon I \times \Omega \to \mathbb{R}$ is convex if for all $\lambda \in [0,1]$ and $u, v \in I$, the inequality

$$X(\lambda u + (1-\lambda)v, \cdot) \le \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot) \quad \text{(a.e.)} \tag{1.2}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process X is Jensen-convex or $\frac{1}{2}$-convex. A stochastic process X is concave if $(-X)$ is convex. Some interesting properties of convex and Jensen-convex processes are presented in [**2, 11**].

Now, we present some results proved by Kotrys [**6**] about Hermite-Hadamard inequality for convex stochastic processes.

Lemma 1. *If $X\colon I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t,\cdot) = A(\cdot)t + B(\cdot)$, where $A, B\colon \Omega \to \mathbb{R}$ are random variables, such that $E\left[A^2\right] < \infty$, $E\left[B^2\right] < \infty$ and $[a,b] \subset I$, then*

$$\int_a^b X(t,\cdot)\mathrm{d}t = A(\cdot)\frac{b^2 - a^2}{2} + B(\cdot)(b-a) \qquad \text{(a.e.)}.$$

Proposition 1. *Let $X\colon I \times \Omega \to \mathbb{R}$ be a convex stochastic process and $t_0 \in \operatorname{int} I$. Then there exists a random variable $A\colon \Omega \to \mathbb{R}$ such that X is supported at t_0 by the process $A(\cdot)(t - t_0) + X(t_0,\cdot)$. That is,*

$$X(t,\cdot) \ge A(\cdot)(t-t_0) + X(t_0,\cdot) \qquad \text{(a.e.)}$$

for all $t \in I$.

Theorem 2. *Let $X\colon I \times \Omega \to \mathbb{R}$ be Jensen-convex stochastic process, mean-square continuous in the interval I. Then for any $u, v \in I$, we have*

$$X\Big(\frac{u+v}{2},\cdot\Big) \le \frac{1}{v-u}\int_u^v X(t,\cdot)\mathrm{d}t \le \frac{X(u,\cdot) + X(v,\cdot)}{2} \quad \text{(a.e.)}. \tag{1.3}$$

The aim of this paper is to extend the classical Hermite-Hadamard-type and Jensen inequalities to convex stochastic processes.

2. Main Results

Theorem 3 (Hermite-Hadamard-type inequality)**.** *Let $X\colon I \times \Omega \to \mathbb{R}$ be Jensen-convex stochastic process, mean-square continuous in the interval I. Then for any*

$u, v \in I$ and for all $\lambda \in [0,1]$, we have

$$X\Big(\frac{u+v}{2},\cdot\Big) \le h(\lambda) \le \frac{1}{v-u}\int_u^v X(t,\cdot)\mathrm{d}t \le H(\lambda) \le \frac{X(u,\cdot)+X(v,\cdot)}{2}, \tag{2.1}$$

where

$$h(\lambda) := \lambda X\Big(\frac{\lambda v+(2-\lambda)u}{2},\cdot\Big) + (1-\lambda)X\Big(\frac{(1+\lambda)v+(1-\lambda)u}{2},\cdot\Big)$$

and

$$H(\lambda) := \frac{1}{2}(X(\lambda v+(1-\lambda)u,\cdot) + \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot)).$$

Proof. We give two different proofs for the theorem.

Firstly, let X be a Jensen-convex stochastic process, mean-square continuous in the interval I. Applying (1.3) on the subinterval $[u, \lambda v+(1-\lambda)u]$ with $\lambda \neq 0$, we get

$$X\Big(\frac{\lambda v+(2-\lambda)u}{2},\cdot\Big) \le \frac{1}{\lambda(v-u)}\int_u^{\lambda v+(1-\lambda)u} X(t,\cdot)\mathrm{d}t \le \frac{X(u,\cdot)+X(\lambda v+(1-\lambda)u,\cdot)}{2}. \tag{2.2}$$

Applying (1.3) on $[\lambda v+(1-\lambda)u, v]$ with $\lambda \neq 1$ again, we get

$$X\Big(\frac{(1+\lambda)v+(1-\lambda)u}{2},\cdot\Big) \le \frac{1}{(1-\lambda)(v-u)}\int_{\lambda v+(1-\lambda)u}^{v} X(t,\cdot)\mathrm{d}t \le \frac{X(\lambda v+(1-\lambda)u,\cdot)+X(v,\cdot)}{2}. \tag{2.3}$$

Multiplying (2.2) by λ, (2.3) by $(1-\lambda)$, and adding the resulting inequalities, we get

$$h(\lambda) \le \frac{1}{v-u}\int_u^v X(t,\cdot)\mathrm{d}t \le H(\lambda), \tag{2.4}$$

where $h(\lambda)$ and $H(\lambda)$ are defined as in Theorem 3.

In [**2**], Nikodem proved that every Jensen-convex stochastic process and continuous in probability is convex. Using this fact, we obtain

$$\begin{aligned} X\Big(\frac{u+v}{2},\cdot\Big) &= X\Big(\lambda\frac{\lambda v+(2-\lambda)u}{2} + (1-\lambda)\frac{(1+\lambda)v+(1-\lambda)u}{2},\cdot\Big) \\ &\le \lambda X\Big(\frac{\lambda v+(2-\lambda)u}{2},\cdot\Big) + (1-\lambda)X\Big(\frac{(1+\lambda)v+(1-\lambda)u}{2},\cdot\Big) \\ &\le \frac{1}{2}(X(\lambda v+(2-\lambda)u,\cdot) + \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot)) \le \frac{X(u,\cdot)+X(v,\cdot)}{2}. \end{aligned} \tag{2.5}$$

Then by (2.4) and (2.5), we get (2.1).

Secondly, since the process X is mean-square continuous, it is continuous in probability. Nikodem [**2**] proved that every Jensen-convex probability stochastic continuous in process is convex. Since X is convex, then by Proposition 1, it has a supporting process at any point $t_0 \in \operatorname{int} I$. Let us take a support at $t_0 = \frac{u+v}{2}$, then we have

$$X(t,\cdot) \geq A(\cdot)\Big(t - \frac{u+v}{2}\Big) + X\Big(\frac{u+v}{2},\cdot\Big) \qquad \text{(a.e.)}.$$

Using Lemma 1 on the subinterval $[u, \lambda v + (1-\lambda)u]$ with $\lambda \neq 0$, we can write

$$\begin{aligned} &\int\limits_u^{\lambda v+(1-\lambda)u} X(t,\cdot)\mathrm{d}t \\ (2.6)\qquad &\geq \int\limits_u^{\lambda v+(1-\lambda)u} \Big[A\Big(\cdot)(t - \frac{u+\lambda v+(1-\lambda)u}{2}\Big) + X\Big(\frac{u+\lambda v+(1-\lambda)u}{2},\cdot\Big)\Big]\mathrm{d}t \\ &= \lambda X\Big(\frac{u+\lambda v+(1-\lambda)u}{2},\cdot\Big)(v-u) \qquad \text{(a.e.)}, \end{aligned}$$

and again on $[\lambda v + (1-\lambda)u, v]$ with $\lambda \neq 1$, we get

$$\begin{aligned} &\int\limits_{\lambda v+(1-\lambda)u}^{v} X(t,\cdot)\mathrm{d}t \\ (2.7)\qquad &\geq \int\limits_{\lambda v+(1-\lambda)u}^{v} \Big[A(\cdot)\Big(t - \frac{v+\lambda v+(1-\lambda)u}{2}\Big) + X\Big(\frac{v+\lambda v+(1-\lambda)u}{2},\cdot\Big)\Big]\mathrm{d}t \\ &= (1-\lambda)X\Big(\frac{v+\lambda v+(1-\lambda)u}{2},\cdot\Big)(v-u) \qquad \text{(a.e.)}. \end{aligned}$$

From (2.6) and (2.7), we obtain

$$\begin{aligned} \frac{1}{v-u}\int\limits_u^v X(t,\cdot)\mathrm{d}t &\geq \lambda X\Big(\frac{\lambda v+(2-\lambda)u}{2},\cdot\Big) + (1-\lambda)X\Big(\frac{(1+\lambda)v+(1-\lambda)u}{2},\cdot\Big) \\ &\geq X\Big(\frac{u+v}{2},\cdot\Big). \end{aligned}$$

It finishes the proof of first two inequalities in (2.1).

If we take $t = \lambda u + (1-\lambda)v$ in inequality (1.2), then $\lambda = \frac{t-v}{u-v}$ and by the convexity of X, we have

$$\begin{aligned} X(t,\cdot) &\leq \frac{t-v}{u-v}X(u,\cdot) + \frac{u-t}{u-v}X(v,\cdot) \\ &= \frac{X(v,\cdot)-X(u,\cdot)}{v-u}t + \frac{X(v,\cdot)u - X(u,\cdot)v}{u-v} \qquad \text{(a.e.)}. \end{aligned}$$

As before, using Lemma 1 on the subinterval $[u, \lambda v + (1-\lambda)u]$ with $\lambda \neq 0$, we can write

$$\begin{aligned} \int_u^{\lambda v+(1-\lambda)u} X(t,\cdot)\mathrm{d}t \leq & \int_u^{\lambda v+(1-\lambda)u} \Big[\frac{X(\lambda v+(1-\lambda)u,\cdot)-X(u,\cdot)}{\lambda(v-u)}t \\ & + \frac{X(\lambda v+(1-\lambda)u,\cdot)u - X(u,\cdot)v}{\lambda(v-u)}\Big]\mathrm{d}t \qquad (2.8) \\ = & \ [X(\lambda v+(1-\lambda)u,\cdot)-X(u,\cdot)]\Big(\frac{\lambda v+(2-\lambda)u}{2}\Big) \\ & - X(\lambda v+(1-\lambda)u,\cdot)u + X(u,\cdot)(\lambda v+(1-\lambda)u) \end{aligned}$$

and again on $[\lambda v + (1-\lambda)u, v]$ with $\lambda \neq 1$, we get

$$\begin{aligned} \int_{\lambda v+(1-\lambda)u}^{v} X(t,\cdot)\mathrm{d}t \leq & \int_{\lambda v+(1-\lambda)u}^{v} \Big[\frac{X(v,\cdot)-X(\lambda v+(1-\lambda)u,\cdot)}{(1-\lambda)(v-u)}t \\ & + \frac{X(\lambda v+(1-\lambda)u,\cdot)v - X(v,\cdot)(\lambda v+(1-\lambda)u)}{(1-\lambda)(v-u)}\Big]\mathrm{d}t \qquad (2.9) \\ = & \ [X(v,\cdot)-X(\lambda v+(1-\lambda)u,\cdot)]\Big(\frac{(1+\lambda)v+(1-\lambda)u}{2}\Big) \\ & + X(\lambda v+(1-\lambda)u,\cdot)v - X(v,\cdot)(\lambda v+(1-\lambda)u). \end{aligned}$$

From (2.8) and (2.9), we obtain

$$\begin{aligned} \frac{1}{v-u}\int_u^v X(t,\cdot)\mathrm{d}t & \leq \frac{1}{2}(X(\lambda v+(1-\lambda)u,\cdot)+\lambda X(u,\cdot)+(1-\lambda)X(v,\cdot)) \\ & \leq \frac{X(u,\cdot)+X(v,\cdot)}{2}, \end{aligned}$$

which completes the proof. □

Now, we start the following important inequality for convex stochastic processes

Theorem 4 (Jensen-type inequality)**.** *If $X\colon I\times\Omega\to\mathbb{R}$ is a convex stochastic process, then we have*

$$X\Big(\frac{1}{b-a}\int_a^b \varphi(t,\cdot)\mathrm{d}t,\cdot\Big) \leq \frac{1}{b-a}\int_a^b X\circ\varphi(t,\cdot)\mathrm{d}t$$

for an arbitrary non-negative integrable stochastic process $\varphi\colon I\times\Omega\to I\subset\mathbb{R}$.

Proof. From Proposition 1, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b X\circ\varphi(t,\cdot)\mathrm{d}t - X\Big(\frac{1}{b-a}\int_a^b \varphi(t,\cdot)\mathrm{d}t,\cdot\Big)\\
&= \frac{1}{b-a}\int_a^b \Big[X\circ\varphi(t,\cdot) - X\Big(\frac{1}{b-a}\int_a^b \varphi(t,\cdot)\mathrm{d}t,\cdot\Big)\Big]\mathrm{d}t\\
&\geq A(\cdot)\Big\{\frac{1}{b-a}\int_a^b \Big[\varphi(t,\cdot) - \frac{1}{b-a}\int_a^b \varphi(t,\cdot)\mathrm{d}t\Big]\mathrm{d}t\Big\}\\
&= A(\cdot)\Big\{\frac{1}{b-a}\int_a^b \varphi(t,\cdot) - \frac{1}{b-a}\int_a^b \varphi(t,\cdot)\mathrm{d}t\Big\}\\
&= 0,
\end{aligned}$$

which completes the proof. □

Theorem 5 (Generalized Hermite–Hadamard-type inequality)**.** *Let* $X\colon I\times\Omega\to\mathbb{R}$ *be a convex stochastic process. If the non-negative integrable stochastic process* $\varphi\colon I\times\Omega\to I\subset\mathbb{R}$ *such that* $X\circ\varphi(t,\cdot)$ *is also convex stochastic process, then we have*

$$\begin{aligned}
X\Big(\frac{1}{b-a}\int_a^b \varphi(t,\cdot)\mathrm{d}t,\cdot\Big) &\leq h(\lambda_1,\dots,\lambda_n) \leq \frac{1}{b-a}\int_a^b X\circ\varphi(t,\cdot)\mathrm{d}t\\
&\leq H(\lambda_1,\dots,\lambda_n) \leq \frac{X\circ\varphi(a,\cdot)+X\circ\varphi(b,\cdot)}{2}
\end{aligned}$$

for $n\in\mathbb{N}$, $\lambda_0=0$, $\lambda_{n+1}=1$ *and arbitrary* $0\leq\lambda_1\leq\dots\leq\lambda_n\leq 1$, *where*

$$h(\lambda_1,\dots,\lambda_n) = \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)X\Bigg(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\varphi(t,\cdot)\mathrm{d}t,\cdot\Bigg),$$

$$\begin{aligned}
H(\lambda_1,\dots,\lambda_n) = \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\Bigg(&\frac{X\circ\varphi((1-\lambda_k)a+\lambda_k b,\cdot)}{2}\\
&+\frac{X\circ\varphi((1-\lambda_{k+1})a+\lambda_{k+1}b,\cdot)}{2}\Bigg).
\end{aligned}$$

Proof. From Theorem 4 and convexity of $X\circ\varphi(t,\cdot)$, we have

$$\begin{aligned}
X\Bigg(\frac{1}{b-a}\int_a^b \varphi(t,\cdot)\mathrm{d}t,\cdot\Bigg) &\leq \frac{1}{b-a}\int_a^b X\circ\varphi(t,\cdot)\mathrm{d}t\\
&\leq \frac{X\circ\varphi(a,\cdot)+X\circ\varphi(b,\cdot)}{2}.
\end{aligned}\tag{2.10}$$

Appliying (2.10) to $[(1-\lambda_k)a+\lambda_k b,(1-\lambda_{k+1})a+\lambda_{k+1}b]$, $k=0,1,\ldots,n$, we have

$$X\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\varphi(t,\cdot)\mathrm{d}t,\cdot\right)$$

$$\leq \frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} X\circ\varphi(t,\cdot)\mathrm{d}t \tag{2.11}$$

$$\leq \frac{X\circ\varphi((1-\lambda_k)a+\lambda_k b,\cdot)+X\circ\varphi((1-\lambda_{k+1})a+\lambda_{k+1}b,\cdot)}{2}.$$

After multiplying each term in (2.11) by $(\lambda_{k+1}-\lambda_k)$ and later summing the result over k from 0 to n, we have

$$\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)X\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\varphi(t,\cdot)\mathrm{d}t,\cdot\right)$$

$$\leq \frac{1}{b-a}\sum_{k=0}^{n}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} X\circ\varphi(t,\cdot)\mathrm{d}t$$

$$\leq \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{X\circ\varphi((1-\lambda_k)a+\lambda_k b,\cdot)+X\circ\varphi((1-\lambda_{k+1})a+\lambda_{k+1}b,\cdot)}{2}.$$

That is,

$$h(\lambda_1,\ldots,\lambda_n)\leq \frac{1}{b-a}\sum_{k=0}^{n}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} X\circ\varphi(t,\cdot)\mathrm{d}t\leq H(\lambda_1,\ldots,\lambda_n).$$

Using the convexity of $X(t,\cdot)$ and $X\circ\varphi(t,\cdot)$ for $\sum\limits_{k=0}^{n}(\lambda_{k+1}-\lambda_k)=1$, we grt

$$X\left(\frac{1}{b-a}\int\limits_a^b\varphi(t,\cdot)\mathrm{d}t,\cdot\right)$$

$$= X\left(\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\varphi(t,\cdot)\mathrm{d}t,\cdot\right)$$

$$\leq \sum_{k=0}^{n}\left(\lambda_{k+1}-\lambda_k)X(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int\limits_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\varphi(t,\cdot)\mathrm{d}t,\cdot\right)$$

$$\le \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{X\circ\varphi((1-\lambda_k)a+\lambda_k b,\cdot)+X\circ\varphi((1-\lambda_{k+1})a+\lambda_{k+1}b,\cdot)}{2}$$
$$\le \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{(1-\lambda_k)X\circ\varphi(a,\cdot)+\lambda_k X\circ\varphi(b,\cdot)}{2}$$
$$+\frac{(1-\lambda_{k+1})X\circ\varphi(a,\cdot)+\lambda_{k+1}X\circ\varphi(b,\cdot)}{2}$$
$$=\frac{1}{2}\sum_{k=0}^{n}((1-\lambda_k)-(1-\lambda_{k+1}))((1-\lambda_k)+(1-\lambda_{k+1}))X\circ\varphi(a,\cdot)$$
$$+\frac{1}{2}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)(\lambda_{k+1}+\lambda_k)X\circ\varphi(b,\cdot)$$
$$=\frac{1}{2}\sum_{k=0}^{n}((1-\lambda_k)^2-(1-\lambda_{k+1})^2)X\circ\varphi(a,.)+\frac{1}{2}\sum_{k=0}^{n}(\lambda_{k+1}^2-\lambda_k^2)X\circ\varphi(b,\cdot)$$
$$=\frac{X\circ\varphi(a,\cdot)+X\circ\varphi(b,\cdot)}{2}.$$

This completes the proof. □

Corollary 1. *Under assumption of Theorem 5 with $\lambda_k=\lambda$ for $k=1,2,\dots,n$, we have the inequality*

$$X\left(\frac{1}{b-a}\int_a^b\varphi(t,\cdot)\mathrm{d}t,\cdot\right)\le h_1(\lambda)\le\frac{1}{b-a}\int_a^b X\circ\varphi(t,\cdot)\mathrm{d}t$$
$$\le H_1(\lambda)\le\frac{X\circ\varphi(a,\cdot)+X\circ\varphi(b,\cdot)}{2}, \tag{2.12}$$

where

$$h_1(\lambda)=\lambda X\left(\frac{1}{\lambda(b-a)}\int_a^{(1-\lambda)a+\lambda b}\varphi(t,\cdot)\mathrm{d}t,\cdot\right)$$
$$+(1-\lambda)X\left(\frac{1}{(1-\lambda)(b-a)}\int_{(1-\lambda)a+\lambda b}^{b}\varphi(t,\cdot)\mathrm{d}t,\cdot\right)$$
$$H_1(\lambda)=\frac{1}{2}(X(\lambda a+(1-\lambda)b,\cdot)+\lambda X(a,\cdot)+(1-\lambda)X(b,\cdot)).$$

Remark 1. If we choose $\varphi(t,\cdot)=t$ in Corollary 1, then inequality (2.12) reduces inequality (2.1).

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