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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF VECTOR SOLUTIONS OF FIRST-ORDER 2-D NEUTRAL DIFFERENCE SYSTEMS

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ABSTRACT. The aim of this paper is to obtain the sufficient conditions for oscillation and nonoscillation of vector solutions of a class of first-order two-dimensional nonautonomous neutral delay difference systems of the form:

$$\Delta \begin{bmatrix} \alpha(\nu)+q(\nu)\alpha(\nu-p) \\ \beta(\nu)+q(\nu)\beta(\nu-p) \end{bmatrix} = \begin{bmatrix} a_1(\nu) & a_2(\nu) \\ a_3(\nu) & a_4(\nu) \end{bmatrix} \begin{bmatrix} \phi(\alpha(\nu-l)) \\ \psi(\beta(\nu-m)) \end{bmatrix} + \begin{bmatrix} \omega_1(\nu) \\ \omega_2(\nu) \end{bmatrix},$$

where $p>0$, $m\geq 0$, $l\geq 0$ are integers, $a_j(\nu)$, $j=1,2,3,4$, $q(\nu),\omega_1(\nu),\omega_2(\nu)$ are real valued sequences for $\nu\in\mathbb{N}(\nu_0)$, and $\phi,\psi\in\mathcal{C}(\mathbb{R},\mathbb{R})$ are bounded functions with the properties $u\phi(u)>0$, $v\psi(v)>0$ for $u\neq 0$, $v\neq 0$. We verify some of our results with illustrative examples.

1. Introduction

In this paper, we study the oscillatory behaviour of solutions of the first-order two-dimensional nonautonomous difference systems of the form:
(NAS1)

$$\Delta \begin{bmatrix} \alpha(\nu)+q(\nu)\alpha(\nu-p) \\ \beta(\nu)+q(\nu)\beta(\nu-p) \end{bmatrix} = \begin{bmatrix} a_1(\nu) & a_2(\nu) \\ a_3(\nu) & a_4(\nu) \end{bmatrix} \begin{bmatrix} \phi(\alpha(\nu-l)) \\ \psi(\beta(\nu-m)) \end{bmatrix} + \begin{bmatrix} \omega_1(\nu) \\ \omega_2(\nu) \end{bmatrix}$$

for $\nu\geq\nu_0>\rho=\max\{l,m,p\}$, where $p>0$, $m\geq 0$, $l\geq 0$ are integers, $a_j(\nu)$, $j=1,2,3,4$, $q(\nu)$, $\omega_1(\nu)$, $\omega_2(\nu)$ are sequences of real numbers for $\nu\in\mathbb{N}(\nu_0)=\{\nu_0,\nu_0+1,\dots\}$, and $\phi,\psi\in\mathcal{C}(\mathbb{R},\mathbb{R})$ are bounded nondecreasing functions with the properties $u\phi(u)>0$, $v\psi(v)>0$ for $u\neq 0$, $v\neq 0$.

In [**15**], the authors Tripathy and Das have studied the first-order neutral autonomous system

$$\text{(NAS2)}\qquad \Delta \begin{bmatrix} \alpha(\nu)-q\alpha(\nu-p) \\ \beta(\nu)-q\beta(\nu-p) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \alpha(\nu-l) \\ \beta(\nu-m) \end{bmatrix}, \quad \nu\geq\nu_0,$$

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where $a_1, a_2, a_3, a_4, q \in \mathbb{R}$, $p > 1$, $l, m \in \mathbb{N}$. Here, the state of art is the construction of characteristic equation along with its qualitative study to locate the roots on $\mathbb{R}$. Another state of art is the linearized oscillation method for the first-order nonlinear neutral nonautonomous difference systems

(NAS3)

$$\Delta \begin{bmatrix} \alpha(\nu) - q(\nu)h_1(\alpha(\nu-p)) \\ \beta(\nu) - q(\nu)h_2(\beta(\nu-p)) \end{bmatrix} + \begin{bmatrix} a_1(\nu) & a_2(\nu) \\ a_3(\nu) & a_4(\nu) \end{bmatrix} \begin{bmatrix} \phi(\alpha(\nu-l)) \\ \psi(\beta(\nu-m)) \end{bmatrix} = 0, \ \nu \geq \nu_0,$$

in the range $1 < q(\nu) < \infty$, where $h_1, h_2, \phi, \psi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. The authors' purpose was to understand the oscillatory and nonoscillatory behaviour of solutions of (NAS2) and (NAS3), and the study of (NAS3) is incomplete as long as $q(\nu) \leq 0$ and $0 < q(\nu) \leq 1$ are concerned.

Very often, two neighbouring countries with their common boundaries involve with common boundary issues, which affects the countries defence budgets. If $\alpha(\nu)$ and $\beta(\nu)$ are the expenditures on arms by two neighbouring countries B and D, respectively, at discrete time ν, then the change in the next defence budget expenditure by the country B has a term proportional to β, since the larger expenditure in arms by D, the larger will be the change of expenditure in arms by B. Also, it has a term proportional to $-\alpha$, since its own arms expenditure has an inhibiting effect on the next expenditure on arms by B. It may also contain a term independent of the expenditures depending on mutual suspicions or mutual goodwill. Taking these considerations into account, the above arms race model for two countries takes the form:

$$\Delta\alpha(\nu) = -a_1\alpha(\nu) + a_2\beta(\nu) + r,$$
$$\Delta\beta(\nu) = a_3\beta(\nu) - a_4\beta(\nu) + s,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$ are positive constants, and $r, s > 0$ (< 0) for mutual suspicions (mutual goodwill). In the matrix form, we have the model

$$\Delta \begin{bmatrix} \alpha(\nu) \\ \beta(\nu) \end{bmatrix} = \begin{bmatrix} -a_1 & a_2 \\ a_3 & -a_4 \end{bmatrix} \begin{bmatrix} \alpha(\nu) \\ \beta(\nu) \end{bmatrix} + \begin{bmatrix} r \\ s \end{bmatrix},$$

which can also be treated as the discrete version of the Richardson's armed race model, and the study of the preceding system have been documented in the literature [**13**] and [**14**] unlike the methods established in [**4, 8, 9**] and [**10**].

Let $\alpha(\nu)$ and $\beta(\nu)$ be the population size of matured male and female, respectively, at time ν. Before maturity, that is at time $(\nu-\tau)$, let $\alpha(\nu-\tau)$ and $\beta(\nu-\tau)$ be the unmatured male and female size, respectively. After the τ period, the growth of population size depends on the so called matured male and female along with the converted unmatured male and female, which can be modelled in the form:

$$\Delta(\alpha(\nu) + q(\nu)\alpha(\nu-\tau)) = a_1(\nu)\phi(\alpha(\nu-\mu)) + a_2(\nu)\psi(\beta(\nu-\varsigma)),$$
$$\Delta(\beta(\nu) + q(\nu)\beta(\nu-\tau)) = a_3(\nu)\phi(\alpha(\nu-\mu)) + a_4(\nu)\psi(\beta(\nu-\varsigma)),$$

where $\tau, \mu, \varsigma \in \mathbb{I}^+$. With this population growth, if there is a possibility of emigration or immigration of some matured male and female say $\omega_1(\nu)$ and $\omega_2(\nu)$, respectively, then the resultant population size could be

$$\Delta \begin{bmatrix} (\alpha(\nu)+q(\nu)\alpha(\nu-\tau)) \\ (\beta(\nu)+q(\nu)\beta(\nu-\tau)) \end{bmatrix} = \begin{bmatrix} a_1(\nu) & a_2(\nu) \\ a_3(\nu) & a_4(\nu) \end{bmatrix} \begin{bmatrix} \phi(\alpha(\nu-\mu)) \\ \psi(\beta(\nu-\varsigma)) \end{bmatrix} + \begin{bmatrix} \omega_1(\nu) \\ \omega_2(\nu) \end{bmatrix},$$

which is similar to (NAS1). More on difference equations and system of difference equations, we refer the monographs [**1, 2, 3, 7**] and some works [**5, 6, 12**].

Qualitative behaviour of solutions of system of differential/difference equations play an important role in Dynamical Systems. Keeping this fact in mind, our object here is to study the necessary and sufficient conditions for oscillatory and asymptotic behaviour of vector solutions of population model (NAS1). We infer the Richardson's armed race model as a special case of (NAS1).

Definition 1.1. By a solution of (NAS1), we mean a vector $\Psi(\nu)=[\alpha(\nu),\beta(\nu)]^T$, which satisfies (NAS1) for $\nu \in \mathbb{N}(-\vartheta) = \{-\vartheta, -\vartheta+1, \dots 0, 1, 2, \dots\}$. We say that the vector solution $\Psi(\nu)$ oscillates componentwise or strongly oscillates or simply oscillates, if each component oscillates. Otherwise, the vector solution $\Psi(\nu)$ is called nonoscillatory, that is, each component of the vector solution is either eventually positive or eventually negative and say strongly nonoscillatory if both components of $\Psi(\nu)$ are nonoscillatory.

Lemma 1.2 (cf. [**11**])**.** *Let $\phi(k)$, $\psi(k)$ and $q(k)$ be real valued functions of discrete arguments defined for $k \geq k_0 \geq 0$ such that $\phi(k) = \psi(k) + q(k)\psi(k-s)$, $k \geq k_0 + s$, where $s \geq 0$ is an integer. Suppose that there exist real numbers ϱ_1, ϱ_2, ϱ_3, ϱ_4 such that $q(\varrho)$ is in one of following ranges:*

1. $-\infty < \varrho_1 \leq q(k) \leq 0$,
2. $0 \leq q(k) \leq \varrho_2 < 1$,
3. $1 < \varrho_3 \leq q(k) \leq \varrho_4 < \infty$.

If $\psi(k) > 0$ for $k \geq k_0$, $\liminf_{k\to\infty} \psi(k) = 0$, and $\lim_{k\to\infty} \phi(k) = d$ exists, then $d = 0$. If $\psi(k) < 0$ for $k \geq k_0$, $\limsup_{k\to\infty} \psi(k) = 0$, and $\lim_{k\to\infty} \phi(k) = d$ exists, then $d = 0$.

Theorem 1.3 (cf. [**3**])**.** *Let $\mathcal{X}$ be a Banach space. Let V be a bounded closed convex subset of $\mathcal{X}$, and let $\mathcal{M}_1, \mathcal{M}_2$ be two maps of V into $\mathcal{X}$ such that $\mathcal{M}_1\alpha + \mathcal{M}_2\beta \in V$ for every pair $\alpha, \beta \in V$. If $\mathcal{M}_1$ is contraction and $\mathcal{M}_2$ is completely continuous, then the equation*

$$\mathcal{M}_1\alpha + \mathcal{M}_2\alpha = \alpha$$

has a solution in V.

2. Necessary and sufficient conditions

In this section, we establish the necessary and sufficient conditions for all vector solutions $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ of the system (NAS1), which either strongly oscillate or tend to zero as $\nu \to 0$.

Lemma 2.1. *Let* $-1 < c_1 \le q(\nu) \le 0$ *and* $a_1(\nu) < 0$, $a_2(\nu) > 0$, $a_3(\nu) > 0$, $a_4(\nu) < 0$ *for any large* ν. *Assume that the following conditions hold:*

(A_1) $\left|\sum_{\nu=0}^{\infty}\omega_1(\nu)\right| < \infty$, $\left|\sum_{\nu=0}^{\infty}\omega_2(\nu)\right| < \infty$,

(A_2) $\sum_{\nu=0}^{\infty} a_2(\nu) < \infty$, $\sum_{\nu=0}^{\infty} a_3(\nu) < \infty$,

(A_3) $\sum_{\nu=0}^{\infty} a_1(\nu) = -\infty$, $\sum_{\nu=0}^{\infty} a_4(\nu) = -\infty$.

Let $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) < 0$, $\alpha(\nu-p) < 0$, $\alpha(\nu-l) < 0$, *and* $\beta(\nu) < 0$, $\beta(\nu-p) < 0$, $\beta(\nu-m) < 0$ *for* $\nu \ge \nu_1 > \nu_0$. *Then* $\lim_{\nu\to\infty}\Psi(\nu) = 0$.

Proof. We have that $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ is a strongly nonoscillatory vector solution of (NAS1) such that $\alpha(\nu) < 0$, $\alpha(\nu-p) < 0$, $\alpha(\nu-l) < 0$, and $\beta(\nu) < 0$, $\beta(\nu-p) < 0$, $\beta(\nu-m) < 0$ for $\nu \ge \nu_1 > \nu_0$. We set

$$P_1(\nu) = \sum_{i=\nu}^{\infty} a_2(i)\psi(\beta(i-m)), \qquad P_2(\nu) = \sum_{i=\nu}^{\infty} a_3(i)\phi(\alpha(i-l));$$

$$z_1(\nu) = \alpha(\nu) + q(\nu)\alpha(\nu-p), \qquad z_2(\nu) = \beta(\nu) + q(\nu)\beta(\nu-p);$$

$$W_1(\nu) = z_1(\nu) + P_1(\nu) - \sum_{i=0}^{\nu-1}\omega_1(i), \qquad W_2(\nu) = z_2(\nu) + P_2(\nu) - \sum_{i=0}^{\nu-1}\omega_2(i)$$

for $\nu \ge \nu_2 > \nu_1 + \rho$. Therefore,

$$\Delta W_1(\nu) = a_1(\nu)\phi(\alpha(\nu-l)) \ge 0, \tag{1}$$

$$\Delta W_2(\nu) = a_4(\nu)\psi(\beta(\nu-m)) \ge 0 \tag{2}$$

for $\nu \ge \nu_2$. Consequently, $W_1(\nu)$ and $W_2(\nu)$ are monotonic for $\nu \ge \nu_2$. Let $W_1(\nu) > 0$ for $\nu \ge \nu_2$. We claim that $\alpha(\nu)$ is bounded. If not, let there exist a subsequence $\alpha(\nu_j)$ of $\alpha(\nu)$ such that $\nu_j \to \infty$ and $\alpha(\nu_j) \to -\infty$ as $j \to \infty$ and $\alpha(\nu_j) = \min\{\alpha(\nu) : \nu_2 \le \nu \le \nu_j\}$. We may choose ν_j sufficiently large such that $\nu_j - p > \nu_2$, and hence,

$$\begin{aligned} W_1(\nu_j) &= \alpha(\nu_j) + q(\nu_j)\alpha(\nu_j - p) + \sum_{i=\nu_j}^{\infty} a_2(i)\psi(\beta(i-m)) - \sum_{i=0}^{\nu_j-1}\omega_1(i) \\ &\le (1+c_1)\alpha(\nu_j) + \sum_{i=\nu_j}^{\infty} a_2(i)\psi(\beta(i-m)) - \sum_{i=0}^{\nu_j-1}\omega_1(i) \to -\infty \text{ as } j \to \infty \end{aligned} \tag{3}$$

implies that $W_1(\nu_j) < 0$ for large ν_j, a contradiction. Thus, our claim holds, and $\lim_{\nu\to\infty} W_1(\nu)$ exists. If $\limsup_{\nu\to\infty}\alpha(\nu) = \delta_1, -\infty < \delta_1 < 0$, then we can find $\eta_1 < 0$

and $\nu_3 > \nu_2$ such that $\alpha(\nu - l) < \eta_1$ for $\nu \geq \nu_3$. It follows from (1) that

$$\sum_{\nu=\nu_3}^{t-1} a_1(\nu)\phi(\alpha(\nu - l)) = \sum_{\nu=\nu_3}^{t-1} \Delta W_1(\nu) = W_1(t) - W_1(\nu_3) \leq W_1(t)$$

implies that

$$\phi(\eta_1) \sum_{\nu=\nu_3}^{\infty} a_1(\nu) < \sum_{\nu=\nu_3}^{\infty} a_1(\nu)\phi(\alpha(\nu - l)) \leq \lim_{t\to\infty} W_1(t) < \infty,$$

which contradicts to (A_3). Ultimately, $\limsup_{\nu\to\infty} \alpha(\nu) = 0$. Now, $\lim_{\nu\to\infty} W_1(\nu)$ exists implies that $\lim_{\nu\to\infty} z_1(\nu)$ exists. Using Lemma 1.2, we get $\lim_{\nu\to\infty} z_1 = 0$, and similar conclusion holds if $W_1(\nu) < 0$ for $\nu \geq \nu_2$. For $-1 < c_1 \leq q(\nu) \leq 0$, we infer that

$$\begin{aligned} 0 = \liminf_{\nu\to\infty} z_1(\nu) &\leq \liminf_{\nu\to\infty} [\alpha(\nu) + c_1\alpha(\nu - p)] \\ &\leq \liminf_{\nu\to\infty} \alpha(\nu) + \limsup_{\nu\to\infty} [c_1\alpha(\nu - p)] \\ &= \liminf_{\nu\to\infty} \alpha(\nu) + c_1 \liminf_{\nu\to\infty} [\alpha(\nu - p)] \\ &= (1 + c_1) \liminf_{\nu\to\infty} \alpha(\nu), \end{aligned}$$

that is, $\liminf_{\nu\to\infty} \alpha(\nu) = 0$, and hence, $\lim_{\nu\to\infty} \alpha(\nu) = 0$. The preceding argument can be made for $W_2(\nu)$ as well, and consequently, $0 = \lim_{\nu\to\infty} \beta(\nu)$. This completes the proof of the lemma. □

Lemma 2.2. *Assume that all the conditions of* Lemma 2.1 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) > 0$, $\alpha(\nu - p) > 0$, $\alpha(\nu - l) > 0$, *and* $\beta(\nu) > 0$, $\beta(\nu - p) > 0$, $\beta(\nu - m) > 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then* $\lim_{\nu\to\infty} \Psi(\nu) = 0$.

Proof. We have that $\Psi(\nu)$ is a strongly nonoscillatory vector solution of (NAS1) such that $\alpha(\nu) > 0$, $\alpha(\nu - p) > 0$, $\alpha(\nu - l) > 0$, and $\beta(\nu) > 0$, $\beta(\nu - p) > 0$, $\beta(\nu - m) > 0$ for $\nu \geq \nu_1$. Setting $\alpha^*(\nu) = -\alpha(\nu)$, $\beta^*(\nu) = -\beta(\nu)$, $\omega_1^*(\nu) = -\omega_1(\nu)$, $\omega_2^*(\nu) = -\omega_2(\nu)$, $\phi^*(\alpha) = -\phi(-\alpha)$, $\psi^*(\alpha) = -\psi(-\beta)$, we see that the vector $[\alpha^*(\nu), \beta^*(\nu)]^T$ is the solution of

(NAS4)
$$\Delta \begin{bmatrix} \alpha^*(\nu) + q(\nu)\alpha^*(\nu - p) \\ \beta^*(\nu) + q(\nu)\beta^*(\nu - p) \end{bmatrix} = \begin{bmatrix} a_1(\nu) & a_2(\nu) \\ a_3(\nu) & a_4(\nu) \end{bmatrix} \begin{bmatrix} \phi^*(\alpha^*(\nu - l)) \\ \psi^*(\beta^*(\nu - m)) \end{bmatrix} + \begin{bmatrix} \omega_1^*(\nu) \\ \omega_2^*(\nu) \end{bmatrix},$$

$\nu \geq \nu_1$, which is equivalent to (NAS1). The rest of the proof follows from Lemma 2.1. This completes the proof of the lemma. □

Lemma 2.3. *Assume that all the conditions of* Lemma 2.1 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) < 0$, $\alpha(\nu - p) < 0$, $\alpha(\nu - l) < 0$, *and* $\beta(\nu) > 0$, $\beta(\nu - p) > 0$, $\beta(\nu - m) > 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then* $\lim_{\nu\to\infty} \Psi(\nu) = 0$.

Proof. Let $\Psi(\nu)$ be a strongly nonoscillatory vector solution of (NAS1) such that $\alpha(\nu) < 0$, $\alpha(\nu-p) < 0$, $\alpha(\nu-l) < 0$, and $\beta(\nu) > 0$, $\beta(\nu-p) > 0$, $\beta(\nu-m) > 0$ for $\nu \geq \nu_1$, then we find

$$\Delta W_1(\nu) = a_1(\nu)\phi(\alpha(\nu - l)) \geq 0, \tag{4}$$

$$\Delta W_2(\nu) = a_4(\nu)\psi(\beta(\nu - m)) \leq 0, \tag{5}$$

in which $W_1(\nu)$ and $W_2(\nu)$ are monotonic. All we are done for $W_1(\nu)$ insisting the fact $\lim_{\nu\to\infty} \alpha(\nu) = 0$. If we put $-\beta(\nu) = v(\nu)$ in (5), then it happens to be

$$\Delta[v(\nu) + q(\nu)v(\nu - p) + \sum_{i=\nu}^{\infty} a_3(i)\phi(\alpha(i - l))] = a_4(\nu)\psi(v(\nu - m)) \geq 0,$$

which is similar to (4), and the argument is analogous to that of Lemma 2.1. □

Lemma 2.4. *Assume that all the conditions of* Lemma 2.1 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) > 0$, $\alpha(\nu - p) > 0$, $\alpha(\nu - l) > 0$, *and* $\beta(\nu) < 0$, $\beta(\nu - p) < 0$, $\beta(\nu - m) < 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then* $\lim_{\nu\to\infty} \Psi(\nu) = 0$.

Lemma 2.5. *Let* $0 \leq q(\nu) \leq c_2 < \infty$ *and* $a_1(\nu) < 0$, $a_2(\nu) > 0$, $a_3(\nu) > 0$, $a_4(\nu) < 0$ *for any large* ν. *Assume that all the conditions of* Lemma 2.1 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) < 0$, $\alpha(\nu-p) < 0$, $\alpha(\nu-l) < 0$, *and* $\beta(\nu) < 0$, $\beta(\nu-p) < 0$, $\beta(\nu-m) < 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then* $\lim_{\nu\to\infty} \Psi(\nu) = 0$.

Proof. Let $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ be a strongly nonoscillatory vector solution of (NAS1). Assume that $\alpha(\nu) < 0$, $\alpha(\nu - p) < 0$, $\alpha(\nu - l) < 0$, and $\beta(\nu) < 0$, $\beta(\nu - p) < 0$, $\beta(\nu - m) < 0$ for $\nu \geq \nu_1$. Setting as in Lemma 2.1, we conclude that $W_1(\nu)$ and $W_2(\nu)$ are monotonic for $\nu \geq \nu_1$. Let $W_1(\nu) > 0$ for $\nu \geq \nu_2 > \nu_1$. As in Lemma 2.1, we can claim the boundedness of $\alpha(\nu)$. Consequently, $\lim_{\nu\to\infty} W_1(\nu)$ exists. Proceeding as in Lemma 2.1, we have $\limsup_{\nu\to\infty} \alpha(\nu) = 0$. As $\lim_{\nu\to\infty} z_1(\nu)$ exists and due to Lemma 1.2, it follows that $\lim_{\nu\to\infty} z_1(\nu) = 0$. If $W_1(\nu) < 0$ for $\nu \geq \nu_2$, then $\lim_{\nu\to\infty} W_1(\nu)$ exists, and so, $\lim_{\nu\to\infty} z_1(\nu)$. Since $z_1(\nu) \leq \alpha(\nu)$ for $\nu \geq \nu_2$, then it follows that $\alpha(\nu)$ is bounded. Again, as in Lemma 2.1, we have $\lim_{\nu\to\infty} z_1(\nu) = 0$. Of course, $z_1(\nu) \leq \alpha(\nu)$ for $\nu \geq \nu_2$ implies that $\lim_{\nu\to\infty} \alpha(\nu) = 0$. The proof is analogous for monotonic $W_2(\nu)$, from which we find that $\lim_{\nu\to\infty} \beta(\nu) = 0$. Hence, the proof of the lemma is completed. □

Lemma 2.6. *Assume that all the conditions of* Lemma 2.5 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) > 0$, $\alpha(\nu - p) > 0$, $\alpha(\nu - l) > 0$, *and* $\beta(\nu) > 0$, $\beta(\nu - p) > 0$, $\beta(\nu - m) > 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then* $\lim_{\nu\to\infty} \Psi(\nu) = 0$.

Lemma 2.7. *Assume that all the conditions of* Lemma 2.5 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) < 0$, $\alpha(\nu - p) < 0$, $\alpha(\nu - l) < 0$, *and* $\beta(\nu) > 0$, $\beta(\nu - p) > 0$, $\beta(\nu - m) > 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then* $\lim_{\nu\to\infty} \Psi(\nu) = 0$.

Lemma 2.8. *Assume that all the conditions of* Lemma 2.5 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) > 0$, $\alpha(\nu - p) > 0$, $\alpha(\nu - l) > 0$, *and* $\beta(\nu) < 0$, $\beta(\nu - p) < 0$, $\beta(\nu - m) < 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then* $\lim_{\nu\to\infty} \Psi(\nu) = 0$.

Lemma 2.9. *Let* $-\infty < c_5 \leq q(\nu) \leq c_6 < -1$ *and* $a_1(\nu) < 0$, $a_2(\nu) > 0$, $a_3(\nu) > 0$, $a_4(\nu) < 0$ *for any large* ν. *Assume that all the conditions of* Lemma 2.1 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) < 0$, $\alpha(\nu-p) < 0$, $\alpha(\nu-l) < 0$, *and* $\beta(\nu) < 0$, $\beta(\nu-p) < 0$, $\beta(\nu-m) < 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then every bounded vector solution* $\Psi(\nu)$ *of* (NAS1) *tends to zero as* $\nu \to \infty$.

Proof. The proof of the lemma follows from Lemma 2.1, and using Lemma 1.2, we have $\lim_{n\to\infty} z_1(\nu) = 0 = \lim_{\nu\to\infty} z_2(\nu)$. Hence,

$$\begin{aligned} 0 = \lim_{\nu\to\infty} z_1(\nu) &= \liminf_{\nu\to\infty}(\alpha(\nu) + q(\nu)\alpha(\nu - p)) \\ &\leq \liminf_{\nu\to\infty}(\alpha(\nu) + c_6\alpha(\nu - p)) \\ &\leq \limsup_{\nu\to\infty} \alpha(\nu) + \liminf_{\nu\to\infty}(c_6\alpha(\nu - p)) \\ &= (1 + c_6) \limsup_{\nu\to\infty} \alpha(\nu) \end{aligned}$$

implies that $\lim_{\nu\to\infty} \alpha(\nu) = 0$. $\lim_{\nu\to\infty} \beta(\nu) = 0$ can similarly be dealt with. This completes the proof of the lemma. □

Lemma 2.10. *Assume that all conditions of* Lemma 2.9 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) > 0$, $\alpha(\nu-p) > 0$, $\alpha(\nu-l) > 0$, *and* $\beta(\nu) > 0$, $\beta(\nu-p) > 0$, $\beta(\nu-m) > 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then every bounded vector solution* $\Psi(\nu)$ *of* (NAS1) *tends to zero as* $\nu \to \infty$.

Lemma 2.11. *Assume that all conditions of* Lemma 2.9 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) < 0$, $\alpha(\nu-p) < 0$, $\alpha(\nu-l) < 0$, *and* $\beta(\nu) > 0$, $\beta(\nu-p) > 0$, $\beta(\nu-m) > 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then every bounded vector solution* $\Psi(\nu)$ *of* (NAS1) *tends to zero as* $\nu \to \infty$.

Lemma 2.12. *Assume that all conditions of* Lemma 2.9 *hold. Let* $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ *be a strongly nonoscillatory vector solution of* (NAS1) *such that* $\alpha(\nu) > 0$, $\alpha(\nu-p) > 0$, $\alpha(\nu-l) > 0$, *and* $\beta(\nu) < 0$, $\beta(\nu-p) < 0$, $\beta(\nu-m) < 0$ *for* $\nu \geq \nu_1 > \nu_0$. *Then every bounded vector solution* $\Psi(\nu)$ *of* (NAS1) *tends to zero as* $\nu \to \infty$.

Theorem 2.13. *Let $a_1(\nu) < 0$, $a_2(\nu) > 0$, $a_3(\nu) > 0$, $a_4(\nu) < 0$ for large ν. Suppose that $\phi, \psi \in \mathrm{Lip}\{l_1, l_2, [a,b], -\infty < a < b < \infty\}$, and* (A$_1$), (A$_2$) *hold. If $-1 < c_1 \le q(\nu) \le 0$, then every vector solution of* (NAS1) *either strongly oscillates or converges to zero as $\nu \to \infty$ if and only if* (A$_3$) *holds.*

Proof. Let $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ be a vector solution of (NAS1). If it oscillates, then there is nothing to prove. Assume that it is a strongly nonoscillatory vector solution of (NAS1). Without loss of generality, we consider here the four possible cases:

Case 1: $\alpha(\nu) < 0$, $\alpha(\nu - p) < 0$, $\alpha(\nu - l) < 0$, and $\beta(\nu) < 0$, $\beta(\nu - p) < 0$, $\beta(\nu - m) < 0$ for $\nu \ge \nu_1 > \nu_0$.

Case 2: $\alpha(\nu) > 0$, $\alpha(\nu - p) > 0$, $\alpha(\nu - l) > 0$, and $\beta(\nu) > 0$, $\beta(\nu - p) > 0$, $\beta(\nu - m) > 0$ for $\nu \ge \nu_1 > \nu_0$.

Case 3: $\alpha(\nu) < 0$, $\alpha(\nu - p) < 0$, $\alpha(\nu - l) < 0$, and $\beta(\nu) > 0$, $\beta(\nu - p) > 0$, $\beta(\nu - m) > 0$ for $\nu \ge \nu_1 > \nu_0$.

Case 4: $\alpha(\nu) > 0$, $\alpha(\nu - p) > 0$, $\alpha(\nu - l) > 0$, and $\beta(\nu) < 0$, $\beta(\nu - p) < 0$, $\beta(\nu - m) < 0$ for $\nu \ge \nu_1 > \nu_0$.

In each of the above cases, we find $\lim_{\nu\to\infty} \Psi(\nu) = 0$ by using Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4, respectively.

Conversely, we assume that

$$\sum_{\nu=0}^{\infty} a_1(\nu) > -\infty, \qquad \sum_{\nu=0}^{\infty} a_4(\nu) > -\infty.$$

Let $\mathcal{B} = l^\infty_{\nu_1}$ be the Banach space of all real bounded sequences $\Psi = [\alpha(\nu), \beta(\nu)]^T$ with the sup norm defined by

$$\mathcal{B} = \{\Psi \colon \mathbb{N} \to \mathbb{R}^2 : \|\Psi\| = \sup_{\nu\in\mathbb{N}} |\Psi| < \infty\}.$$

For a fixed interval I, define

$$\upsilon_1 = \{\Psi \in \mathcal{B} : \alpha(\nu), \beta(\nu) \in I \times I, \nu \ge \nu_1\},$$

where $I = [\frac{1+c_1}{8}, 1]$. Clearly, $\upsilon_1 \subset \mathcal{B}$ is closed, bounded and convex. By (A$_1$), there exists $\nu_1 > 0$ such that

$$\sum_{i=\nu}^{\infty} |a_1(i)| < \frac{(1+c_1)}{12 l_1}, \qquad \sum_{i=\nu}^{\infty} a_2(i) < \frac{(1+c_1)}{12 l_2},$$
$$\sum_{i=\nu}^{\infty} a_3(i) < \frac{(1+c_1)}{12 l_1}, \qquad \sum_{i=\nu}^{\infty} |a_4(i)| < \frac{(1+c_1)}{12 l_2}$$

and

$$\left|\sum_{i=\nu}^{\infty} \omega_1(i)\right| < \frac{(1+c_1)}{24}, \qquad \left|\sum_{i=\nu}^{\infty} \omega_2(i)\right| < \frac{(1+c_1)}{24}$$

for $\nu \ge \nu_1$, where $l_1 = \max\{u_1, \phi(1)\}$, $l_2 = \max\{u_2, \psi(1)\}$, and u_1, u_2 are the Lipschitz constants of ϕ and ψ, respectively, on $[\frac{1+c_1}{8}, 1]$.

Define the maps $\mathcal{T}, \mathbb{T}\colon \upsilon_1 \to \mathcal{B}$ such that

$$(\mathcal{T}\Psi)(\nu) = \begin{bmatrix} \dfrac{(1+c_1)}{4} - q(\nu)\alpha(\nu-p) - \sum_{i=\nu}^{\infty} a_1(i)\phi(\alpha(i-l)) - \sum_{i=\nu}^{\infty}\omega_1(i) \\ \dfrac{(1+c_1)}{4} - q(\nu)\beta(\nu-p) - \sum_{i=\nu}^{\infty} a_4(i)\psi(\beta(i-m)) - \sum_{i=\nu}^{\infty}\omega_2(i) \end{bmatrix}, \quad \nu \geq \nu_1,$$

$(\mathcal{T}\Psi)(\nu) = (\mathcal{T}\Psi)(\nu_1)$ for $\nu_1 - \eta < \nu < \nu_1$,

and

$$(\mathbb{T}\Psi)(\nu) = \begin{bmatrix} -\sum_{i=\nu}^{\infty} a_2(i)\psi(\beta(i-m)) \\ -\sum_{i=\nu}^{\infty} a_3(i)\phi(\alpha(i-l)) \end{bmatrix} \quad \text{for } \nu \geq \nu_1,$$

$$(\mathbb{T}\Psi)(\nu) = (\mathbb{T}\Psi)(\nu_1) \quad \text{for } \nu_1 - \eta < \nu < \nu_1.$$

We note that

$$\mathcal{T} = \begin{bmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{bmatrix}, \quad \mathbb{T} = \begin{bmatrix} \mathbb{T}_1 \\ \mathbb{T}_2 \end{bmatrix}.$$

Let $\Psi, \Upsilon \in \upsilon_1$. Then for $\nu \geq \nu_1$,

$$\begin{aligned}
(\mathcal{T}_1\Psi)(\nu) + (\mathbb{T}_1\Upsilon)(\nu) &= \frac{(1+c_1)}{4} - q(\nu)\alpha(\nu-p) - \sum_{i=\nu}^{\infty} a_1(i)\phi(\alpha(i-l)) \\
&\quad - \sum_{i=\nu}^{\infty}\omega_1(i) - \sum_{i=\nu}^{\infty} a_2(i)\psi(\beta(i-m)) \\
&\leq \frac{(1+c_1)}{4} - c_1 + \sum_{i=\nu}^{\infty} |a_1(i)||\phi(\alpha(i-l))| + \left|\sum_{i=\nu}^{\infty}\omega_2(i)\right| \\
&\quad + \sum_{i=\nu}^{\infty} |a_2(i)||\psi(\beta(i-m)) - \psi(0)| \\
&\leq \frac{(1+c_1)}{4} - c_1 + \phi(1)\frac{(1+c_1)}{12l_1} + \frac{(1+c_1)}{24} + \psi(1)\frac{(1+c_1)}{12l_2} \\
&\leq \frac{(1+c_1)}{4} - c_1 + \frac{(1+c_1)}{6} + \frac{(1+c_1)}{24} = \frac{11-13c_1}{24} < 1
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{T}_1\Psi)(\nu) + (\mathbb{T}_1\Upsilon)(\nu) &= \frac{(1+c_1)}{4} - q(\nu)\alpha(\nu-p) - \sum_{i=\nu}^{\infty} a_1(i)\phi(\alpha(i-l)) \\
&\quad - \sum_{i=\nu}^{\infty}\omega_1(i) - \sum_{i=\nu}^{\infty} a_2(i)\psi(\beta(i-m)) \\
&\geq \frac{(1+c_1)}{4} - \left|\sum_{i=\nu}^{\infty}\omega_1(i)\right| - \sum_{i=\nu}^{\infty} |a_2(i)||\psi(\beta(i-m)) - \psi(0)| \\
&\geq \frac{(1+c_1)}{4} - \frac{(1+c_1)}{24} - \psi(1)\frac{(1+c_1)}{12l_2} = \frac{(1+c_1)}{8}.
\end{aligned}$$

Similarly, $(\mathcal{T}_2\Psi)(\nu)+(\mathbb{T}_2\Upsilon)(\nu)\in \upsilon_1$ for $\nu\geq \nu_1$, and so, $\mathcal{T}\Psi+\mathbb{T}\Upsilon\in\upsilon_1$. For Ψ_1 and $\Psi_2\in\upsilon_1$,

$$\begin{aligned}|(\mathcal{T}_1\Psi_1)(\nu)-(\mathcal{T}_1\Psi_2)(\nu)| &\leq |q(\nu)||\alpha_1(\nu-m)-\alpha_2(\nu-m)| \\ &\quad+\sum_{i=\nu}^{\infty}|a_1(i)||\phi(\alpha_1(i-l))-\phi(\alpha_2(i-l))| \\ &\leq -c_1\|\alpha_1-\alpha_2\|+\frac{(1+c_1)}{12l_1}(l_1\|\alpha_1-\alpha_2\|) \\ &=\left[-c_1+\frac{1+c_1}{12}\right]\|\alpha_1-\alpha_2\|=\frac{1-11c_1}{12}\|\alpha_1-\alpha_2\|\end{aligned}$$

and also,

$$|(\mathcal{T}_2\Psi_1)(\nu)-(\mathcal{T}_2\Psi_2)(\nu)|\leq\frac{1-11c_1}{12}\|\beta_1-\beta_2\|$$

for $\nu\geq\nu_1$ implies that

$$\|\mathcal{T}\Psi_1-\mathcal{T}\Psi_2\|\leq\frac{1-11c_1}{12}\|\Psi_1-\Psi_2\|,$$

that is, $\mathcal{T}$ is a contraction mapping.

Next, we show that $\mathbb{T}$ is continuous. Let $\Psi_j(\nu)=[\alpha_j(\nu),\beta_j(\nu)]^T\in\upsilon_1$ for any $j\in\mathbb{N}$. Let $\Psi_j(\nu)$ be such that $\alpha_j(\nu)\to\alpha(\nu)$ and $\beta_j(\nu)\to\beta(\nu)$ as $j\to\infty$. If we choose $\Psi(\nu)=[\alpha(\nu),\beta(\nu)]^T$, then $\Psi_j\in\upsilon_1$ implies that $\Psi\in\upsilon_1$, and hence, $\alpha(\nu),\beta(\nu)\in I$ for $\nu\geq\nu_1$. Therefore,

$$\begin{aligned}|(\mathbb{T}_1\Psi_j)(\nu)-(\mathbb{T}_1\Psi)(\nu)| &\leq\sum_{i=\nu}^{\infty}|a_2(i)||\psi(\beta_j(i-m))-\psi(\beta(i-m))| \\ &\leq l_2\sum_{i=\nu_1}^{\infty}|a_2(i)||\beta_j(i-m)-\beta(i-m)|\to 0\quad\text{as } j\to\infty\end{aligned}$$

and

$$\begin{aligned}|(\mathbb{T}_2\Psi_j)(\nu)-(\mathbb{T}_2\Psi)(\nu)| &\leq\sum_{i=\nu}^{\infty}|a_3(i)||\phi(\alpha_j(i-l))-\phi(\alpha(i-l))| \\ &\leq l_1\sum_{i=\nu_1}^{\infty}|a_3(i)||\alpha_j(i-m)-\alpha(i-l)|\to 0\quad\text{as } j\to\infty\end{aligned}$$

imply that

$$\|(\mathbb{T}\Psi_j)-(\mathbb{T}\Psi)\|\to 0\; as\; j\to\infty,$$

that is, $\mathbb{T}$ is continuous. Next, we show that $\mathbb{T}\upsilon_1$ is uniformly Cauchy. Let $\epsilon>\frac{(1+c_1)}{12}>0$ be given. We can find $\nu_2>\nu_1$ such that for $\nu\geq\nu_2$,

$$\begin{aligned}&\sum_{i=\nu}^{\infty}a_2(i)|\psi(\beta(i-m))|<\epsilon, \\ &\sum_{i=\nu}^{\infty}a_3(i)|\phi(\alpha(i-m))|<\epsilon\end{aligned}$$

due to (A_2) and the definitions of $P_1(\nu)$ and $P_2(\nu)$. For $\nu_4 > \nu_3 > \nu_2$, it follows that

$$\begin{aligned}|(\mathbb{T}_1\Psi)(\nu_4) - (\mathbb{T}_1\Psi)(\nu_3)| &= \Big|\sum_{i=\nu_4}^{\infty} a_2(i)\psi(\beta(i-m)) - \sum_{i=\nu_3}^{\infty} a_2(i)\psi(\beta(i-m))\Big| \\ &= \Big|\sum_{i=\nu_3}^{\nu_4} a_2(i)\psi(\beta(i-m))\Big| \\ &\le \sum_{i=\nu_3}^{\infty} |a_2(i)||\psi(\beta(i-m))| < \epsilon\end{aligned}$$

and

$$\begin{aligned}|(\mathbb{T}_2\Psi)(\nu_4) - (\mathbb{T}_2\Psi)(\nu_3)| &= \Big|\sum_{i=\nu_4}^{\infty} a_3(i)\phi(\alpha(i-l)) - \sum_{i=\nu_3}^{\infty} a_3(i)\phi(\alpha(i-l))\Big| \\ &= \Big|\sum_{i=\nu_3}^{\nu_4} a_3(i)\phi(\alpha(i-l))\Big| \\ &\le \sum_{i=\nu_3}^{\infty} |a_3(i)||\phi(\alpha(i-l))| < \epsilon,\end{aligned}$$

that is, $\mathbb{T}\upsilon_1$ is uniformly Cauchy for every $\nu_3, \nu_4 > \nu_2$. Hence by the Krasnoselskii's fixed point theory, there exists a solution $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ of (NAS1) in υ_1 such that $(\mathcal{T}\Psi)(\nu) + (\mathbb{T}\Psi)(\nu) = \Psi(\nu)$ for $\nu \ge \nu_2$. Looking at the fact that

$$(\mathcal{T}_1\Psi)(\nu) + (\mathbb{T}_1\Psi)(\nu) = \alpha(\nu), \qquad (\mathcal{T}_2\Psi)(\nu) + (\mathbb{T}_2\Psi)(\nu) = \beta(\nu)$$

for $\nu \ge \nu_2$, it is easy to verify that $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ is the required vector solution of (NAS1). This completes the proof of the theorem. □

Theorem 2.14. *Let $0 \le q(\nu) \le c_2 < 1$ for large ν. Suppose that $\phi, \psi \in$ Lip$\{l_1, l_2, [t_1, t_2], 0 < t_1 < t_2 < \infty\}$. If* ($A_1$) *and* ($A_2$) *hold, then every vector solution of* (NAS1) *strongly oscillates or converges to zero as $\nu \to \infty$ if and only if* (A_3) *holds.*

Proof. Let $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ be a vector solution of (NAS1). If it oscillates, then there is nothing to prove. With (A_3), let $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ be a strongly nonoscillatory vector solution of (NAS1), then proceeding like Theorem 2.13, we can undertake four similar cases. In each case, we can also find $\lim_{\nu\to\infty} \Psi(\nu) = 0$ by using Lemmas 2.5, 2.6, 2.7 and 2.8, respectively.

Conversely, we assume that (A_3) fails to hold. Let $\mathcal{B} = l^{\infty}_{\nu_1}$ be the Banach space of all real bounded sequences $\Psi = [\alpha(\nu), \beta(\nu)]^T$ with the sup norm

$$\mathcal{B} = \{\Psi\colon \mathbb{N} \to \mathbb{R}^2 : \|\Psi\| = \sup_{\nu\in\mathbb{N}} |\Psi| < \infty\}.$$

Define

$$\upsilon_2 = \{\Psi \in \mathcal{B} : \alpha(\nu), \beta(\nu) \in I \times I, \nu \ge \nu_1\},$$

where $I = [\frac{1-c_2}{10}, 1]$. It is easy to see that $\upsilon_2 \subset \mathcal{B}$ is closed, bounded and convex. By using (A_1), we can find $\nu_1 > 0$ such that

$$\sum_{i=\nu}^{\infty} |a_1(i)| < \frac{(1-c_2)}{5l_1}, \qquad \sum_{i=\nu}^{\infty} a_2(i) < \frac{(1-c_2)}{5l_2},$$
$$\sum_{i=\nu}^{\infty} a_3(i) < \frac{(1-c_2)}{5l_1}, \qquad \sum_{i=\nu}^{\infty} |a_4(i)| < \frac{(1-c_2)}{5l_2}$$

and

$$\Big|\sum_{i=\nu}^{\infty} \omega_1(i)\Big| < \frac{(1-c_2)}{10}, \qquad \Big|\sum_{i=\nu}^{\infty} \omega_2(i)\Big| < \frac{(1-c_2)}{10}$$

for $\nu \geq \nu_1$, where $l_1 = \max\{u_1, \phi(1)\}$, $l_2 = \max\{u_2, \psi(1)\}$, and u_1, u_2 are the Lipschitz constants of ϕ and ψ, respectively, on $[\frac{1-c_2}{10}, 1]$. We define the maps $\mathcal{T}, \mathbb{T} \colon \upsilon_2 \to \mathcal{B}$ such that

$$(\mathcal{T}\Psi)(\nu) = \begin{bmatrix} \dfrac{(2+3c_2)}{5} - q(\nu)\alpha(\nu-p) - \displaystyle\sum_{i=\nu}^{\infty} a_1(i)\phi(\alpha(i-l)) - \sum_{i=\nu}^{\infty} \omega_1(i) \\ \dfrac{(2+3c_2)}{5} - q(\nu)\beta(\nu-p) - \displaystyle\sum_{i=\nu}^{\infty} a_4(i)\psi(\beta(i-m)) - \sum_{i=\nu}^{\infty} \omega_2(i) \end{bmatrix}, \quad \nu \geq \nu_1,$$
$$(\mathcal{T}\Psi)(\nu) = (\mathcal{T}\Psi)(\nu_1) \quad \text{for } \nu_1 - \eta < \nu < \nu_1$$

and

$$(\mathbb{T}\Psi)(\nu) = \begin{bmatrix} -\displaystyle\sum_{i=\nu}^{\infty} a_2(i)\psi(\beta(i-m)) \\ -\displaystyle\sum_{i=\nu}^{\infty} a_3(i)\phi(\alpha(i-l)) \end{bmatrix} \quad \text{for } \nu \geq \nu_1,$$
$$(\mathbb{T}\Psi)(\nu) = (\mathbb{T}\Psi)(\nu_1) \quad \text{for } \nu_1 - \eta < \nu < \nu_1.$$

Without any loss of generality,

$$\mathcal{T} = \begin{bmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{bmatrix}, \qquad \mathbb{T} = \begin{bmatrix} \mathbb{T}_1 \\ \mathbb{T}_2 \end{bmatrix}.$$

The rest of the proof of the theorem follows from the proof of Theorem 2.13, and hence the details are omitted. □

Theorem 2.15. *Let* $1 \leq c_3 \leq q(\nu) \leq c_4 < \frac{(c_3)^2}{2} < \infty$ *for large* ν. *Suppose that* $\phi, \psi \in \mathrm{Lip}\{l_1, l_2, [t_3, t_4], -\infty < t_3 < t_4 < \infty\}$. *If* (A_1) *and* (A_2) *hold, then every vector solution of* (NAS1) *either strongly oscillates or converges to zero as* $\nu \to \infty$ *if and only if* (A_3) *holds.*

Proof. The first half of the proof follows from Theorem 2.14. Here, also in each case, by using same Lemmas followed by Theorem 2.14, we can find $\lim_{\nu\to\infty} \Psi(\nu) = 0$. For the remaining half, $\mathcal{B} = l^{\infty}_{\nu_1}$ is the Banach space of all real bounded sequences $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ with the sup norm

$$\mathcal{B} = \{\Psi \colon \mathbb{N} \to \mathbb{R}^2 : \|\Psi\| = \sup_{\nu\in\mathbb{N}} |\Psi| < \infty\},$$

and for a fixed interval I,

$$\upsilon_3 = \{\Psi \in \mathcal{B} : (\alpha(\nu), \beta(\nu)) \in I \times I, \nu \geq \nu_1\},$$

where $I = [\frac{c_3-1}{8c_3c_4}, 1]$ and $\upsilon_3 \subset \mathcal{B}$ is closed, bounded and convex. Due to (A_1), we can find $\nu_1 > 0$ such that

$$\sum_{i=\nu}^{\infty} |a_1(i)| < \frac{(c_3-1)}{4c_3l_1}, \qquad \sum_{i=\nu}^{\infty} a_2(i) < \frac{(c_3-1)}{8c_4l_2},$$
$$\sum_{i=\nu}^{\infty} a_3(i) < \frac{(c_3-1)}{8c_4l_1}, \qquad \sum_{i=\nu}^{\infty} |a_4(i)| < \frac{(c_3-1)}{4c_3l_2}$$

and

$$\left|\sum_{i=\nu}^{\infty} \omega_1(i)\right| < \frac{(c_3-1)}{8c_3c_4}, \qquad \left|\sum_{i=\nu}^{\infty} \omega_2(i)\right| < \frac{(c_3-1)}{8c_3c_4}$$

for $\nu \geq \nu_1$, where $l_1 = \max\{u_1, \phi(1)\}$, $l_2 = \max\{u_2, \psi(1)\}$, and u_1, u_2 are the Lipschitz constants of ϕ and ψ, respectively, on $[\frac{c_3-1}{8c_3c_4}, 1]$.

Define the maps $\mathcal{T}, \mathbb{T} \colon \upsilon_3 \to \mathcal{B}$ such that

$$(\mathcal{T}\Psi)(\nu) = \begin{bmatrix} \frac{(2c_3^2+c_3-1)}{4c_3q(\nu+p)} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} a_1(i)\phi(\alpha(i-l)) - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} \omega_1(i) \\ \frac{(2c_3^2+c_3-1)}{4c_3q(\nu+p)} - \frac{\beta(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} a_4(i)\psi(\beta(i-m)) - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} \omega_1(i) \end{bmatrix},$$

for $\nu \geq \nu_1$,

$$(\mathcal{T}\Psi)(\nu) = (\mathcal{T}\Psi)(\nu_1) \quad \text{for } \nu_1 - \eta < \nu < \nu_1,$$

and

$$(\mathbb{T}\Psi)(\nu) = \begin{bmatrix} -\frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} a_2(i)\psi(\beta(i-m)) \\ -\frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} a_3(i)\phi(\alpha(i-l)) \end{bmatrix} \quad \text{for } \nu \geq \nu_1,$$
$$(\mathbb{T}\Psi)(\nu) = (\mathbb{T}\Psi)(\nu_1) \quad \text{for } \nu_1 - \eta < \nu < \nu_1.$$

Indeed,

$$\mathcal{T} = \begin{bmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{bmatrix}, \qquad \mathbb{T} = \begin{bmatrix} \mathbb{T}_1 \\ \mathbb{T}_2 \end{bmatrix}.$$

Let $\Psi, \Upsilon \in \upsilon_3$. For $\nu \geq \nu_1$,

$$\begin{aligned}(\mathcal{T}_1\Psi)(\nu) + (\mathbb{T}_1\Upsilon)(\nu) &= \frac{(2c_3^2+c_3-1)}{4c_3q(\nu+p)} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} a_1(i)\phi(\alpha(i-l)) \\ &\quad - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} \omega_1(i) - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} a_2(i)\psi(\beta(i-m))\end{aligned}$$

$$\le \frac{(2c_3^2+c_3-1)}{4c_3^2} + \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} |a_1(i)\|\phi(\alpha(i-l)) - \phi(0)|$$

$$+ \frac{1}{q(\nu+p)}\left| \sum_{i=\nu+p}^{\infty} \omega_1(i)\right| + \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} |a_2(i)\|\psi(\beta(i-m)) - \psi(0)|$$

$$\le \frac{(2c_3^2+c_3-1)}{4c_3^2} + \phi(1)\frac{1}{c_3} \sum_{i=\nu+p}^{\infty} |a_1(i)| + \frac{1}{c_3}\left[\frac{c_3-1}{8c_3c_4}\right]$$

$$+ \psi(1)\frac{1}{c_3} \sum_{i=\nu+p}^{\infty} |a_2(i)|$$

$$\le \frac{(2c_3^2+c_3-1)}{4c_3^2} + \frac{c_3-1}{4c_3^2} + \frac{c_3-1}{8c_3^2} + \frac{c_3-1}{8c_3^2}$$

$$= \frac{4c_3^2+6c_3-6}{8c_3^2} < 1$$

and

$$(\mathcal{T}_1\Psi)(\nu) + (\mathbb{T}_1\Upsilon)(\nu) = \frac{(2c_3^2+c_3-1)}{4c_3 q(\nu+p)} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} a_1(i)\phi(\alpha(i-l))$$

$$- \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} \omega_1(i) - \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} a_2(i)\psi(\beta(i-m))$$

$$\ge \frac{(2c_3^2+c_3-1)}{4c_3c_4} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)}\left| \sum_{i=n+p}^{\infty} \omega_1(i)\right|$$

$$- \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} |a_2(i)||\psi(\beta(i-m)) - \psi(0)|$$

$$\ge \frac{(2c_3^2+c_3-1)}{4c_3c_4} - \frac{1}{c_3} - \frac{1}{c_3}\left[\frac{c_3-1}{8c_3c_4}\right] - \psi(1)\frac{1}{c_3}\left[\frac{c_3-1}{8c_4l_2}\right]$$

$$\ge \frac{(2c_3^2+c_3-1)}{4c_3c_4} - \frac{1}{c_3} - \frac{c_3-1}{8c_3c_4} - \frac{c_3-1}{8c_3c_4}$$

$$= \frac{4c_3^2-8c_4}{8c_3c_4} > \frac{c_3-1}{8c_3c_4},$$

that is, $(\mathcal{T}_1\Psi)(\nu)+(\mathbb{T}_1\Upsilon)(\nu) \in \upsilon_3$, and similar observation follows when $(\mathcal{T}_2\Psi)(\nu)+(\mathbb{T}_2\Upsilon)(\nu) \in \upsilon_3$. For Ψ_1 and $\Psi_2 \in \upsilon_3$,

$$|\mathcal{T}_1\Psi_1(\nu) - \mathcal{T}_1\Psi_2(\nu)|$$

$$\le \frac{1}{c_3}\|\alpha_1(\nu+p) - \beta_2(\nu+p)\| + \frac{1}{q(\nu+p)} \sum_{i=\nu+p}^{\infty} |a_1(i)||\phi(\alpha_1(i-l)) - \phi(\alpha_2(i-l))|$$

$$\le \frac{1}{c_3}\|\alpha_1 - \alpha_2\| + \frac{1}{c_3}\left[\frac{c_3-1}{4c_3l_1}\right] |\phi(\alpha_1(i-l)) - \phi(\alpha_2(i-l))|$$

$$\leq \frac{1}{c_3}\|\alpha_1-\alpha_2\| + \left[\frac{c_3-1}{4c_3^2 l_1}\right]\|\alpha_1-\alpha_2\|$$
$$\leq \left[\frac{1}{c_3}+\frac{c_3-1}{4c_3^2}\right]\|\alpha_1-\alpha_2\| = \frac{5c_3-1}{4c_3^2}\|\alpha_1-\alpha_2\|$$

and so also, we can find that $|\mathcal{T}_2\Psi_1(\nu) - \mathcal{T}_2\Psi_2(\nu)| \leq \frac{5c_3-1}{4c_3^2}\|\beta_1-\beta_2\|$ for $\nu \geq \nu_1$. As a result,

$$\|\mathcal{T}\Psi_1 - \mathcal{T}\Psi_2\| \leq \frac{5c_3-1}{4c_3^2}\|\Psi_1-\Psi_2\|$$

shows that $\mathcal{T}$ is a contraction mapping.

For continuity of $\mathbb{T}$, we let $\Psi_j(\nu) = [\alpha_j(\nu), \beta_j(\nu)]^T \in \upsilon_3$ for any $j \in \mathbb{N}$, and $\Psi_j(\nu)$ has the property: $\alpha_j(\nu) \to \alpha(\nu)$, $\beta_j(\nu) \to \beta(\nu)$ as $j \to \infty$. If we choose $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$, then $\Psi_j(\nu) \in \upsilon_3$ if and only if $\Psi(\nu) \in \upsilon_3$, and hence, $\alpha(\nu), \beta(\nu) \in \left[\frac{c_3-1}{8c_3c_4}, 1\right]$ for $\nu \geq \nu_1$. The rest of this part follows from Theorem 2.14. Hence, the details are omitted. So, the theorem is proved. □

Theorem 2.16. *Let $-\infty < c_5 \leq q(\nu) \leq c_6 < -1$ for large ν. Suppose that $\phi, \psi \in \mathrm{Lip}\{l_1, l_2, [r_1, r_2], -\infty < r_1 < r_2 < \infty\}$, and let* (A$_1$), (A$_2$) *hold. Then every bounded vector solution of* (NAS1) *either strongly oscillates or converges to zero if and only if* (A$_3$) *holds.*

Proof. The sufficient part follows from Theorem 2.13. Also, in each case, we can obtain $\lim_{\nu\to\infty} \Psi(\nu) = 0$ by using Lemmas 2.9, 2.10, 2.11 and 2.12, respectively.

Conversely, we assume that

$$\sum_{\nu=0}^{\infty} a_1(\nu) > -\infty, \qquad \sum_{\nu=0}^{\infty} a_4(\nu) > -\infty.$$

Let $\mathcal{B} = l^{\infty}_{\nu_1}$ be the Banach space of all bounded sequence in $\mathbb{R}^2$ with the sup norm

$$\mathcal{B} = \{\Psi\colon \mathbb{N} \to \mathbb{R}^2 : |\Psi| = \sup_{\nu\in\mathbb{N}} |\Psi| < \infty\},$$

and define

$$\upsilon_4 = \{\Psi \in \mathcal{B} : (\alpha(\nu), \beta(\nu)) \in I \times I, \nu \geq \nu_1\},$$

where $I = [\frac{-c_6}{M-c_6}, \Lambda]$. Indeed, $\upsilon_4 \subset \mathcal{B}$ is closed, bounded and convex. Because of (A$_1$), we choose

$$M > \max\left\{-c_5, c_6 + \frac{c_6}{1+c_6}\right\}, \qquad \Lambda = \frac{2M-(M+1)c_6}{(c_6-M)(1+c_6)} > 0$$

such that

$$\sum_{\nu=\nu_1}^{\infty} |a_1(\nu)| < \frac{(-c_6)}{2(M-c_6)l_1}, \qquad \sum_{\nu=\nu_1}^{\infty} a_2(\nu) < \frac{(-c_6)}{2(M-c_6)l_2},$$
$$\sum_{\nu=\nu_1}^{\infty} a_3(\nu) < \frac{(-c_6)}{2(M-c_6)l_1}, \qquad \sum_{\nu=\nu_1}^{\infty} |a_4(\nu)| < \frac{(-c_6)}{2(M-c_6)l_2}$$

and

$$\Big|\sum_{\nu=\nu_1}^{\infty}\omega_1(\nu)\Big| < \frac{(-c_6)}{2(M-c_6)}, \qquad \Big|\sum_{\nu=\nu_1}^{\infty}\omega_2(\nu)\Big| < \frac{(-c_6)}{2(M-c_6)}$$

for $\nu \geq nu_1$, where l_1, l_2 are Lipschitz constants of ϕ and ψ, respectively, on $[\frac{-c_6}{(M-c_6)}, \Lambda]$. Define the maps $\mathcal{T}, \mathbb{T}\colon \upsilon_4 \to \mathcal{B}$ such that

$$(\mathcal{T}\Psi)(\nu) = \begin{bmatrix} \frac{-(2-c_6)M}{(M-r_3)q(\nu+p)} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_1(\nu)\phi(\alpha(i-l)) - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty}\omega_1(i) \\ \frac{-(2-c_6)M}{(M-c_6)q(\nu+p)} - \frac{\beta(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_4(\nu)\psi(\beta(i-m)) - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty}\omega_2(i) \end{bmatrix}$$

for $\nu \geq \nu_1$,

$(\mathcal{T}\Psi)(\nu) = (\mathcal{T}\Psi)(\nu_1)$ for $\nu_1 - \eta < \nu < \nu_1$,

and

$$(\mathbb{T}\Psi)(\nu) = \begin{bmatrix} -\frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_2(i)\psi(\beta(i-m)) \\ -\frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_3(i)\phi(\alpha(i-l)) \end{bmatrix} \quad \text{for } \nu \geq \nu_1,$$

$$(\mathbb{T}\Psi)(\nu) = (\mathbb{T}\Psi)(\nu_1) \quad \text{for } \nu_1 - \eta < \nu < \nu_1.$$

We note that

$$\mathcal{T} = \begin{bmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{bmatrix}, \qquad \mathbb{T} = \begin{bmatrix} \mathbb{T}_1 \\ \mathbb{T}_2 \end{bmatrix}.$$

Let $\Psi, \Upsilon \in \upsilon_4$. Then for $\nu \geq \nu_1$,

$$\begin{aligned}
&(\mathcal{T}_1\Psi)(n) + (\mathbb{T}_1\Upsilon)(\nu) \\
&= \frac{-(2-c_6)M}{(M-c_6)q(\nu+p)} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_1(i)\phi(\alpha(i-l)) \\
&\quad - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty}\omega_1(i) - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_2(i)\psi(\beta(i-m)) \\
&\leq \frac{-(2-c_6)M}{(M-c_6)c_6} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty}\omega_1(i) \\
&\quad - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_2(i)\psi(\beta(i-m)) \\
&\leq \frac{-(2-c_6)M}{(M-c_6)(-c_5)} - \frac{1}{c_5}\frac{(-c_6)}{(M-c_6)} - \frac{1}{q(\nu+p)}\Big|\sum_{i=\nu+p}^{\infty}\omega_1(i)\Big| \\
&\quad - \frac{1}{q(n+p)}\sum_{i=n+p}^{\infty}|a_2(i)||\psi(\beta(i-m)) - \psi(0)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{-(2-c_6)M}{(M-c_6)c_6} - \frac{L}{c_6} + \frac{1}{2(M-c_6)} + \frac{1}{2(M-c_6)}\\
&\leq \frac{-(2-c_6)M}{(M-c_6)c_6} - \frac{L}{c_6} + \frac{1}{(M-c_6)}\\
&\leq \frac{-(2-c_6)M - L(M-c_6) + c_6}{(M-c_6)c_6}\\
&\leq -\left[\frac{2M-(M+1)c_6\{1-\frac{1}{-(1+c_6)}\}}{(M-c_6)c_6}\right]\\
&= \left[\frac{2M-(M+1)c_6\{1-\frac{1}{-(1+c_6)}\}}{(c_6-M)(1+c_6)}\right] = \Lambda,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathcal{T}_1\Psi)(\nu) + (\mathbb{T}_1\Upsilon)(\nu)\\
&= \frac{-(2-c_6)M}{(M-c_6)q(\nu+p)} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_1(i)\phi(\alpha(i-l))\\
&\quad - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty}\omega_1(i) - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_2(i)\psi(\beta(i-m))\\
&\geq \frac{-(2-c_6)M}{(M-c_6)q(\nu+p)} - \frac{\alpha(\nu+p)}{q(\nu+p)} - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty} a_1(i)\phi(\alpha(i-l))\\
&\quad - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty}\omega_1(i)\\
&\geq \frac{-(2-c_6)M}{(M-c_6)(-c_5)} - \frac{1}{c_5}\frac{(-c_6)}{(M-c_6)} - \frac{1}{q(\nu+p)}\Big|\sum_{i=\nu+p}^{\infty}\omega_1(i)\Big|\\
&\quad - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty}|a_1(i)||\phi(\alpha(i-l)) - \phi(0)|\\
&\geq \frac{-(2-c_6)M}{(M-c_6)c_5} + \frac{c_6}{(M-c_6)c_5} - \frac{1}{(-c_6)}\frac{(-c_6)}{2(M-c_6)} - \frac{1}{(-c_6)}\frac{(-c_6)\Lambda}{2(M-c_6)}\\
&\geq \frac{-(2-r_3)M}{(M-c_6)c_5} + \frac{c_6}{(M-c_6)c_5} - \frac{1}{(M-c_6)}\\
&\geq \frac{-Mc_6}{-(M-c_6)c_5} = -\frac{c_6}{(M-c_6)}.
\end{aligned}
$$

Comparable findings can be acquired for $(\mathcal{T}_2\Psi)(\nu) + (\mathbb{T}_2\Upsilon)(\nu)$, $\nu \geq \nu_1$. Hence, $\mathcal{T}\Psi + \mathbb{T}\Upsilon \in \upsilon_4$.

For Ψ_1 and $\Psi_2 \in \upsilon_4$, it is easy to verify that

$$\begin{aligned}|(\mathcal{T}_1\Psi_1)(\nu) - (\mathcal{T}_1\Psi_2)(\nu)| &\leq -\frac{1}{c_6}|\alpha_1(\nu+p) - \alpha_2(\nu+p)| \\ &\quad - \frac{1}{q(\nu+p)}\sum_{i=\nu+p}^{\infty}|a_1(i)||\phi(\alpha_1(i-l)) - \phi(\alpha_2(i-l))| \\ &\leq -\frac{1}{c_6}|\alpha_1 - \alpha_2| + \frac{c_6}{(M-c_6)}\|\alpha_1 - \alpha_2\| \\ &\leq -\frac{1}{c_6}|\alpha_1 - \alpha_2| + \frac{c_6}{M-c_6}\|\alpha_1 - \alpha_2\| \\ &\leq \left[\frac{1}{M-c_6} - \frac{1}{c_6}\right]\|\alpha_1 - \alpha_2\|\end{aligned}$$

and

$$|(\mathcal{T}_2\Psi_1)(\nu) - (\mathcal{T}_2\Psi_2)(\nu)| \leq \left[\frac{1}{M-c_6} - \frac{1}{c_6}\right]\|\beta_1 - \beta_2\|$$

for $\nu \geq \nu_1$, that is,

$$\|\mathcal{T}\Psi_1 - \mathcal{T}\Psi_2\| \leq \left[\frac{1}{M-c_6} - \frac{1}{c_6}\right]\|\Psi_1 - \Psi_2\|.$$

Therefore, $\mathcal{T}$ is a contraction mapping.

The rest of the proof follows from the proof of Theorem 2.13. Therefore, we can prove that $\mathbb{T}$ is continuous whereas $\mathbb{T}\upsilon_4$ is uniformly Cauchy. So, by using the Krasnoselskii's fixed point theorem, there exists a solution $\Psi(\nu) = [\alpha(\nu), \beta(\nu)]^T$ of (NAS1) in υ_4 such that $(\mathcal{T}\Psi)(\nu) + (\mathbb{T}\Psi)(\nu) = \Psi(\nu)$ for $\nu \geq \nu_1$. This completes the proof of the theorem. □

Remark. The prototype of ϕ and ψ in Theorems 2.13, 2.14 and 2.15 could be of the form

$$\begin{bmatrix}\phi(\alpha)\\ \psi(\beta)\end{bmatrix} = \begin{bmatrix}\dfrac{\alpha}{\tau^2+\alpha^2}\\ \dfrac{\beta}{\rho^2+\beta^2}\end{bmatrix},$$

and for Theorem 2.16, ϕ and ψ could be of the form

$$\begin{bmatrix}\phi(\alpha)\\ \psi(\beta)\end{bmatrix} = \begin{bmatrix}|\alpha|^{\tau_1}\operatorname{sgn}\alpha\\ |\beta|^{\tau_2}\operatorname{sgn}\beta\end{bmatrix},$$

where τ_1 and τ_2 are the ratio of odd positive integers.

3. Discussion and examples

In this effort, we have discussed the oscillatory, asymptotic behaviour and existence of nonoscillatory vector solutions of (NAS1) in which $a_1(\nu) < 0$, $a_2(\nu) > 0$, $a_3(\nu) > 0$, $a_4(\nu) < 0$ for all large ν along with the hypotheses (A_1), (A_2) and (A_3). But, the results of Section 2 are not applicable to the following example.

Example 3.1. Consider a two-dimensional nonlinear nonautonomous neutral difference system of the form:
(NAS5)

$$\Delta\begin{bmatrix}\alpha(\nu)+\mathrm{e}^{-\nu}\alpha(\nu-2)\\ \beta(\nu)+\mathrm{e}^{-\nu}\beta(\nu-2)\end{bmatrix}=\begin{bmatrix}a_1(\nu) & a_2(\nu)\\ a_3(\nu) & a_4(\nu)\end{bmatrix}\begin{bmatrix}\dfrac{\alpha(\nu-2)}{1+\alpha^2(\nu-2)}\\ \dfrac{\beta(\nu-2)}{1+\beta^2(\nu-2)}\end{bmatrix}+\begin{bmatrix}(-1)^\nu\mathrm{e}^{-\nu}\\ (-1)^\nu\mathrm{e}^{-2\nu}\end{bmatrix}$$

for $\nu > 2$, where

$$\begin{aligned}
a_1(\nu) &= (1+\mathrm{e}^{-2(\nu-2)})(\mathrm{e}^{-(\nu+2)}+\mathrm{e}^{-\nu}),\\
a_2(\nu) &= -(1+4\mathrm{e}^{-2(\nu-2)})(\frac{1}{2}\mathrm{e}^{-3}+\mathrm{e}^{-2}+\mathrm{e}^{-(\nu+2)}+\mathrm{e}^{-\nu}),\\
a_3(\nu) &= -(1+\mathrm{e}^{-2(\nu-2)})(2\mathrm{e}^{-3}+2\mathrm{e}^{-2}+5\mathrm{e}^{-(\nu+2)}+2\mathrm{e}^{-\nu}),\\
a_4(\nu) &= (1+4\mathrm{e}^{-2(\nu-2)})(\mathrm{e}^{-(\nu+2)}).
\end{aligned}$$

Indeed,

$$(\mathrm{A}_4)\ \sum_{\nu=0}^{\infty}a_1(\nu)<\infty,\quad \sum_{\nu=0}^{\infty}a_4(\nu)<\infty,$$

$$(\mathrm{A}_5)\ \sum_{\nu=0}^{\infty}a_2(\nu)=-\infty,\quad \sum_{\nu=0}^{\infty}a_3(\nu)=-\infty.$$

Even if, the conditions of Theorem 2.13 are not satisfied, still (NAS5) admits a solution $\Psi(\nu)=[(-1)^\nu\mathrm{e}^{-\nu},2(-1)^\nu\mathrm{e}^{-\nu}]^T$, which is oscillatory and converge to zero as well.

Hence, with the above example, we are partially satisfied with the following results:

Theorem 3.2. *Let $a_1(\nu)>0$, $a_2(\nu)<0$, $a_3(\nu)<0$, $a_4(\nu)>0$ and $-1<c_1\le q(\nu)\le 0$ for large ν. Suppose that $\phi,\psi\in\mathrm{Lip}\{l_1,l_2,[a,b],-\infty<a<b<\infty\}$ and (A_1) and (A_4) hold. Then every bounded vector solution of* (NAS1) *either strongly oscillates or converges to zero if and only if (A_5) holds.*

Theorem 3.3. *Let $0\le q(\nu)\le c_2<1$ for large ν, and let (A_1) and (A_4) hold. Suppose that $\phi,\psi\in\mathrm{Lip}\{l_1,l_2,[\gamma_1,\gamma_2],0<\gamma_1<\gamma_2<\infty\}$. Then every bounded vector solution of* (NAS1) *strongly oscillates or converges to zero as $\nu\to\infty$ if and only if (A_5) holds.*

Theorem 3.4. *Let $1\le c_3\le q(\nu)\le c_4<\frac{(c_3)^2}{2}<\infty$ for large ν. Assume that $\phi,\psi\in\mathrm{Lip}\{l_1,l_2,[\kappa_1,\kappa_2],0<\kappa_1<\kappa_2<\infty\}$. If (A_1), (A_4) and (A_5) hold, then the conclusion of* Theorem 3.3 *holds true.*

Theorem 3.5. *Let $-\infty<c_5\le q(\nu)\le c_6<-1$ for large ν. Assume that $\phi,\psi\in\mathrm{Lip}\{l_1,l_2,[\varpi_1,\varpi_2],-\infty<\varpi_1<\varpi_2<\infty$, and let (A_1), (A_4) and (A_5) hold, then the conclusion of the theorem remains intact.*

We conclude this section with the following examples to verify our main results of Section 2.

Example 3.6. Consider a two-dimensional nonlinear nonautonomous neutral difference system of the form:
(NAS6)

$$\Delta \begin{bmatrix} \alpha(\nu) + \mathrm{e}^{-\nu}\alpha(\nu-2) \\ \beta(\nu) + \mathrm{e}^{-\nu}\beta(\nu-2) \end{bmatrix} = \begin{bmatrix} a_1(\nu) & a_2(\nu) \\ a_3(\nu) & a_4(\nu) \end{bmatrix} \begin{bmatrix} \dfrac{\alpha(\nu-2)}{1+\alpha^2(\nu-2)} \\ \dfrac{\beta(\nu-2)}{1+\beta^2(\nu-2)} \end{bmatrix} + \begin{bmatrix} (-1)^{\nu}\mathrm{e}^{-\nu} \\ (-1)^{\nu}\mathrm{e}^{-2\nu} \end{bmatrix}$$

for $\nu > 2$, where

$$\begin{aligned} a_1(\nu) &= -(1+\mathrm{e}^{-2(\nu-2)})(\mathrm{e}^{-3} + 2\mathrm{e}^{-2} + 2\mathrm{e}^{-(\nu+2)} + \mathrm{e}^{-\nu}), \\ a_2(\nu) &= (1+4\mathrm{e}^{-2(\nu-2)})(\frac{1}{2}\mathrm{e}^{-(\nu+2)}), \\ a_3(\nu) &= (1+\mathrm{e}^{-2(\nu-2)})(\mathrm{e}^{-(\nu+2)}), \\ a_4(\nu) &= -(1+4\mathrm{e}^{-2(\nu-2)})(\mathrm{e}^{-3} + \mathrm{e}^{-2} + 2\mathrm{e}^{-(\nu+2)} + \mathrm{e}^{-\nu}). \end{aligned}$$

After verifying $(\mathrm{A}_1), (\mathrm{A}_2)$ and (A_3) of Theorem 2.13, we conclude that every vector solution of (NAS6) either strongly oscillates or converges to zero. In particular, $\Psi(\nu) = [(-1)^{\nu}\mathrm{e}^{-\nu}, 2(-1)^{\nu}\mathrm{e}^{-\nu}]^T$ is such a vector solution of (NAS6). Since our analysis deals with the components of the vector solution, we verify the solution of the above problem in the following Figure 1.

However, when our analysis is anticipated by means of vectors, we verify the problem through Figure 2.

Example 3.7. Consider a two-dimensional nonlinear nonautonomous neutral difference system of the form:

(NAS7)
$$\begin{aligned} &\Delta \begin{bmatrix} \alpha(\nu) + (-5-\mathrm{e}^{-\nu})\alpha(\nu-1) \\ \beta(\nu) + (-5-\mathrm{e}^{-\nu})\beta(\nu-1) \end{bmatrix} \\ &\qquad = \begin{bmatrix} a_1(\nu) & a_2(\nu) \\ a_3(\nu) & a_4(\nu) \end{bmatrix} \begin{bmatrix} \alpha^3(\nu-4) \\ \beta^3(\nu-4) \end{bmatrix} + \begin{bmatrix} (-1)^{\nu}\mathrm{e}^{-\nu} \\ 2(-1)^{\nu}\mathrm{e}^{-\nu} \end{bmatrix} \end{aligned}$$

for $\nu > 2$, where $a_1(\nu) = -(12 + 3\mathrm{e}^{-\nu} + \mathrm{e}^{-(\nu+1)})$, $a_2(\nu) = \frac{1}{8}\mathrm{e}^{-\nu}$, $a_3(\nu) = 2\mathrm{e}^{-(\nu+1)}$, $a_4(\nu) = -(3 + \frac{1}{2}\mathrm{e}^{-\nu} + \frac{1}{2}\mathrm{e}^{-(\nu+1)})$. After verifying $(\mathrm{A}_1), (\mathrm{A}_2)$ and (A_3) of Theorem 2.16, we conclude that every bounded vector solution of (NAS7) either strongly oscillates or converges to zero. In particular, $\Psi(\nu) = [(-1)^{\nu}, 2(-1)^{\nu}]^T$ is such a bounded vector solution of (NAS7), and we verify the solution componentwise in the following Figure 3.

However, when our analysis is anticipated by means of vectors, we verify the problem through Figure 4.

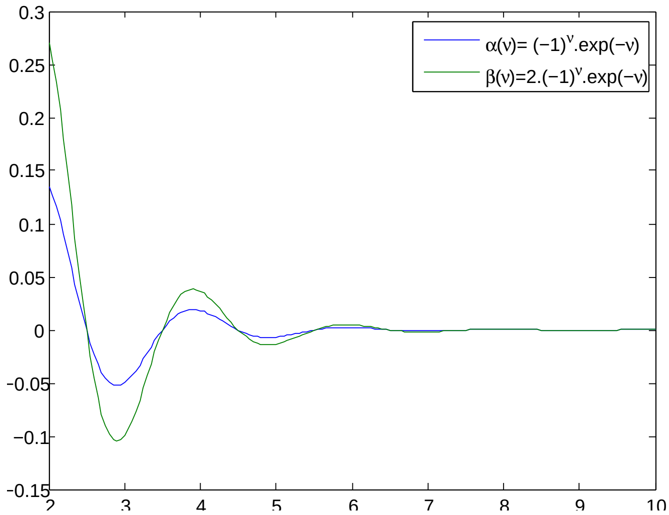

Figure 1. Componentwise solutions

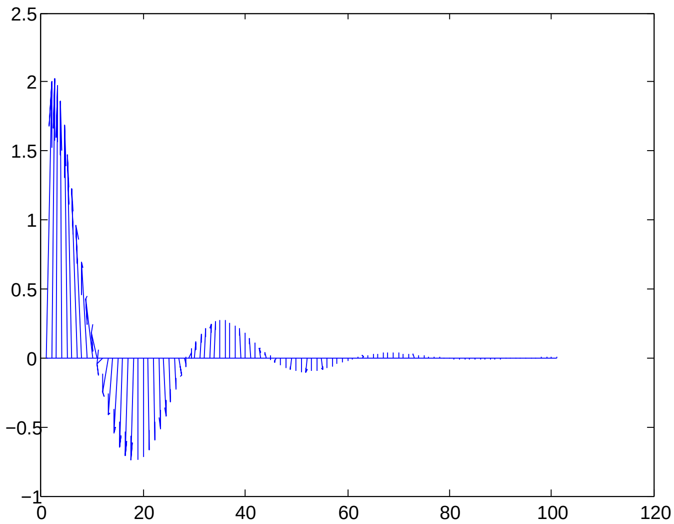

Figure 2. Vector solutions

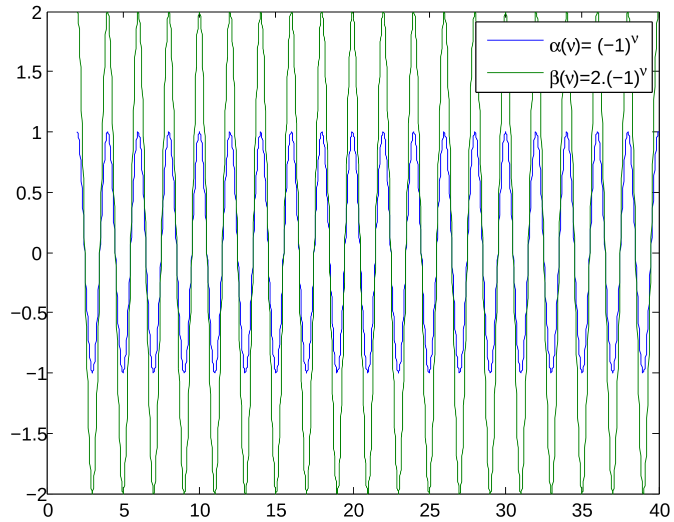

Figure 3. Componentwise solutions

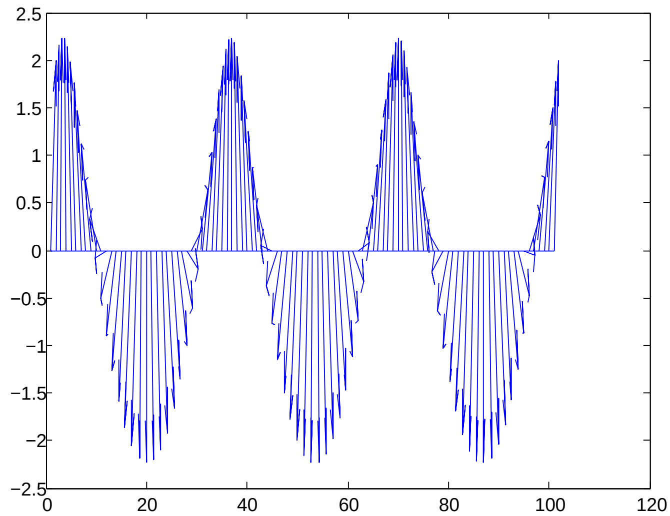

Figure 4. Vector solutions

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