Acta Math. Univ. Comenianae
Vol. LXXXV, 2 (2016), pp. 205–217

SOME PROPERTIES OF DIFFERENTIAL ROOT AND THEIR APPLICATIONS

G. A. GRIGORIAN

Abstract. The definition of differential root as a solution of the Riccati equation with a special initial value is given. A variety of properties of this solution is established, some of which generalize properties of the arithmetic square root. By using these properties, we prove some boundedness and stability criteria for second order linear ordinary differential equations.

1. Some properties of differential root

Let $x(t)$ and $x_1(t)$ be real valued continuous functions on $[t_0;+\infty)$. Consider Riccati equations

$$y'(t)+y^2(t)=x(t), \qquad t\geq t_0, \tag{1.1}$$

$$y'(t)+y^2(t)=x_1(t), \qquad t\geq t_0. \tag{1.2}$$

Theorem 1.1. *Let $y_0(t)$ be a real valued solution of Eq.* (1.1) *on $[t_0;+\infty)$, and let $x_1(t)\geq x(t)$, $t\geq t_0$. Then for every $y_{(0)}\geq y_0(t_0)$, Eq.* (1.2) *has the solution $y_1(t)$ on $[t_1;+\infty)$, satisfying the initial condition $y_1(t_0)=y_{(0)}$; moreover $y_1(t)\geq y_0(t)$, $t\geq t_0$.*

The proof of a more general theorem is presented in [**1**].

Remark 1.1. The following assertion ([**2**, p. 129]) is a consequence of the Theorem 1.1.

Theorem 1.2. *If $K_0>K_1>0$ and $y_0(t)>0$, $y_1(t)>0$ are solutions of Eq.* (1.1) *and* (1.2) *on $[t_0;+\infty)$, where $K_1\leq x_1(t)\leq x(t)<K_0$, $t\geq t_0$, $y_0(t_0)\leq\sqrt{K_0}$, $y_1(t_0)\geq\sqrt{K_1}$, then $y_1(t)\geq\sqrt{K_1}$, $y_0(t)\leq\sqrt{K_0}$, $t\geq t_0$. In particular, if $\alpha\in(0;1)$ and $K_1>\alpha K_0$, then $y_1(t)>\sqrt{\alpha}y_0(t)$.*

Since $y_0(t)\equiv 0$ is a solution of the equation

$$y'(t)+y^2(t)=0, \qquad t\geq t_0,$$

then from Theorem 1.1, we immediately obtain the following corollary.

Received April 29, 2015; revised December 08, 2015.
2010 *Mathematics Subject Classification.* Primary 34C10, 34C11, 34C99.
Key words and phrases. Riccati equation; differential root; nonnegativity; monotonicity; bounbdedness of solutions; Liapunov stability; asymptotic stability.

Corollary 1.1. *Let $x(t) \geq 0,\ \ t \geq t_0$. Then for every $y_{(0)} \geq 0$, equation* (1.1) *has the solution $y_0(t)$ on $[t_0; +\infty)$, satisfying the initial condition $y_0(t_0) = y_{(0)}$, and $y_0(t) \geq 0$, $t \geq t_0$.*

Remark 1.2. Theorem 1.1 does not follow either from Theorem 4.1 (Corollary 4.2) or from Wintners theorem, proved in [**3**, pp. 40–45] for more general equations. In particular, in Theorem 4.1, the solution of an equation is compared with the maximum or minimum solution of another one while in Theorem 1.1, maximum or minimum conditions are not imposed on any of the compared solutions. Conditions of Wintner theorem for equation (1.1) (may be) cannot be fulfilled.

Definition 1.3. A solution $y(t)$ of equation (1.1) with nonnegative right hand side, satisfying the initial value condition $y(t_0) = \sqrt{x(t_0)}$, is said to be a differential root of $x(t)$ and denoted by $y_x(t)$.

Remark 1.3. The definition of differential root is new.

In the sequel the denotation $y_A(t)$ mean, that $A(t) \geq 0$, $t \geq t_0$. From Corollary 1.1, it follows that a differential root is defined on $[t_0; +\infty)$ and is nonnegative.

Theorem 1.4. *The differential root has the following properties:*

I. *If $x_1(t) \leq x_2(t),\ \ t \geq t_0$, then $y_{x_1}(t) \leq y_{x_2}(t)$, $t \geq t_0$.*

II. $y_{x_1+x_2}(t) \leq y_{x_1}(t) + y_{x_2}(t), \qquad t \geq t_0.$

III. $y_{x_2-x_1}(t) \geq y_{x_2}(t) - y_{x_1}(t), \qquad t \geq t_0.$

IV. *If $\alpha \in (0;1)$, then $y_{\alpha x_1+(1-\alpha)x_2}(t) \geq \alpha y_{x_1}(t) + (1-\alpha) y_{x_2}(t), \qquad t \geq t_0$.*

V. *If $\alpha \in (0;1)$, then $y_{\alpha x}(t) \geq \alpha y_x(t), \qquad t \geq t_0$.*

VI. *If $\alpha \in R \setminus [0;1]$, then $y_{\alpha x_1+(1-\alpha)x_2}(t) \leq \alpha y_{x_1}(t) + (1-\alpha) y_{x_2}(t), \quad t \geq t_0$.*

VII. *If $\alpha > 1$, then $y_{\alpha x}(t) \leq \alpha y_x(t), \qquad t \geq t_0$.*

VIII. *If $x(t)$ is nondecreasing (nonincreasing function) on $[t_0; +\infty)$ then $y_x(t)$ is the same and $y_x(t) \leq \sqrt{x(t)}$ $(\geq \sqrt{x(t)})$, $\qquad t \geq t_0$.*

IX. *If $x(t)$ is nondecreasing (nonincreasing function) on $[t_0; +\infty)$ and if $0 < \alpha < 1 (\alpha > 1)$, then $y_{\alpha x}(t) \leq \sqrt{\alpha} y_x(t), \qquad t \geq t_0$.*

X. *If $x(t)$ is nondecreasing (nonincreasing function) on $[t_0; +\infty)$ and if $\alpha > 1 (0 < \alpha < 1)$, then $y_{\alpha x}(t) \geq \sqrt{\alpha} y_x(t), \qquad t \geq t_0$.*

Proof. The property I. immediately follows from the Theorem 1.1, the property III. is a consequence of II., V. is a consequence of IV., VII. is a consequence of VI., and IX. is a consequence of X. Let us prove II. From the evident equality

$$[y_{x_1}(t) + y_{x_2}(t)]' + [y_{x_1}(t) + y_{x_2}(t)]^2 = x_1(t) + x_2(t) + 2 y_{x_1}(t) y_{x_2}(t), \qquad t \geq t_0,$$

it follows that the function $\widetilde{y}(t) \equiv y_{x_1}(t) + y_{x_2}(t)$ is a solution of the equation

$$y'(t) + y^2(t) = \widetilde{x}(t), \qquad t \geq t_0,$$

where $\widetilde{x}(t) \equiv x_1(t) + x_2(t) + 2y_{x_1}(t)y_{x_2}(t)$, $t \ge t_0$. Since

$$\widetilde{y}(t_0) = y_{x_1}(t_0) + y_{x_2}(t_0) = \sqrt{x_1(t_0)} + \sqrt{x_2(t_0)} = \sqrt{\widetilde{x}(t_0)},$$

then

$$\widetilde{y}(t) = y_{\widetilde{x}}(t), \qquad t \ge t_0. \tag{1.3}$$

By virtue of Corollary 1.1, $y_{x_j}(t) \ge 0$, $t \ge t_0$, $j = 1, 2$. Then $x_1(t) + x_2(t) \le \widetilde{x}(t)$, $t \ge t_0$. By virtue of I. from here and (1.3), it follows II. Let us prove IV. It is easy to check that for every number α, the following equality holds:

$$\begin{aligned}&[\alpha y_{x_1}(t) + (1-\alpha)y_{x_2}(t)]' + [\alpha y_{x_1}(t) + (1-\alpha)y_{x_2}(t)]^2\\ &= \alpha x_1(t) + (1-\alpha)x_2(t) - \alpha(1-\alpha)[y_{x_1}(t) - y_{x_1}(t)]^2, \qquad t \ge t_0.\end{aligned} \tag{1.4}$$

So the function $y_{[\alpha]}(t) \equiv \alpha y_{x_1}(t) + (1-\alpha)y_{x_2}(t)$ is a solution of the equation

$$y'(t) + y^2(t) = x_{[\alpha]}(t), \qquad t \ge t_0,$$

where $x_{[\alpha]}(t) \equiv \alpha x_1(t) + (1-\alpha)x_2(t) - \alpha(1-\alpha)[y_{x_1}(t) - y_{x_1}(t)]^2$. Let $\alpha \in (0; 1)$. Then

$$x_{[\alpha]}(t) \le \alpha x_1(t) + (1-\alpha)x_2(t), \qquad t \ge t_0. \tag{1.5}$$

Since

$$\begin{aligned}y_{[\alpha]}(t_0) &= \alpha y_{x_1}(t_0) + (1-\alpha)y_{x_2}(t_0) = \alpha\sqrt{x_1(t_0)} + (1-\alpha)\sqrt{x_2(t_0)}\\ &= \sqrt{\alpha x_1(t_0) + (1-\alpha)x_2(t_0) - \alpha(1-\alpha)[\sqrt{x_1(t_0)} - \sqrt{x_1(t_0)}]^2}\\ &\le \sqrt{\alpha x_1(t_0) + (1-\alpha)x_2(t_0)},\end{aligned}$$

then by virtue of the Theorem 1.1 from (1.5), it follows IV. To prove VI., we note that for $\alpha \in R \setminus [0; 1]$, the following inequalities hold:

$$x_{[\alpha]}(t) \ge \alpha x_1(t) + (1-\alpha)x_2(t), \quad t \ge t_0, \qquad y_{[\alpha]}(t_0) \ge \sqrt{\alpha x_1(t_0) + (1-\alpha)x_2(t_0)}$$

(it is assumed that $\alpha x_1(t) + (1-\alpha)x_2(t)$, $t \ge t_0$). By virtue of Theorem 1.1 from here, it follows VI. Let us prove VIII. First we show that

$$y_x(t) \le \sqrt{x(t)}(\ge \sqrt{x(t)}), \qquad t \ge t_0. \tag{1.6}$$

Suppose the contrary. Then there exists $t_1 > t_0$ such that

$$y_x(t_1) > \sqrt{x(t_1)} \quad (< \sqrt{x(t_1)}). \tag{1.7}$$

Let $\bar{t}_1 = \sup\{t \in [t_0; t_1] : y_x(t) - \sqrt{x(t)}\}$. By virtue of (1.7) the following inequality holds:

$$y_x(t) > \sqrt{x(t)} \quad (< \sqrt{x(t)}), \qquad t \in (\bar{t}_1; t_1]. \tag{1.8}$$

From (1.7), it follows

$$y'_x(t_*) > 0 \quad (< 0) \tag{1.9}$$

for some $t_* \in (\bar{t}_1; t_1]$. Indeed, suppose that $y_x'(t) \le 0 \;\; (\ge 0)$. Then since $x(t)$ is a nondecreasing (nonincreasing) function, then

$$y_x(t_1) - \sqrt{x(t_1)} \le y_x(t_1) - \sqrt{x(\bar{t}_1)} = \int\limits_{\bar{t}_1}^{t_1} y_x'(s)\mathrm{d}s \le 0$$

$$(y_x(t_1) - \sqrt{x(t_1)} \ge y_x(t_1) - \sqrt{x(\bar{t}_1)} = \int\limits_{\bar{t}_1}^{t_1} y_x'(s)\mathrm{d}s \ge 0),$$

which contradicts (1.7). This contradiction proves (1.9). From (1.9), it follows

$$y_x(t_*) < \sqrt{x(t_*)} \quad (> \sqrt{x(t_*)}).$$

But, on the other hand, the opposite inequality follows from (1.8)

$$y_x(t_*) > \sqrt{x(t_*)} \quad (< \sqrt{x(t_*)}).$$

The obtained contradiction proves (1.6). From (1.6), it follows $y_x'(t) \ge 0 \;\; (\le 0)$, $t \ge t_0$. It means that $y_x(t)$ is a nondecreasing (nonincreasing) function. The property VIII. is proved. It remains to prove IX. We observe that

$$\sqrt{\alpha} y_x'(t) + \alpha y_x^2(t) = \alpha x(t) + (\sqrt{\alpha} - \alpha) y_x'(t), \;\; t \ge t_0, \qquad \alpha > 0.$$

Therefore, $\sqrt{\alpha} y_x(t)$ is a solution of the equation

$$y'(t) + y^2(t) = \widetilde{x}_{[\alpha]}(t), \qquad t \ge t_0,$$

where

$$\widetilde{x}_{[\alpha]}(t) \equiv \alpha x(t) + (\sqrt{\alpha} - \alpha y_x'(t)) \ge \alpha x(t), \qquad t \ge t_0, \tag{1.10}$$

for $\alpha \in (0;1)$ (for $\alpha > 1$), since by virtue of VIII. $y_x'(t) \ge 0 \;\; (\le 0)$, $t \ge t_0$. It is not difficult to check that $\sqrt{\alpha} y_x(t_0) = \sqrt{\widetilde{x}_{[\alpha]}(t_0)}$. By virtue of I. from here and (1.10), it follows IX. The proof of the theorem is completed. $\square$

Let us compare Theorem 1.4 with the following result proved in [**2**].

Theorem 1.5. *If $\alpha \in (0;1)$, $0 < \alpha x(t) < x_1(t) < x(t)$, $t \ge t_0$, $y_0(t)$ and $y_1(t)$ are solutions of* (1.1) *and* (1.2)*, respectively, then from the inequalities $y_1(t_0) > \alpha y_0(t_0) > 0$, it follows $y_1(t) > \alpha y_0(t)$, $t \ge t_0$ (see* [**2**, p. 130]*).*

If $y_1(t_0) \ge \sqrt{x_1(t_0)}$ and $0 < y_0(t_0) \le \sqrt{x(t_0)}$, then Theorem 1.5 follows from V. For this case the condition $x_1(t) < x(t)$ is no longer required. Indeed, by virtue of V, for $\alpha \in (0;1)$, the following inequality holds $y_{\alpha x}(t) \ge \alpha y_x(t)$, $t \ge t_0$. Then since $y_1(t_0) \ge y_{x_1}(t_0) \ge \alpha y_x(t_0)$ and $y_1(t_0) > y_0(t_0)$, then by virtue of I. from relation $x_1(t) > \alpha x(t)$, $t \ge t_0$, it follows $y_1(t) > y_0(t)$, $t \ge t_0$.

2. Preliminary lemmas

Let $p(t)$ be a continuous differentiable function on $[t_0; +\infty)$ and $q(t)$ be a continuous function on $[t_0; +\infty)$. Consider the following equation

$$\phi''(t) + p(t)\phi'(t) + q(t)\phi(t) = 0, \qquad t \geq t_0. \tag{2.1}$$

Here we establish some properties of specially constructed solutions $\phi_\pm(t)$ of this equation connected with differential root of the "discriminant" $D(t)$ of equation (2.1) by certain relations and establish some additional properties of differential root. On the basis of the obtained we establish correlation between boundedness and stability of solutions of equation (2.1) and properties of $\phi_\pm(t)$. The preliminary result obtained in this paragraph together with the properties of differential root, which were obtained in the previous paragraph, we use in the next paragraph to prove boundedness and stability criteria for equation (2.1).

In equation (2.1), we make the following change

$$\phi(t) = E(t)\psi(t), \qquad t \geq t_0, \tag{2.2}$$

where $E(t) \equiv \exp\Big\{-\frac{1}{2}\int_{t_0}^t p(\tau)\mathrm{d}\tau\Big\}$, $t \geq t_0$. We get

$$\psi''(t) - D(t)\psi(t) = 0, \qquad t \geq t_0, \tag{2.3}$$

where $D(t) \equiv \frac{p'(t)}{2} + \frac{p^2(t)}{4} - q(t)$, $t \geq t_0$. In the sequel we assume that $D(t) \geq 0$, $t \geq t_0$, and without restriction of generalization (not counting the trivial case $D(t) \equiv 0$) we take that $D(t_0) > 0$. From (2.2), it follows

$$\phi'(t) = -\frac{p(t)}{2}\phi(t) + E(t)\psi'(t), \qquad t \geq t_0. \tag{2.4}$$

Consider the equation

$$y'(t) + y^2(t) = D(t), \qquad t \geq t_0. \tag{2.5}$$

The following correlation

$$\psi(t) = \lambda_0 \exp\Big\{\int_{t_0}^{t} y(\tau)\mathrm{d}\tau\Big\}, \qquad t \geq t_0, \quad \lambda_0 = \text{const.} \neq 0 \tag{2.6}$$

connects solutions $y(t)$ to equation (2.3) with solutions $\psi(t)$ to the equation (2.5).

From here and (2.4), it follows

$$\phi'(t) = \left[y(t) - \frac{p(t)}{2}\right]\phi(t), \qquad t \geq t_0. \tag{2.7}$$

Consider the integral

$$\nu_D(t) \equiv \int_t^{+\infty} \exp\Big\{-2\int_t^{\tau} y_D(s)\mathrm{d}s\Big\}d\tau, \qquad t \geq t_0.$$

Lemma 2.1. *For every $t \ge t_0$, the integral $\nu_D(t)$ is convergent and*

$$\nu_D(t) \le \frac{1}{y_D(t)}, \qquad t \ge t_0. \tag{2.8}$$

Proof. For every $t_1 \ge t_0$ and for every $s \ge t_1$, the following inequality holds

$$y_D(s) \ge \frac{y_D(t_1)}{y_D(t_1)(s-t_1)+1} \stackrel{def}{=} y_{(t_1)}(s). \tag{2.9}$$

Indeed, the function $y_{(t_1)}(s)$ (t_1 is fixed) is a solution of equation

$$y'(s) + y^2(s) = 0, \qquad s \ge t_1,$$

and $y_{(t_1)}(t_1) = y_D(t_1)$. By virtue of the Theorem 1.1 from here, it follows (2.9). From (2.9), it follows

$$\begin{aligned} y_D(t) &\le \int\limits_t^{+\infty} \exp\Big\{-\int\limits_t^{\tau} \frac{2y_D(t)\mathrm{d}s}{y_D(t)(s-t)+1}\Big\}\mathrm{d}\tau \\ &= \int\limits_t^{+\infty} \exp\Big\{-2\ln[y_D(t)(\tau-t)+1]\Big\}\mathrm{d}\tau = \frac{1}{y_D(t)}, \end{aligned}$$

$t \ge t_0$. The lemma is proved. □

From (2.9), it follows

$$\int\limits_{t_0}^{t} y_D(\tau)d\tau \ge \ln[1+\sqrt{D(t_0)}(t-t_0)], \qquad t \ge t_0. \tag{2.10}$$

It is not difficult to check, that $y_*(t) \equiv y_D(t) - \frac{1}{\nu_D(t)}$, $t \ge t_0$, is a solution of equation (2.5) (see. [**4**]). From (2.8) it follows:

$$y_*(t) \le 0, \qquad t \ge t_0. \tag{2.11}$$

Consider the functions

$$\psi_+(t) \equiv \exp\Big\{\int\limits_{t_0}^{t} y_D(\tau)\mathrm{d}\tau\Big\}, \quad \psi_-(t) \equiv \exp\Big\{\int\limits_{t_0}^{t} y_*(\tau)\mathrm{d}\tau\Big\}, \quad \phi_\pm(t) \equiv E(t)\psi_\pm(t),$$

where $t \ge t_0$. By virtue of (2.6), $\psi_\pm(t)$ are solutions of equation (2.3). Therefore, by virtue of (2.2), $\phi_\pm(t)$ are solutions of equation (2.1).

Lemma 2.2. *All solutions of equation* (2.1) *are bounded (vanish on the* $+\infty$*) if and only if the solution* $\phi_+(t)$ *is bounded (vanishes on the* $+\infty$*).*

Proof. From (2.10) and (2.11), it follows that $\lim_{t\to+\infty} \frac{\psi_-(t)}{\psi_+(t)} = 0$. Therefore, $\psi_\pm(t)$ are linearly independent. By virtue of (2.2), $\phi_\pm(t)$ are solutions of equation (2.1). Since $\psi_\pm(t)$ are linearly independent, then $\phi_\pm(t)$ are the same, and since $y_*(t) < y_D(t)$, $t \ge t_0$, then

$$|\phi_-(t)| < |\phi_+(t)|, \qquad t \ge t_0. \tag{2.12}$$

Let all solutions of equation (2.1) be bounded (vanish on the $+\infty$). Then $\phi_+(t)$ is bounded (vanishes on the $+\infty$). Let $\phi_+(t)$ bounded (vanish on the $+\infty$). Then by virtue of linear independence of $\phi_\pm(t)$ from (2.12), it follows that all solutions of equation (2.1) are bounded (vanish on the $+\infty$). The lemma is proved. □

Lemma 2.3. *Equation* (2.1) *is stable in the sense of Liapunov (asymptotically) if and only if the functions $\phi_+(t)$, $\phi'_+(t)$ are bounded (vanish on the $+\infty$).*

Proof. By virtue of (2.7), the following equalities hold:

$$\phi'_+(t) = \Big[y_D(t) - \frac{p(t)}{2}\Big]\phi_+(t), \qquad t \geq t_0, \tag{2.13}$$

$$\phi'_-(t) = \Big[y_*(t) - \frac{p(t)}{2}\Big]\phi_-(t), \qquad t \geq t_0. \tag{2.14}$$

From (2.13), it follows

$$|\phi'_+(t)| \leq \Big[y_D(t) + \frac{|p(t)|}{2}\Big]|\phi_+(t)|, \qquad t \geq t_0, \tag{2.15}$$

From (2.14), we have

$$\psi'_-(t) = \Big[y_D(t) - \frac{p(t)}{2}\Big]\phi_-(t) - \frac{E(t)}{\nu_D(t_0)}\exp\Big\{\int_{t_0}^{t}\Big[y_D(\tau) - \frac{1}{\nu_D(\tau)}\Big]\mathrm{d}\tau - \ln\frac{\nu_D(t)}{\nu_D(t_0)}\Big\}$$

$$= \phi'_+(t)\exp\Big\{-\int_{t_0}^{t}\frac{\mathrm{d}\tau}{\nu_D(\tau)}\Big\} - \frac{E(t)}{\nu_D(t_0)}\exp\Big\{\int_{t_0}^{t}\Big[y_D(\tau) - \frac{1}{\nu_D(\tau)} - \frac{\nu'_D(\tau)}{\nu_D(\tau)}\Big]\mathrm{d}\tau\Big\},$$

where $t \geq t_0$. Hence from the easily verifiable equality $\frac{\nu'_D(\tau)}{\nu_D(\tau)} = 2y_D(t) - \frac{1}{\nu_D(t)}$, $t \geq t_0$, it follows

$$\phi'_-(t) = \exp\Big\{-\int_{t_0}^{t}\frac{\mathrm{d}\tau}{\nu_D(\tau)}\Big\}\phi'_+(t) - \frac{E(t)}{\nu_D(t_0)}\exp\Big\{-\int_{t_0}^{t} y_D(\tau)\mathrm{d}\tau\Big\}, \qquad t \geq t_0.$$

Therefore,

$$|\phi'_-(t)| \leq |\phi'_+(t)| + \frac{1}{\nu_D(t_0)}|\phi_+(t)|, \qquad t \geq t_0.$$

From here and from Lemma 2.2, it follows that equation (2.1) is Liapunov stable (asymptotically stable) if and only if the functions $\phi_+(t), \phi'_+(t)$ are bounded (vanish on the $+\infty$). The lemma is proved. □

3. Criteria for the boundedness and stability

For investigation of stability properties of linear systems of ordinary differential equations (in particular, of equation (2.1)) there are mainly used different estimation methods of solutions of systems (in particular, of solutions of equation (2.1)) as well as some asymptotic methods. The main estimation methods include the Liapunov estimate method, the freezing method, and the Bogdanov, Wazevsky and

Lozinsky estimate methods (see [**5**, pp. 40–98, 132–145]. The asymptotic methods include mainly methods based on the Liuville transformation (see [**6**, pp. 131, 152–153] of the Russian translation; [**7**, pp. 32–35, 55–61] WKB estimates. All these methods and other methods (e.g., see [**3**], pp. 392, 393; [8]) permit one to single out wide classes of stable and unstable systems (in particular, equation (2.1)) in terms of their coefficients. However, none of them can completely describe the class of stable and unstable systems (in particular, equation (2.1)) in terms of their coefficients.

Let $p_j(t)$ $(j = 1, 2)$ be continuously differentiable, and let $q_j(t)$ $(j = 1, 2)$ be continuous functions on $[t_0; +\infty)$. Let us consider the equations

$$\phi''(t) + p_j(t)\phi'(t) + q_j(t)\phi(t) = 0, \qquad t \geq t_0, \tag{3.1$_j$}$$

$j = 1, 2$. Let $E_j(t) \equiv \exp\Big\{-\frac{1}{2}\int_{t_0}^t p_j(\tau)\mathrm{d}\tau\Big\}$, $D_j(t) \equiv \frac{p_j'(t)}{2} + \frac{p_j^2(t)}{4} - q_j(t)$, $t \geq t_0$, $j = 1, 2$. In the sequel we assume that $D_j(t) \geq 0$, $t \geq t_0$, $j = 1, 2$.

Theorem 3.1. *Let all solutions of equation* (3.1_1) *be bounded (vanish on the* $+\infty$*), and let* $D(t) \leq D_1(t)$, $t \geq t_0$, $\operatorname{Re}\int_{t_0}^t [p_1(\tau) - p(\tau)]\mathrm{d}\tau$ *be upper bounded. Then all solutions of equation* (2.1) *are bounded (vanish on the* $+\infty$*). Moreover, if in addition, the functions* $D(t)$ *and* $p(t)\exp\Big\{\frac{1}{2}\int_{t_0}^t [p_1(\tau) - p(\tau)]\mathrm{d}\tau\Big\}$ *are bounded, then equation* (2.1) *is Liapunov stable (asymptotically stable).*

Proof. It is evident, that

$$\phi_+(t) = \exp\Big\{\int_{t_0}^t [y_D(\tau) - y_{D_1}(\tau) + \frac{1}{2}(p_1(\tau) - p(\tau))]\mathrm{d}\tau\Big\}\phi_1(t), \qquad t \geq t_0. \tag{3.2}$$

Since $D(t) \leq D_1(t)$, $t \geq t_0$, by virtue of I., $y_D(t) \leq y_{D_1}(t)$, $t \geq t_0$. From here and (3.2), it follows that

$$|\phi_+(t)| \leq \exp\Big\{\frac{1}{2}\operatorname{Re}\int_{t_0}^t [p_1(\tau) - p(\tau)]\mathrm{d}\tau\Big\}|\phi_1(t)|, \qquad t \geq t_0. \tag{3.3}$$

Let all solutions of equation (3.1_1) be bounded (vanish on the $+\infty$). Taking into account that $\frac{1}{2}\operatorname{Re}\int_{t_0}^t [p_1(\tau) - p(\tau)]\mathrm{d}\tau$ is bounded from above, from here and (3.3), we get that $\phi_+(t)$ is bounded (vanishes on $+\infty$). Therefore, by virtue of Lemma 2.2, all solutions of equation (2.1) are bounded (vanish on the $+\infty$). Further, from (2.15) and (3.2), it follows

$$|\phi_+'(t)| \leq \Big[y_D(t) + \frac{|p(t)|}{2}\Big]\exp\Big\{\frac{1}{2}\operatorname{Re}\int_{t_0}^t [p_1(\tau) - p(\tau)]\mathrm{d}\tau\Big\}|\phi_1(t)|, \quad t \geq t_0. \tag{3.4}$$

Let $D(t)$ and $p(t)\exp\Big\{\frac{1}{2}\int_{t_0}^t [p_1(\tau) - p(\tau)]\mathrm{d}\tau\Big\}$ are bounded. By virtue of I. from the boundedness of $D(t)$, it follows $y_D(t) \leq \sup_{t \geq t_0}\sqrt{D(t)} < +\infty$. From here, from (3.4) and from boundedness of $p(t)\exp\Big\{\frac{1}{2}\int_{t_0}^t[p_1(\tau)-p(\tau)]\mathrm{d}\tau\Big\}$, it follows that $\phi'_+(t)$ is bounded (vanishes on the $+\infty$). According to Lemma 2.3, it follows from here that equation (2.1) is Liapunov stable (asymptotically stable). The proof of the theorem is complete. □

Corollary 3.1. *Let the following conditions hold:*

a) $q(t)\geq 0$ $(D(t)\geq 0)$, $t\geq t_0$;

b) $\int_{t_0}^{+\infty}\exp\Big\{-\int_{t_0}^{\tau}p(s)\mathrm{d}s\Big\}\mathrm{d}\tau<+\infty$.

Then all solutions of equation (2.1) *are bounded. Moreover, if in addition,* $p(t)$ *and* $D(t)$ *are bounded, then equation* (2.1) *is Liapunov stable.*

Proof. We put $p_1(t)=p(t)$, $t\geq t_0$, $q_1(t)\equiv 0$. Then

$$\phi_1(t)\equiv 1 \quad\text{and}\quad \phi_2(t)\equiv\int_{t_0}^{t}\exp\Big\{-\int_{t_0}^{\tau}p(s)\mathrm{d}s\Big\}\mathrm{d}\tau,\quad t\geq t_0,$$

are linearly independent solutions of equation (3.1_1). From b), it follows that $\phi_2(t)$ is bounded. Therefore, all solutions of equation (3.1_1) are bounded. From the condition a), it follows that $D(t)\leq D_1(t)$, $t\geq t_0$. By virtue of the Theorem 3.1 from here and from boundedness of all solutions of equation (3.1_1), it follows that boundedness all the solutions of equation (2.1). If besides of a) and b), the functions $p(t)$ and $D(t)$ are bounded, then (because of $p_1(t)=p(t)$, $t\geq t_0$) by virtue of Theorem 3.1, equation (2.1) is Liapunov stable. The proof of the corollary is complete. □

Theorem 3.2. *Let all solutions of equation* (3.1_j) $(j=1,2)$ *be bounded (all solutions of equation* (3.1_1) *be bounded, and all solutions of equation* (3.1_2) *vanish on the* $+\infty$*), and let* $D(t)\leq D_1(t)+D_2(t)$, $t\geq t_0$*, the function* $\operatorname{Re}\int_{t_0}^t[p_1(\tau)+p_2(\tau)-p(\tau)]\mathrm{d}\tau$ *be upper bounded. Then all solutions of equation* (2.1) *are bounded (vanish on the* $+\infty$*). Moreover, if in addition,* $D(t)$ *and* $p(t)\exp\Big\{\frac{1}{2}\int_{t_0}^t[p_1(\tau)+p_2(\tau)-p(\tau)]\mathrm{d}\tau\Big\}$ *are bounded, then equation* (2.1) *is Liapunov stable (asymptotically stable).*

Proof. It is evident that

$$\phi_+(t)=\exp\Big\{\int_{t_0}^{t}\Big[\frac{1}{2}\Big(p_1(\tau)+p_2(\tau)-p(\tau)\Big)+y_D(\tau)-y_{D_1}(\tau)-y_{D_2}(\tau)\Big]\mathrm{d}\tau\Big\}\times\phi_1(t)\phi_2)(t),\qquad t\geq t_0. \tag{3.5}$$

By virtue of II, from the conditions $D(t)\leq D_1(t)+D_2(t)$, $t\geq t_0$, it follows $y_D(t)-y_{D_1}(t)-y_{D_2}(t)\leq 0$, $t\geq t_0$. From here and (3.5), we get

$$|\phi_+(t)|\leq\exp\Big\{\frac{1}{2}\operatorname{Re}\int_{t_0}^{t}[p_1(\tau)+p_2(\tau)-p(\tau)]\,\mathrm{d}\tau\Big\}|\phi_1(t)||\phi_2(t)|,\qquad t\geq t_0. \tag{3.6}$$

From here and from upper boundedness of function $\operatorname{Re}\int_{t_0}^{t}[p_1(\tau)+p_2(\tau)-p(\tau)]\mathrm{d}\tau$, it follows that

$$|\phi_+(t)| \le M|\phi_1(t)||\phi_2(t)|, \quad t \ge t_0, \quad M = \text{const} < +\infty. \tag{3.7}$$

Let all solutions of equations (3.1_j) $(j = 1, 2)$ be bounded (all solutions of equation (3.1_1) be bounded, and all solutions of equation (3.1_2) vanish on the $t \to +\infty$). Then from (3.7)b it follows that $\phi_+(t)$ is bounded (vanishes on the $+\infty$). By virtue of Lemma 2.2 from here, it follows that all solutions of equation (2.1) are bounded (vanish on the $+\infty$). From (2.15) and (3.6), it follows

$$|\phi_+'(t)| \le \Big[y_{_D}(t) + \frac{|p(t)|}{2}\Big]\exp\Big\{\frac{1}{2}\operatorname{Re}\int_{t_0}^{t}\Big[p_1(\tau)+p_2(\tau)-p(\tau)\Big]\mathrm{d}\tau\Big\}|\phi_1(t)||\phi_2(t)|,$$

$t \ge t_0$. Let $D(t)$ and $p(t)\exp\Big\{\frac{1}{2}\int_{t_0}^{t}[p_1(\tau)+p_2(\tau)-p(\tau)]\mathrm{d}\tau\Big\}$ be bounded. Then taking into account I. from the last inequality, we have

$$|\phi_+'(t)| \le M_1|\phi_1(t)||\phi_2(t)|, \quad t \ge t_0, \qquad M_1 = \text{const} < +\infty.$$

Therefore, $\phi_+'(t)$ is bounded (vanishes on the $+\infty$). By virtue of the Lemma 2.3 from this and from boundedness of $\phi_+(t)$ (from the property, that $\phi_+(t) \to 0$ when t tends to $+\infty$), it follows that equation (2.1) is Liapunov stable (asymptotically stable). The proof of the theorem is complete. □

Theorem 3.3. *Let equation* (3.1_1) *have unbounded solution, and let for some* $\alpha \in (0;1)$, $D(t) \ge \alpha D_1(t) + (1-\alpha)D_2(t)$, $t \ge t_0$; *the function* $f_{[\alpha]}(t) \equiv \operatorname{Re}\int_{t_0}^{t}[\alpha p_1(\tau)+(1-\alpha)p_2(\tau)-p(\tau)]\mathrm{d}\tau$ *be bounded from below, but* $\operatorname{Re}\int_{t_0}^{t}p_2(\tau)d\tau$ *be bounded from above. Then equation* (2.1) *has unbounded solution.*

Proof. It is evident, that

$$\phi_+(t) = \exp\Big\{\int_{t_0}^{t}\Big[\frac{\alpha p_1(\tau)+(1-\alpha)p_2(\tau)}{2} + y_{_D}(\tau) - \alpha y_{_{D_1}}(\tau) - (1-\alpha)y_{_{D_2}}(\tau)\Big]\mathrm{d}\tau\Big\} \times (\phi_1(t))^{\alpha}(\phi_2(t))^{1-\alpha}, \qquad t \ge t_0. \tag{3.8}$$

Since $\operatorname{Re}\int_{t_0}^{t}p_2(\tau)d\tau$ is upper bounded, then for $\alpha \in (0;1)$, the following inequality holds

$$|\phi_2(t)|^{1-\alpha} \ge \varepsilon > 0, \qquad t \ge t_0. \tag{3.9}$$

By virtue of I. and IV. from the inequality $D(t) \ge \alpha D_1(t) + (1-\alpha)D_2(t)$, $t \ge t_0$, it follows, that $y_{_D}(t) - \alpha y_{_{D_1}}(t) - (1-\alpha)y_{_{D_2}}(t) \ge 0$, $t \ge t_0$. From here, from (3.8) and (3.9), we get

$$\phi_+(t) \ge \varepsilon\exp\Big\{\operatorname{Re}\int_{t_0}^{t}\frac{\alpha p_1(\tau)+(1-\alpha)p_2(\tau)-p(\tau)}{2}\Big\}|\phi_1(t)|^{\alpha}, \qquad t \ge t_0.$$

From here and lower boundedness of $f_{[\alpha]}(t)$, it follows

$$\phi_+(t)| \geq \varepsilon_1 |\phi_1(t)|^\alpha, \qquad t \geq t_0, \tag{3.10}$$

for some $\varepsilon_1 > 0$. Since equation (3.1_1) has unbounded solution, then by virtue of the Lemma 2.2, $\phi_1(t)$ is unbounded. From here and (3.10), it follows that $\phi_+(t)$ is unbounded. The proof of the theorem is complete. □

Theorem 3.4. *Let all solutions of equation* (3.1_j) $(j = 1, 2)$ *be bounded (vanish on the* $+\infty$*), and let for some* $\alpha > 1$, $D(t) \leq \alpha D_1(t) + (1-\alpha)D_2(t)$, $t \geq t_0$; $f_{[\alpha]}(t)$ *and* $\mathrm{Re}\int_{t_0}^t p_2(\tau)\mathrm{d}\tau$ *be the upper bounded. Then all solutions of equation* (2.1) *are bounded (vanish on the* $+\infty$*). Moreover, if in addition,* $D(t)$ *and* $p(t)\exp\bigl\{\frac{1}{2}\int_{t_0}^t [\alpha p_1(\tau) + (1-\alpha)p_2(\tau) - p(\tau)]\,\mathrm{d}\tau\bigr\}$ *are bounded, then equation* (2.1) *is Liapunov stable (asymptotically stable).*

Proof. By virtue of I. and VI. from the inequality $D(t) \leq \alpha D_1(t) + (1-\alpha)D_2(t)$, $t \geq t_0$, it follows that $y_D(t) - \alpha y_{D_1}(t) - (1-\alpha)y_{D_2}(t) \leq 0,\ t \geq t_0$. From here and (3.9), we get

$$|\phi_1(t)| \leq \exp\Bigl\{\frac{1}{2} f_{[\alpha]}(t)\Bigr\} |\phi_1(t)|^\alpha |\phi_2(t)|^{1-\alpha}, \qquad t \geq t_0. \tag{3.11}$$

Since $y_{D_2}(t) \geq 0,\ t \geq t_0,\ \alpha > 1$ and $Re\int\limits_{t_0}^{t} p_2(\tau)d\tau$ are upper bounded, then from (3.11), it follows

$$|\phi_+(t)| \leq M_2 \exp\Bigl\{\frac{1}{2} f_{[\alpha]}(t)\Bigr\} |\phi_1(t)|^\alpha, \qquad t \geq t_0, \quad M_2 = \mathrm{const} < +\infty. \tag{3.12}$$

Let all solutions of equation (3.1_1) be bounded (vanish on the $+\infty$). Then $\phi_1(t)$ is bounded (vanishes on the $+\infty$). By virtue of Lemma 2.2 from here and (3.12), it follows that all solutions of equation (2.1) are bounded (vanish on the $+\infty$). From (2.15) and (3.12), it follows:

$$|\phi'_+(t)| \leq M_2 \Bigl[y_D(t) + \frac{|p(t)|}{2}\Bigr] \exp\Bigl\{\frac{1}{2} f_{[\alpha]}(t)\Bigr\} |\phi_1(t)|^\alpha, \qquad t \geq t_0.$$

Let $p(t)\exp\bigl\{\frac{1}{2}\int_{t_0}^t \bigl[\alpha p_1(\tau) + (1-\alpha)p_2(\tau) - p(\tau)\bigr]\mathrm{d}\tau\bigr\}$ and $D(t)$ be bounded. Then taking into account I. from the last inequality, we will have

$$|\phi'_+(t)| \leq M_3 |\phi_1(t)|^\alpha, \qquad t \geq t_0, \quad M_3 = \mathrm{const} < +\infty.$$

Therefore, $\phi'_+(t)$ is bounded (vanishes on the $+\infty$). By virtue of Lemma 2.3 from here and from boundedness of $\phi_+(t)$ (from the property that $\phi_+(t) \to 0$ when t tends to $+\infty$), it follows that equation (2.1) is Liapunov stable (asymptotically stable). The proof of the theorem is complete. □

Theorem 3.5. *Let* $D(t)$ *be a nondecreasing function, and let* $f(t) \equiv \int_{t_0}^t \bigl[\sqrt{D(\tau)} - \mathrm{Re}\,\frac{p(\tau)}{2}\bigr]\mathrm{d}\tau$, $t \geq t_0$, *is upper bounded* $\bigl(I \equiv \int_{t_0}^{+\infty} \bigl[\sqrt{D(\tau)} - \mathrm{Re}\,\frac{p(\tau)}{2}\bigr]\mathrm{d}\tau = -\infty\bigr)$. *Then all solutions of equation* (2.1) *are bounded (vanish on the* $+\infty$*). If* $f_1(t) \equiv f(t) + \ln\bigl[\frac{|p(t)|}{2} + \sqrt{D(t)}\bigr]$ *is upper bounded* $(I_1 \equiv \lim_{t\to+\infty} f_1(t) = -\infty)$, *then equation* (2.1) *is Liapunov stable (asymptotically stable).*

Proof. Since $D(t)$ is a nondecreasing function on $[t_0; +\infty)$, then by virtue of VIII., the following inequality holds $y_D(t) \le \sqrt{D(t)}$, $t \ge t_0$. Therefore,

$$|\phi_+(t)| \le \exp\{f(t)\}, \qquad t \ge t_0. \tag{3.13}$$

From here and from upper boundedness of $f(t)$ (from equality $I = -\infty$), it follows that $\phi_+(t)$ is bounded (vanishes on the $+\infty$). By virtue of Lemma 2.2 from here, it follows that all solutions of equation (2.1) are bounded (vanish on the $+\infty$). Let $f_1(t)$ be bounded from above ($I_1 = -\infty$). Then since $\sqrt{D(t)}$ is nondecresing function, then $f(t)$ is bounded from above ($I = -\infty$). Therefore, as proven above, $\phi_+(t)$ is bounded (vanishes on the $+\infty$). It means that by virtue of Lemma 2.3 to complete the proof of the theorem, it remains to show that $\phi'_+(t)$ is bounded (vanishes on the $+\infty$). From (2.15), (3.13) and from inequality $y_D(t) \le \sqrt{D(t)}$, $t \ge t_0$, it follows

$$|\phi'_+(t)| \le \exp\{f_1(t)\}, \qquad t \ge t_0.$$

Further since $f_1(t)$ is upper bounded ($I_1 = -\infty$), then $\phi'_+(t)$ is bounded (vanishes on the $+\infty$). The proof of the theorem is complete. □

Theorem 3.6. *Let $D(t)$ be a non increasing function and let*

$$\overline{\lim_{t\to+\infty}} \int\limits_{t_0}^{t} \Big[\sqrt{D(\tau)} - \operatorname{Re}\frac{p(\tau)}{2}\Big] d\tau = +\infty. \tag{3.14}$$

Then equation (2.1) *has unbounded solution.*

Proof. Since $D(t)$ is a non increasing function, then by virtue of VIII., the following inequality $y_D(t) \ge \sqrt{D(t)}$, $t \ge t_0$ holds. Therefore,

$$|\phi_+(t)| \ge \exp\Big\{\int\limits_{t_0}^{t} \Big[\sqrt{D(\tau)} - \operatorname{Re}\frac{p(\tau)}{2}\Big] d\tau\Big\}, \qquad t \ge t_0.$$

From here and (3.14), it follows that $\phi_+(t)$ is unbounded. The proof of the theorem is complete. □

Example 3.1. Consider the equation

$$\phi''(t) + 2(\lambda + \sin^2 t)\phi'(t) + (\lambda^2 + \sin 2t)\phi(t) = 0, \qquad t \ge t_0, \tag{3.15}$$

$\lambda \ge 1$. Here $D(t) = (\lambda + \sin^2(t))^2 - \lambda^2 \ge 0$, $t \ge t_0$. It is easy to see that all the other conditions of the Corollary 3.1 for (3.15) are also satisfied. Therefore, equation (3.15) is Liapunov stable. Note that the condition

$$\int\limits_{t_0}^{+\infty} \left|\frac{D''(t)}{D^{3/2}(t)} - \frac{5}{4}\frac{(D'(t))^2}{D^{5/2}(t)}\right| dt < +\infty$$

of the WKB approximation method (see [**7**, p. 55]) in our case is not fulfilled. Therefore, the last one is not applicable for establishing stability of equation (3.15).

References

1. Grigorian G. A., *On two comparison tests for second-order linear ordinary differential equations* (Russian), Differ. Uravn. **47(9)** (2011), 1225–1240. Translation in Differ. Equ. **47(9)** (2011), 1237–1252.
2. Gerber M., Hasselblatt B. and Kessing D., *The Riccati Equation: Pinching of Forsing and Solutions*, Experimental Mathematics, **12(2)** (2003), 129–134.
3. Hartman Ph., *Ordinary differential equations*, Mir, Moscow 1970.
4. Grigorian G. A., *On some properties of solutions of the Riccati equations* (Russian), Izv. Nats. Akad. Nauk Armenii Math. **42(4)** (2007), 184–197.
5. Adrianova L. Ya., *Vvedenie v teoriyu system differential'nikh uravnenij* (*Introduction to the Theory of Linear Systems of Differential Equations*) Izv. St., Peterburgskij Univ. St. Peterburg 1992.
6. Bellman R., *Stability Theory of Differential Equations*, New York, 1953. Translated *Teoriya ustoichivosti reshenij differentialnich uravnenij*, Inostrannaya Literatura, Moskow 1954.
7. Fedoryuk M. V., *Asymptotic Methods in the Theory of Ordinary linear differential equations*, Nauka, Moskow 1983 [in Russian].
8. Soboľ I. M., Investigation with the Aid of Polar Coordinates of the Asymptotic Behavior of Solutions of a Linear Differential equations of the Second Order, Mat. Sb., **28(70)**, (1951), 707–714.

G. A. Grigorian, Institute of mathematics NAS of Armenia, Armenia,

e-mail: `mathphys2@instmath.sci.am`