

GORENSTEIN INJECTIVE COVERS AND ENVELOPES OVER RINGS THAT SATISFY THE AUSLANDER CONDITION

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ABSTRACT. It was recently proved [17] that the class of Gorenstein injective left R -modules is both covering and enveloping over a two-sided noetherian ring R with the property that the character modules of the Gorenstein injective left R -modules are Gorenstein flat. It was also proved that over the same type of rings, the class of Gorenstein flat right R -modules is preenveloping [16]. We prove here that if R is a two-sided noetherian ring R such that R satisfies the Auslander condition and has finite finitistic left injective dimension, then R has the desired property; the character module of any Gorenstein injective is Gorenstein flat.

1. INTRODUCTION

The class of Gorenstein injective modules is one of the key elements in Gorenstein homological algebra. So it is natural to consider the question of the existence of the Gorenstein injective covers and envelopes.

Their existence over the Gorenstein rings is known (Enochs-Jenda, [8], 2000). Then Holm and Jørgensen proved [12] that the class of Gorenstein injective modules, $\mathcal{GI}$, is covering over commutative noetherian rings with dualizing complexes (2011). More recently (2015), we proved [7] that the class of Gorenstein injective modules is both covering and enveloping over any commutative noetherian ring R such that the character modules of Gorenstein injective modules are Gorenstein flat. Then, in [17], we extended these results to two-sided noetherian rings with the property that G^+ is Gorenstein flat whenever ${}_RG$ is Gorenstein injective.

Another essential component of the Gorenstein homological algebra is the class of Gorenstein flat modules. While the existence of the Gorenstein flat precovers over arbitrary rings is known, it is still an open question what is the most general type of ring over which the class of Gorenstein flat modules is preenveloping. In [16], we proved the existence of the Gorenstein flat preenvelopes over two-sided noetherian rings R such that the character modules of Gorenstein injective left R -modules are Gorenstein flat.

Examples of such rings include the Gorenstein rings, the commutative noetherian rings with dualizing complexes, as well as the two-sided noetherian rings R that are left n -perfect for some positive integer n and such that the pair (R, R)

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has a dualizing bimodule ${}_R V_R$ ([16]). We also proved in [16] that any two-sided noetherian ring R such that the injective dimension of R_R is finite, has the desired property.

We prove here that any two-sided noetherian ring R such that R satisfies the Auslander condition and has finite finitistic left injective dimension, has the property that the character module of any Gorenstein injective left R -module is Gorenstein flat. Consequently, the class of Gorenstein injective left R -modules, $\mathcal{GI}$, is both covering and enveloping over such a ring, and the class of Gorenstein flat right R -modules, $\mathcal{GF}$, is preenveloping over such rings.

2. PRELIMINARIES

We consider associative rings with identity. Unless otherwise specified, by an R -module we mean a left R -module.

We recall that a module M is Gorenstein injective if there exists an exact and $\text{Hom}(\text{Inj}, -)$ exact complex of injective modules $\mathbf{I} = \dots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \dots$ such that $M = \text{Ker}(E_0 \rightarrow E_{-1})$.

We use the notation $\mathcal{GI}$ for the class of Gorenstein injective modules.

A stronger notion is that of strongly Gorenstein injective module. By [1], a module M is strongly Gorenstein injective if there exists an exact and $\text{Hom}(\text{Inj}, -)$ exact sequence $\dots \rightarrow E \rightarrow E \rightarrow E \rightarrow \dots$ such that E is an injective R -module and $M = \text{Ker}(E \rightarrow E)$.

It is known ([1]) that an R -module M is Gorenstein injective if and only if it is a direct summand of a strongly Gorenstein injective module.

We also recall the definitions for Gorenstein injective precovers and covers.

Definition 1. A homomorphism $\phi : G \rightarrow M$ is a Gorenstein injective precover of M if G is Gorenstein injective and if for any Gorenstein injective module G' and any $\phi' \in \text{Hom}(G', M)$, there exists $u \in \text{Hom}(G'; G)$ such that $\phi' = \phi u$. A Gorenstein injective precover ϕ is said to be a cover if any $v \in \text{End}_R(G)$ such that $\phi v = \phi$ is an automorphism of G .

The importance of the Gorenstein injective (pre)covers comes from the fact that they allow defining the Gorenstein injective (minimal) left resolutions, i.e., if the ring R is such that every R -module M has a Gorenstein injective precover, then for every M there exists a $\text{Hom}(\mathcal{GI}, -)$ exact complex $\dots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ with $G_0 \rightarrow M$ and $G_i \rightarrow \text{Ker}(G_{i-1} \rightarrow G_{i-2})$ Gorenstein injective precovers. Such a complex is called a Gorenstein injective left resolution of M ; it is unique up to homotopy, so it can be used to compute derived functors of Hom .

If the ring R is such that $\mathcal{GI}$ is covering, then for each ${}_R M$ there exists a $\text{Hom}(\mathcal{GI}, -)$ exact complex $\dots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ with $G_0 \rightarrow M$ and $G_i \rightarrow \text{Ker}(G_{i-1} \rightarrow G_{i-2})$ Gorenstein injective covers. Such a complex is called a *minimal* Gorenstein injective left resolution of M and is unique up to an isomorphism.

The Gorenstein injective preenvelopes, envelopes, and right resolutions are defined dually.

We also use Gorenstein flat modules. They are defined in terms of the tensor product.

Definition 2. A module G is Gorenstein flat if there exists an exact complex of flat modules $\mathbf{F} = \dots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \dots$ such that $I \otimes \mathbf{F}$ is still exact for any injective right R -module I and $G = \text{Ker}(F_0 \rightarrow F_{-1})$.

We use $\mathcal{GF}$ to denote the class of Gorenstein flat modules.

The Gorenstein flat preenvelopes, precovers, left and right resolutions are defined in a similar manner with the Gorenstein injective ones (simply replace $\mathcal{GI}$ with $\mathcal{GF}$ in the definitions).

We recall that the character module of a left R -module M is the right R -module $M^+ = \text{Hom}_Z(M, Q/Z)$.

It is known that the character module of a Gorenstein flat module is Gorenstein injective. If the ring R is right coherent, then a left R -module M is Gorenstein flat if and only if M^+ is a Gorenstein injective right R -module ([13, Theorem 3.6]).

Given a class of R -modules $\mathcal{L}$, as usual by $\mathcal{L}^\perp$, we denote its right orthogonal class, $\mathcal{L}^\perp = \{C : \text{Ext}^1(L, C) = 0, \text{ for all } L \in \mathcal{L}\}$.

3. MAIN RESULT

Bass proved in [2], that a commutative noetherian ring R is a Gorenstein ring if and only if the flat dimension of the i th term in a minimal injective resolution of R is at most $i - 1$ for all $i \geq 1$. In the non-commutative case, Auslander proved that this condition is left-right symmetric ([10, Theorem 3.7]). In this case, the ring is said to satisfy the *Auslander condition*.

In [14] the author introduces the notion of modules satisfying the Auslander condition. We recall the next definition.

Definition 3 ([14]). Given a left noetherian ring R , a left R -module M is said to satisfy the Auslander condition if the flat dimension of the i -th term in the minimal injective resolution of M is at most $i - 1$ for any $i \geq 1$.

We also recall the following theorem.

Theorem (This is part of [14, Theorem 1.3]). *If R is a left noetherian ring, then the following are equivalent:*

1. ${}_R R$ satisfies the Auslander condition.
2. $fd_R E^0(M) \leq fd_R M$ for any ${}_R M$, where $E^0(M)$ is the injective envelope of M .

If moreover R , is left and right noetherian, then the statements above are also equivalent to

3. *the opposite version of (i) ($1 \leq i \leq 2$).*

Remark. By [14, Theorem 1.3], if the left noetherian ring R satisfies the Auslander condition, then $E^0(R)$ is a flat R -module.

We recall that a module N is said to be *strongly cotorsion* if $\text{Ext}^1(X, N) = 0$ for any module X of finite flat dimension. In particular, any injective R -module is strongly cotorsion.

By [18], ${}_R N$ is strongly cotorsion if and only if $\text{Ext}^i(X, N) = 0$ for all $i \geq 1$ and for any ${}_R X$ of finite flat dimension.

It is known that if R is a two-sided noetherian ring, then the following statements are equivalent [6, Theorem 4.4]:

- 1) $E^0(R)$ is flat.
- 2) $E^0(F)$ is flat for any flat R -module F .
- 3) $F_0(M)$ is injective for any strongly cotorsion module ${}_R M$ where $F_0(M)$ denotes the flat cover of M .

It is known that if the class of Gorenstein injective left R -modules is closed under arbitrary direct sums, then the ring R is left noetherian [4]. But whether or not the converse holds, this is still an open question. We proved [17] that if R is two-sided noetherian such that the character modules of Gorenstein injective modules are Gorenstein flat, then the class $\mathcal{GI}$ is closed under direct sums and it is a covering class. In fact, $\mathcal{GI}$ is also enveloping over such rings. Our main result here gives a sufficient condition in order for a two-sided noetherian ring have the desired property: $G^+ \in \mathcal{GF}$ for any $G \in \mathcal{GI}$. We prove that if R is a two-sided noetherian ring such that R satisfies the Auslander condition and has finite finitistic left injective dimension, then the character module of every Gorenstein injective left R -module is Gorenstein flat. Consequently, over such a ring R , the class of Gorenstein injective left R -modules, $\mathcal{GI}$, is both covering and enveloping, and the class of Gorenstein flat right R -modules, $\mathcal{GF}$, is preenveloping.

We start with the following lemma.

Lemma 1. *Let R be a two-sided noetherian ring and let M be a Gorenstein injective module. Then M^{++} is a strongly cotorsion module.*

Proof. We prove first that the result holds for a strongly Gorenstein injective module M .

By definition there exists an exact sequence $0 \rightarrow M \rightarrow \bar{E} \rightarrow M \rightarrow 0$ with $\bar{E}$ injective. This gives an exact sequence $0 \rightarrow M^{++} \rightarrow E \rightarrow M^{++} \rightarrow 0$ with $E = \bar{E}^{++}$ an injective R -module (since R is noetherian).

Therefore we have that $\text{Ext}^i(-, M^{++}) \simeq \text{Ext}^1(-, M^{++})$ for any $i \geq 1$.

Consider an R -module X of finite flat dimension. Since M^{++} is cotorsion and there is an exact sequence $0 \rightarrow F_d \rightarrow \dots \rightarrow F_0 \rightarrow X \rightarrow 0$ with all F_j flat modules, it follows that $\text{Ext}^i(X, M^{++}) = 0$ for all $i \geq d + 1$.

Then by the above, $\text{Ext}^i(X, M) = 0$ for all $i \geq 1$. Thus M^{++} is strongly cotorsion.

Let M' be a Gorenstein injective module. By [1], there exists a strongly Gorenstein injective module M such that $M = M' \oplus N$. Then $M^{++} = M'^{++} \oplus N^{++}$. So $0 = \text{Ext}^i(X, M^{++}) = \text{Ext}^i(X, (M')^{++}) \oplus \text{Ext}^i(X, N^{++})$, and therefore, $\text{Ext}^i(X, (M')^{++}) = 0$ for any R -module X of finite flat dimension. Thus M' is strongly cotorsion. $\square$

We will also use the following result,

Proposition 1. *Let R be a two-sided noetherian ring such that R satisfies the Auslander condition. Then for any strongly Gorenstein injective module M , the module M^{++} has an exact left injective resolution.*

Proof. Since M is strongly Gorenstein injective, there exists an exact sequence $\dots \rightarrow \overline{E} \rightarrow \overline{E} \rightarrow \overline{E} \rightarrow \dots$ with $\overline{E}$ an injective module and with $M = \text{Ker}(\overline{E} \rightarrow \overline{E})$. This gives an exact sequence $\dots \rightarrow E \rightarrow E \rightarrow E \rightarrow \dots$ with $E = (\overline{E})^{++}$ an injective R -module and with $M^{++} = \text{Ker}(E \rightarrow E)$. Since $E \rightarrow M^{++}$ is surjective with E an injective module, it follows that M^{++} has a surjective injective cover. So there exists an exact sequence $0 \rightarrow J \rightarrow I \rightarrow M^{++} \rightarrow 0$ with I injective and with $J \in \text{Inj}^\perp$. Since $\text{Ext}^1(E, J) = 0$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^{++} & \longrightarrow & E & \xrightarrow{f} & M^{++} \longrightarrow 0 \\ & & \downarrow u & & \downarrow u & & \parallel \\ 0 & \longrightarrow & J & \longrightarrow & I & \xrightarrow{g} & M^{++} \longrightarrow 0 \end{array}$$

and therefore, we have an exact sequence $0 \rightarrow M^{++} \xrightarrow{\alpha} J \oplus E \xrightarrow{\delta} I \rightarrow 0$ with $\alpha(x) = (u(x), x)$ and $\delta(x, y) = x - u(y)$.

Since M^{++} is strongly cotorsion and I is injective hence strongly cotorsion, it follows [18] that $J \oplus E$ is strongly cotorsion. Thus J is strongly cotorsion. Then by [6], the flat cover of J , $F_0(J)$ is injective. So there exists a surjection $F_0(J) \rightarrow J$ with $F_0(J)$ an injective module, and therefore, the injective cover of J is surjective. Thus there is an exact sequence $0 \rightarrow J_0 \rightarrow I_0 \rightarrow J \rightarrow 0$ with I_0 injective and $J_0 \in \text{Inj}^\perp$. Since also $J \in \text{Inj}^\perp$, we have $\text{Ext}^1(A, J_0) = \text{Ext}^2(A, J_0) = 0$ for any injective R -module A .

We show that J_0 is a strongly cotorsion module. Let X be an R -module of finite flat dimension. Since ${}_R R$ satisfies the Auslander condition, we have that $E^0(X)$ is a module of finite flat dimension, too (by [14, Theorem 1.3]). The exact sequence $0 \rightarrow X \rightarrow E^0(X) \rightarrow T \rightarrow 0$ gives that T also has finite flat dimension. Since $\text{Ext}^1(E^0(X), J_0) = \text{Ext}^2(E^0(X), J_0) = 0$, we have an exact sequence

$$0 = \text{Ext}^1(E^0(X), J_0) \rightarrow \text{Ext}^1(X, J_0) \rightarrow \text{Ext}^2(T, J_0) \rightarrow \text{Ext}^2(E^0(X), J_0) = 0,$$

Thus $\text{Ext}^1(X, J_0) \simeq \text{Ext}^2(T, J_0)$.

We also have the exact sequence $0 \rightarrow J_0 \rightarrow I_0 \rightarrow J \rightarrow 0$. This gives an exact sequence $0 = \text{Ext}^1(T, I_0) \rightarrow \text{Ext}^1(T, J) \rightarrow \text{Ext}^2(T, J_0) \rightarrow \text{Ext}^2(T, I_0) = 0$. So $\text{Ext}^2(T, J_0) \simeq \text{Ext}^1(T, J) = 0$ (since T has finite flat dimension and J is strongly cotorsion). It follows that $\text{Ext}^1(X, J_0) \simeq \text{Ext}^2(T, J_0) = 0$ for any module X of finite flat dimension. Therefore, J_0 is a strongly cotorsion module. Then, as above, $F_0(J_0)$ is an injective module and therefore, there exists an exact sequence $0 \rightarrow J_1 \rightarrow I_1 \rightarrow J_0 \rightarrow 0$ with I_1 injective and with J_1 such that $\text{Ext}^1(A, J_1) = \text{Ext}^2(A, J_1) = \text{Ext}^3(A, J_1) = 0$ for any injective ${}_R A$. The same argument as above shows that J_1 is a strongly cotorsion module. Continuing we obtain an exact and

$\text{Hom}(\text{Inj}, -)$ sequence $\dots \rightarrow I_1 \rightarrow I_0 \rightarrow I \rightarrow M^{++} \rightarrow 0$ with I and all I_j injective modules. $\square$

Theorem 1. *Let R be a two-sided noetherian ring such that R satisfies the Auslander condition. Assume moreover that R has finite finitistic left injective dimension. Then the character module of any Gorenstein injective left R -module M is Gorenstein flat.*

Proof. First we prove the result for strongly Gorenstein injective modules.

Let M be strongly Gorenstein injective. By Proposition 1, M^{++} has an exact left injective resolution $I = \dots \rightarrow I_2 \xrightarrow{l_2} I_1 \xrightarrow{l_1} I_0 \xrightarrow{l_0} M^{++} \rightarrow 0$.

Also, since M is strongly Gorenstein injective, there exists an exact complex $E' = \dots \rightarrow E \rightarrow E \rightarrow M^{++} \rightarrow 0$ with E an injective R -module.

Since I is $\text{Hom}(\text{Inj}, -)$ exact, we have a commutative diagram

$$\begin{array}{ccccccccc} E' = \dots & \longrightarrow & E & \longrightarrow & E & \longrightarrow & E & \longrightarrow & M^{++} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \\ I = \dots & \longrightarrow & I_2 & \longrightarrow & I_1 & \longrightarrow & I_0 & \longrightarrow & M^{++} & \longrightarrow & 0 \end{array}$$

This gives an exact sequence $\dots \rightarrow I_2 \oplus E \xrightarrow{\alpha_2} I_1 \oplus E \xrightarrow{\alpha_1} I_0 \rightarrow 0$.

Thus $\text{id}_R(\text{Im}(\alpha_i)) < \infty$ for all $i \geq 1$. Since R has finite finitistic injective dimension, we have that for every $i \geq 1$, $\text{id}_R(\text{Im} \alpha_i) \leq d$ where $d = FID(R)$.

Then the exact sequence $0 \rightarrow \text{Ker } \alpha_{d+2} \rightarrow I_{d+2} \oplus E \rightarrow I_{d+1} \oplus E \rightarrow \dots \rightarrow I_2 \oplus E \rightarrow \text{Ker } \alpha_1 \rightarrow 0$ gives that $\text{Ker } \alpha_1$ is injective. Similarly $\text{Ker } \alpha_i$ is injective for every i .

The map of complexes $v: E' \rightarrow I$ gives an exact sequence: $0 \rightarrow I \rightarrow c(v) \rightarrow E'[1] \rightarrow 0$ where $c(v)$ is the mapping cone. The complex E' is exact, so for each i , we have an exact sequence $0 \rightarrow \text{Im}(l_i) \rightarrow \text{Im}(\alpha_i) \rightarrow M^{++} \rightarrow 0$. By the above, $\text{Im}(\alpha_i)$ is an injective R -module for any $i \geq 1$. Also, since $\text{Im}(l_i) \in \text{Inj}^\perp$ for all $i \geq 1$, we have that $\text{Ext}^1(A, \text{Im } l_i) = \text{Ext}^2(A, \text{Im } l_i) = 0$ for all $i \geq 2$, and for any injective R -module A .

Thus for any $i \geq 2$ and each injective module A , we have an exact sequence $0 \rightarrow \text{Im}(l_i) \rightarrow \text{Im}(\alpha_i) \rightarrow M^{++} \rightarrow 0$ with $\text{Im}(\alpha_i)$ injective and with $\text{Ext}^1(A, \text{Im } l_i) = \text{Ext}^2(A, \text{Im } l_i) = 0$. This gives an exact sequence:

$$0 = \text{Ext}^1(A, \text{Im}(\alpha_i)) \rightarrow \text{Ext}^1(A, M^{++}) \rightarrow \text{Ext}^2(A, \text{Im}(l_i)) \rightarrow 0 = \text{Ext}^2(A, \text{Im}(\alpha_i)).$$

Thus $\text{Ext}^1(\text{Inj}, M^{++}) \simeq \text{Ext}^2(A, \text{Im}(l_i)) = 0$. By the proof of Lemma 1 we have that $\text{Ext}^i(\text{Inj}, M^{++}) = 0$ for all $i \geq 1$.

Since M^{++} has an exact left injective resolution and also $\text{Ext}^i(\text{Inj}, M^{++}) = 0$ for all $i \geq 1$, it follows that M^{++} is Gorenstein injective ([8]).

Since $(M^+)^+$ is Gorenstein injective it follows that the module M^+ is Gorenstein flat.

We prove that the result holds for any Gorenstein injective module M' . By [1], M' is a direct summand of a strongly Gorenstein injective module M . Then M'^{++} is a direct summand of M^{++} . By the above, M^{++} is a Gorenstein injective module, so M'^{++} is a Gorenstein injective module. Since $(M'^+)^+$ is Gorenstein injective it follows that the module M'^+ is Gorenstein flat. $\square$

Theorem 1 and [17, Theorem 4 and Corollary 1] give the following result.

Theorem 2. *Let R be a two-sided noetherian ring such that R satisfies the Auslander condition and has finite finitistic left injective dimension. Then the class of Gorenstein injective modules, $\mathcal{GI}$, is both covering and enveloping in $R\text{-Mod}$.*

Also, Theorem 1 and [16, Proposition 1], give the following

Theorem 3. *Let R be a two-sided noetherian ring such that ${}_R R$ satisfies the Auslander condition and has finite finitistic injective dimension. Then the class of Gorenstein flat right R -modules is preenveloping in the category of right R -modules.*

We recall that the invariant $\text{sfi } R$ was introduced in [5] as the supremum of the flat lengths of injective left R -modules. It is easily seen that $\text{sfi } R < \infty$ if and only if any injective left R -module has finite flat dimension.

We prove that if R is a right coherent ring such that $\text{sfi } R < \infty$, then the character modules of Gorenstein injective modules are Gorenstein flat.

Theorem 4. *Let R be a right coherent ring such that $\text{sfi } R < \infty$. Then M^+ is a Gorenstein flat right R -module for any Gorenstein injective ${}_R M$.*

Proof. Because R is right coherent, it is enough to show that M^{++} is Gorenstein injective.

Consider a strongly Gorenstein injective module M . Then there exists an exact complex $\mathbf{I} = \dots \rightarrow E \rightarrow E \rightarrow E \rightarrow \dots$ with E injective and such that $M^{++} = \text{Ker}(E \rightarrow E)$. The short exact sequence $0 \rightarrow M^{++} \rightarrow E \rightarrow M^{++} \rightarrow 0$ gives that $\text{Ext}^i(A, M^{++}) \simeq \text{Ext}^1(A, M^{++})$ for all ${}_R A$. Since $\text{sfi } R < \infty$ and since M^{++} is cotorsion, for every injective R -module A , there exists $d \geq 1$ such that $\text{Ext}^i(A, M^{++}) = 0$ for all $i \geq d$. By the above, $\text{Ext}^i(A, M^{++}) = 0$ for all $i \geq 1$ and any injective ${}_R A$. Thus $\mathbf{I}$ is a totally acyclic complex of injective modules, and so M^{++} is Gorenstein injective.

If M' is a Gorenstein injective module, then M' is a direct summand of a strongly Gorenstein module M . Then M'^{++} is Gorenstein injective as a direct summand of the Gorenstein injective module M^{++} . $\square$

By Theorem 4, if R is two-sided noetherian such that $\text{sfi } R < \infty$ then the class of Gorenstein injective left R -modules is both covering and enveloping, and the class of Gorenstein flat right R -modules is preenveloping.

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