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APPLICATIONS OF A RATHER GENERAL MEAN VALUE THEOREM FOR INTEGRALS

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Abstract. In this paper, we present mean value results for Volterra's operator and for integral of Flett's type. These are based on the following mean value theorem for integrals, due to the author:

> Let a and b be real numbers such that $a < b$, and let $\mathfrak{f}\colon [a,b] \longrightarrow \mathbb{R}$ be a gauge integrable function with $\int_a^b \mathfrak{f} = 0$. Then, for every ascending function $\mathfrak{z}\colon [a,b] \longrightarrow [0,\infty)$ which is nonconstant on (a,b), there exists $c \in (a,b)$ that satisfies $\int_a^c \mathfrak{z}\,\mathfrak{f} = 0$.

We also generalize several recently published problems to their natural setting.

1. Introduction

In this paper, which complements [**6**], we present a few mean value results that involve Volterra's operator and integral of Flett's type as well. These are based on a rather general integral mean value theorem (Theorem 2.3 below), and they extend results presented in [**2**], [**3**] and [**4**].

Before stating our contribution, let us fix some notations and definitions.

Henceforth, we adhere to the following terminology. A real-valued function $\mathfrak{h}$ defined on a nonempty set $\mathrm{D} \subseteq \mathbb{R}$ is termed *ascending* (respectively, *increasing*) if $\mathfrak{h}(x_1) \leq \mathfrak{h}(x_2)$ (respectively, $\mathfrak{h}(x_1) < \mathfrak{h}(x_2)$) whenever $x_1 < x_2$ are points of D. The notions of descending and decreasing functions are defined analogously.

Unless explicitly mentioned, every monotone function, which is either ascending or descending, is defined on a nondegenerate interval of $\mathbb{R}$ and is nonconstant on the set of interior points of its defining interval.

Let a and b be real numbers with $a < b$. The Volterra operator is defined on $\mathrm{L}^2[a,b]$ as follows: for any function $\mathfrak{f} \in \mathrm{L}^2[a,b]$, define $\mathrm{V}\mathfrak{f}$ by

$$\mathrm{V}\mathfrak{f}(x) = \int_a^x \mathfrak{f} \quad \text{for any } x \in [a,b].$$

It is known that V is a bounded linear operator from the Hilbert space $\mathrm{L}^2[a,b]$ into itself (with operatorial norm $2/\pi$) whose Hermitian adjoint $\mathrm{V}^\star\mathfrak{f}$ is given by

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the following formula:

$$\mathrm{V}^{\star}\mathfrak{f}(x) = \int_x^b \mathfrak{f}(t)\mathrm{d}t.$$

Volterra's operator has been studied intensively in the last decades because it is the simplest operator that exhibits a range of phenomena which can arise when one leaves the finite-dimensional real Euclidean spaces.

In this paper, instead of Riemann or Lebesgue integrability, we shall use the more general gauge integrability. A clear presentation of this theory, very accessible to undergraduate students, is given in Demailly's notes [**1**]. For instance, derivatives are gauge integrable.

Recall that if $\mathfrak{f}\colon [a,b] \longrightarrow \mathbb{R}$ is a gauge integrable function, then its indefinite integral $\mathrm{F}\colon [a,b] \longrightarrow \mathbb{R}$ is defined for any $x \in [a,b]$ by $\mathrm{F}(x) = \int_a^x \mathfrak{f}$. Note that the indefinite integral F is always continuous.

Let $\mathfrak{z}\colon [a,b] \longrightarrow \mathbb{R}$ be a monotone function. The above notions of $\mathrm{V}\mathfrak{f}$ and $\mathrm{V}^*\mathfrak{f}$ can be naturally extended as "weighted" Volterra operator $\mathrm{V}_{\mathfrak{z}}\mathfrak{f}$ with Hermitian adjoint $\mathrm{V}_{\mathfrak{z}}^*\mathfrak{f}$, respectively, by the following rules. For any $t \in [a,b]$, define

$$\mathrm{V}_{\mathfrak{z}}\mathfrak{f}(t) = \int_a^t \mathfrak{f}\mathfrak{z} \qquad \text{and} \qquad \mathrm{V}_{\mathfrak{z}}^*\mathfrak{f}(t) = \int_t^b \mathfrak{f}\mathfrak{z}.$$

Their existence is guaranteed by [**6**, Proposition 2].

2. Preliminaries

In this section, we collect a few results from [**6**] that will be used later. The following theorem may be regarded as an integral Rolle type result that pertains to an integral Flett theorem.

Theorem 2.1. *Let $\mathfrak{f}\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function with $\mathfrak{f}(a) = \mathfrak{f}(b)$. Then, for any ascending function $\mathfrak{z}\colon [a,b] \longrightarrow [0,\infty)$, there exists $c \in (a,b)$ that satisfies*

$$\int_a^c \mathfrak{z}\,\mathrm{d}\mathfrak{f} = 0.$$

This implies the following Riemann-Stieltjes integral Flett theorem (cf. [**6**]) which results by partial integration in the above the formula.

Theorem 2.2. *Let $\mathfrak{f}\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function with $\mathfrak{f}(a) = \mathfrak{f}(b)$. Then, for any monotone function $\mathfrak{z}\colon [a,b] \longrightarrow \mathbb{R}$, there exists $c \in (a,b)$ such that*

$$\int_a^c \mathfrak{f}\,\mathrm{d}\mathfrak{z} = (\mathfrak{z}(c) - \mathfrak{z}(a))\mathfrak{f}(c).$$

By choosing above $\mathfrak{z}(x) = x$ for all $x \in [a,b]$, we recover Wayment's result [**7**].

For the most part of this paper, the following mean value theorem for integrals [**6**], which is a consequence of Theorem 2.1, will be used extensively.

Theorem 2.3. *Let* $\mathfrak{h}\colon [a,b] \longrightarrow \mathbb{R}$ *be a gauge integrable function with* $\int_a^b \mathfrak{h} = 0$. *Then, for any ascending function* $\mathfrak{z}\colon [a,b] \longrightarrow [0,\infty)$, *there is* $c \in (a,b)$ *that satisfies*

$$\int_a^c \mathfrak{h}\,\mathfrak{z} = 0.$$

For $[a,b] = [-1,1]$, $\mathfrak{h}(x) = x$ for $x \in [-1,1]$, and $\mathfrak{z}$ is either $\mathfrak{h}$ (which has also negative values), or the constant function 1, or $\max\{-x,0\}$ for $x \in [-1,1]$ (which is descending and nonnegative), one has $\int_0^x \mathfrak{h}\mathfrak{z} \neq 0$ for all $x \in (-1,1)$, hence each hypothesis on $\mathfrak{z}$ in the statement of Theorem 2.3 is necessary.

3. Applications

The first package of applications concerns Volterra's operator. In the set-up fixed in the first section, the second mean value theorem for Riemann integrable functions has a nice interpretation which we quote from [**6**].

Theorem 3.1. *Let* $\mathfrak{f}\colon [a,b] \longrightarrow \mathbb{R}$ *be a gauge integrable function. Then, for any monotone function* $\mathfrak{z}\colon [a,b] \longrightarrow [0,\infty)$, *there exists* $c \in (a,b)$ *such that*

$$\mathrm{V}_{\mathfrak{z}}\mathfrak{f}(b) = \mathfrak{z}(a)\mathrm{V}\mathfrak{f}(c) + \mathfrak{z}(b)\mathrm{V}^*\mathfrak{f}(c).$$

In the same vein, we give the following mean value theorem.

Theorem 3.2. *Let* $\mathfrak{h}\colon [a,b] \longrightarrow \mathbb{R}$ *be a gauge integrable function with* $\int_a^b \mathfrak{h} = 0$. *Then there exist* $c, c_* \in (a,b)$ *such that*

$$\mathrm{V}_{\mathfrak{z}}\mathfrak{h}(c) = \mathfrak{z}(a)\cdot \mathrm{V}\mathfrak{h}(c) \qquad \textit{and} \qquad \mathrm{V}^*_{\mathfrak{z}}\mathfrak{h}(c_*) = \mathfrak{z}(b)\cdot \mathrm{V}^*\mathfrak{h}(c_*).$$

Proof. We consider only the first part of the conclusion. (The second part follows similarly by using the change of variable $[a,b] \ni x \mapsto a+b-x \in [a,b]$ and routine arguments.)

Then observe that, without loss in generality, we may assume that $\mathfrak{z}$ is ascending (otherwise consider $-\mathfrak{z}$ instead of $\mathfrak{z}$). Then apply Theorem 2.3 for $\mathfrak{z}-\mathfrak{z}(a)$ to obtain the desired result. □

Theorem 3.2 extends the main result in [**2**] (Theorem 2.1), which can be recovered when the function $\mathfrak{h}$ is continuous and $\mathfrak{z}$ is smooth of class $\mathcal{C}^1$ such that $\mathfrak{z}'(x) \neq 0$ for all $x \in [a,b]$.

In the same vein, the following results extend Theorem 2.4 and Corollary 2.6 from [**2**], respectively.

Proposition 3.3. *Let* $\mathfrak{f}, \mathfrak{g}\colon [a,b] \longrightarrow \mathbb{R}$ *be gauge integrable functions. Then there exists* $c \in (a,b)$ *that satisfies*

$$\mathrm{V}_{\mathfrak{z}}\mathfrak{f}(c)\int_a^b \mathfrak{g} - \mathrm{V}_{\mathfrak{z}}\mathfrak{g}(c)\int_a^b \mathfrak{f} = \mathfrak{z}(a)\Big(\mathrm{V}\mathfrak{f}(c)\int_a^b \mathfrak{g} - \mathrm{V}\mathfrak{g}(c)\int_a^b \mathfrak{f}\Big).$$

Proof. Consider the function $\mathfrak{h}\colon [a,b] \longrightarrow \mathbb{R}$ defined by setting for $x \in [a,b]$,

$$\mathfrak{h}(x) = \mathfrak{f}(x)\int_a^b \mathfrak{g} - \mathfrak{g}(x)\int_a^b \mathfrak{f},$$

and apply Theorem 2.3. □

The second package of applications extends mean value results that have recently appeared in [**2**], [**3**] and [**4**].

First, we improve Corollary 3.2 in [**3**], which is recovered from the following result for φ of class $\mathcal{C}^1$ and $\mathfrak{f}$ continuous.

Corollary 3.4. *Let $\varphi\colon [0,1] \longrightarrow \mathbb{R}$ be a differentiable increasing function with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then, for any gauge integrable function $\mathfrak{f}\colon [0,1] \longrightarrow \mathbb{R}$ such that*

$$\int_0^1 \varphi = \int_0^1 \varphi\mathfrak{f},$$

there exists $c \in (0,1)$ that satisfies

$$\int_0^c \varphi\mathfrak{f} = 0.$$

Proof. Following [**3**], let $\mathrm{F}\colon [0,1] \longrightarrow \mathbb{R}$ be the indefinite integral of $\mathfrak{f}$, that is $\mathrm{F}(x) = \int_0^x \mathfrak{f}$ for all $x \in [0,1]$. Then F is continuous [**1**], and by [**7**] we derive that

$$\int_0^1 \varphi\mathfrak{f} = \int_0^1 \varphi\,\mathrm{dF}.$$

Now, by partial integration in the right-hand equality above, the hypothesis of the corollary implies that

$$\int_0^1 \mathrm{F}\,\mathrm{d}\varphi = 0.$$

From this we infer that there exists $\alpha \in (0,1)$ such that $\mathrm{F}(\alpha) = 0$. Otherwise, the continuity of F and the intermediate value property for continuous functions imply that F has constant sign on $(0,1)$.

Suppose first that $\mathrm{F} > 0$ on $(0,1)$ and see what happens. Obviously, there exists $\delta > 0$ such that $\mathrm{F} \geq \delta$ on $[1/3, 1/2]$. Since φ is increasing, we have

$$0 = \int_0^1 \mathrm{F}\,\mathrm{d}\varphi \geq \int_{1/3}^{1/2} \mathrm{F}\,\mathrm{d}\varphi \geq \delta \cdot (\varphi(1/2) - \varphi(1/3)) > 0,$$

which is absurd.

The case $\mathrm{F} < 0$ on $(0,1)$ follows similarly, hence we omit the proof.

Therefore, there exists $\alpha \in (0,1)$ with $\mathrm{F}(\alpha) = 0$, that is, $\int_0^\alpha \mathfrak{f} = 0$.

Now, since φ is increasing *a fortiori* φ is ascending, by Theorem 2.3 there exists $c \in (0,\alpha)$ which verifies the conclusion. □

Corollary 3.5. *Let $\mathfrak{f}\colon [0,1] \longrightarrow \mathbb{R}$ be a gauge integrable function with $\int_0^1 \mathfrak{f} = 0$. Then, for any $n \in \mathbb{N}$, there exists $c \in (0,1)$ such that*

$$\int_0^c x^n \mathfrak{f}(x)\,\mathrm{d}x = 0.$$

In the same circle of ideas, we give further applications. Consider a gauge integrable function $\mathfrak{f}\colon [0,\pi/2] \longrightarrow \mathbb{R}$ with $\int_0^{\pi/2} \mathfrak{f} = 0$. Then there exists $c \in (0,\pi/2)$ such that

$$\int_8^c \mathfrak{f} \cdot \sin = 0.$$

In the same vein, if $a, b \in \mathbb{R}$, $0 \le a < b$, and $\mathfrak{f}\colon [a,b] \longrightarrow \mathbb{R}$ is gauge integrable such that $\int_a^b \mathfrak{f} = 0$, then there exists $c \in (a,b)$ which fulfills

$$\int_a^c \mathfrak{f} \cdot \arctan = 0.$$

Granting Theorem 2.3, we generalize the main result in [**3**] (Theorem 2.1) which is recovered from the following assertion if the function $\mathfrak{g}$ is smooth of class $\mathcal{C}^1$.

Corollary 3.6. *Let* $\omega\colon [0,1] \longrightarrow [0,\infty)$ *be an ascending function. Then, for any differentiable function* $\mathfrak{g}\colon [0,1] \longrightarrow \mathbb{R}$ *such that* $\mathfrak{g}(0) = \mathfrak{g}(1)$*, there exists* $c \in (0,1)$ *such that*

$$\int_0^c \mathfrak{g}'\omega = 0.$$

Proof. Apply Theorem 2.3 for $[a,b] = [0,1]$, $\mathfrak{h} = \mathfrak{g}'$ and $\mathfrak{z} = \omega$. □

The following corollary improves the main result in [**4**] and also several results from [**2**] and [**3**].

Corollary 3.7. *Let* $\mathfrak{f}, \mathfrak{g}\colon [a,b] \longrightarrow \mathbb{R}$ *be two gauge integrable functions. Then, for any ascending function* $\mathfrak{z}\colon [a,b] \longrightarrow [0,\infty)$*, there exists* $c \in (a,b)$ *with*

$$\int_a^b \mathfrak{f} \cdot \int_a^c \mathfrak{g}\mathfrak{z} = \int_a^b \mathfrak{g} \cdot \int_a^c \mathfrak{f}\mathfrak{z}.$$

Proof. Apply Theorem 2.3 to the gauge integrable function $\mathfrak{h}$ defined on $[a,b]$ by setting for any $x \in [a,b]$,

$$\mathfrak{h}(x) = \mathfrak{f}(x)\int_a^b \mathfrak{g} - \mathfrak{g}(x)\int_a^b \mathfrak{f}.$$ □

In particular, the following result implies Corollary 2.3 in [**3**] when the functions $\mathfrak{f}$ and $\mathfrak{g}$ are smooth of class $\mathcal{C}^1$.

Corollary 3.8. *Let* $\mathfrak{f}, \mathfrak{g}\colon [0,1] \longrightarrow \mathbb{R}$ *be two differentiable functions such that* $\mathfrak{f}(0) = \mathfrak{g}(0)$*. Then, for any ascending function* $\omega\colon [0,1] \longrightarrow [0,\infty)$*, there exists* $c \in (0,1)$ *such that*

$$\mathfrak{f}(1)\int_0^c \mathfrak{g}'\omega = \mathfrak{g}(1)\int_0^c \mathfrak{f}'\omega.$$

Here is a mean value Flett type result.

Theorem 3.9. *Let* $\mathfrak{f}\colon [a,b] \longrightarrow \mathbb{R}$ *be a continuous function such that* $\int_a^b \mathfrak{f} = 0$*. Let* $\mathfrak{z}\colon [a,b] \longrightarrow \mathbb{R}$ *be a continuous function such that* $\mathfrak{z}(a) = 0$*. Suppose that* $\mathfrak{z}$ *is* *differentiable on* (a, b) *and satisfies* $\mathfrak{z}'(x) \neq 0$ *for all* $x \in (a, b)$ *and* $\mathfrak{z}(x)/\mathfrak{z}'(x) \to 0$ *as* $x \to a^+$*. Then there exists a point* $c \in (a, b)$ *such that*

$$\mathfrak{z}'(c) \int_a^c \mathfrak{f}\mathfrak{z} = \mathfrak{z}^2(c)\mathfrak{f}(c).$$

Proof. Since $\mathfrak{z}'$ has Darboux property, $\mathfrak{z}'$ has constant sign, hence, without loss of generality, let $\mathfrak{z}' > 0$ on (a, b). As in [**3**], let $\mathrm{F}\colon [a, b] \longrightarrow \mathbb{R}$ be 0 at a and the quotient of $\int_a^x \mathfrak{f}\mathfrak{z}$ over $\mathfrak{z}(x)$ for $x \in (a, b]$. By Bernoulli-l'Hôpital's rule and the hypothesis, F results continuous at a. Then, by Theorem 2.3, there exists $b_* \in (a, b)$ with $\mathrm{F}(b_*) = 0$. Since F is differentiable on (a, b), *a fortiori* on (a, b_*), and $\mathrm{F}(a) = \mathrm{F}(b_*) = 0$, by Rolle's lemma, there is $c \in (a, b_*)$ such that $\mathrm{F}'(c) = 0$. Then the proof concludes by routine computations. □

Theorem 3.9 can be applied for $[a, b] = [0, 1]$ and $\mathfrak{z}(x) = x^n$ for $n \in \mathbb{N}$ in order to recover Corollary 3.2 in [**3**] (see also [**5**]) which we quote for reader's convenience.

Corollary 3.10. *Let* $\mathfrak{f}\colon [0, 1] \longrightarrow \mathbb{B}$ *be a continuous function such that* $\int_0^1 \mathfrak{f} = 0$*. Then, for any* $n \in \mathbb{N}$*, there exists* $c \in (0, 1)$ *that verifies*

$$n \int_0^c x^n \mathfrak{f}(x)\mathrm{d}x = c^{n+1}\mathfrak{f}(c).$$

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