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FROM EPH UPPER HALF PLANES TO $\mathbb{RP}^2$: A STUDY OF PROJECTIVE ACTIONS OF SL$(3,\mathbb{R})$ SUBGROUPS

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Abstract. In this paper, we investigate the geometric aspects of the action of SL$(3,\mathbb{R})$ subgroups in the real projective plane from an Erlangen perspective. Specifically, the study investigates the projective action of the subgroups of SL$(3,\mathbb{R})$ on the non-degenerate conics in $\mathbb{RP}^2$, which is obtained as the two-dimensional homogeneous space of SL$(3,\mathbb{R})$. By investigating the subgroup of SL$(3,\mathbb{R})$ that preserves the projective unit circle, a connection is established between this subgroup and the Lie group SL$(2,\mathbb{R})$. Expanding upon V. Kisil's research on the EPH classification of geometries related to the Möbius action of SL$(2,\mathbb{R})$, this paper extends the analysis to the projective action of SL$(3,\mathbb{R})$, with the adjoint representation of SL$(2,\mathbb{R})$ playing a significant role. Mappings from the elliptic, parabolic, and hyperbolic (abbreviated as EPH) upper half plane to the interior, boundary, and exterior of the projective unit circle are investigated. A unified expression for isomorphisms facilitates a cohesive framework for analyzing these mappings. Furthermore, the study investigates Möbius-invariant cycle images, demonstrating that elliptic and hyperbolic cycles correspond to ellipses and hyperbolas, respectively, while parabolic cycles map to the projective unit circle.

1. Introduction

Geometry explores the interrelations between objects in the plane or space. Defined by Felix Klein's Erlangen program, it characterizes geometry through invariants under specific group actions, seeking to unify all geometries into a cohesive concept. Furthermore, with a historical significance dating back to ancient times, geometry holds a pivotal position in mathematics. Notably, conics within geometry, particularly in projective geometry, have consistently captured special attention, as seen in [**12, 14, 29, 32, 33, 36**]. In this direction, numerous authors have conducted substantial research on the geometry in terms of invariants and used the Erlangen approach to construct the geometry of one and two-dimensional spaces, [**1, 5, 10, 22, 23, 28**]. In this trend, the geometry associated with the action of $G = \mathrm{SL}(2,\mathbb{R})$ on the one dimensional homogeneous space G/H represented by Möbius transformation was extensively studied and investigated by several authors [**3, 13, 27**]. In the works [**2, 7, 20**], authors considered Möbius action

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of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{RP}^1$ and described various properties of Möbius-invariant cycles. In the same line, $\mathrm{SL}(2,\mathbb{R})$ invariant geodesic curves and metrics were studied in [**4, 6, 9, 24**].

This series of works has led to several natural and effective generalizations, prompting the exploration of higher dimensions. Furthermore, since projective geometry is a rich and fascinating field that provides a geometric foundation for a variety of disciplines, the study of the projective action of $\mathrm{SL}(3,\mathbb{R})$ on projective space is a crucial part of this field and provides a powerful tool for exploring geometric relationships and structures. In this work, we take the transformation group to be $\mathrm{SL}(3,\mathbb{R})$ and focus on invariant objects. In this manner, we intend to add further invariants to the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$ geometries. To achieve this, we explore the isomorphism between the Poincaré disk (elliptic unit disk) and the projective unit disk. Inspired by V. Kisil's work on the EPH classification of geometries related to the $\mathrm{SL}(2,\mathbb{R})$ action [**26**], this investigation encompasses two additional scenarios: the parabolic and hyperbolic cases. In particular, we derive a unified expression for correspondences

$$f(u,v)=\left(\frac{u}{v},\frac{1+\sigma v^2-u^2}{2v},\frac{1-\sigma v^2+u^2}{2v}\right),$$

that maps the elliptic upper half plane ($\sigma=-1$) to the interior, the hyperbolic upper half plane ($\sigma=1$) to the exterior, and the parabolic upper half plane ($\sigma=0$) to the boundary of the projective unit circle. Through extending the well-established theory of functions from the elliptic (complex) case to parabolic and hyperbolic contexts, this research aims to contribute to the advancement of function theory in these settings, thereby facilitating a more comprehensive understanding of these geometries.

Furthermore, projective geometry augmented with Clifford algebra offers a unified algebraic framework for characterizing points, lines, planes, etc., and their transformations, as exemplified in [**19, 18**]. By utilizing representations of the $\mathrm{SL}(2,\mathbb{R})$ group in Clifford valued function spaces and employing three different Clifford algebras $\mathrm{Cl}(e)$, $\mathrm{Cl}(p)$, and $\mathrm{Cl}(h)$ corresponding to the elliptic, parabolic, and hyperbolic cases in $n+1$ dimensions respectively, we anticipate analogous constructions of these findings for the multidimensional case of $\mathbb{RP}^n$. Hence, our approach for $n=2$ also provides a robust framework for extending the results to multidimensional cases in EPH settings.

2. PRELIMINARIES

The purpose of this section is to present the fundamental notions and results required for the development of this work.

The term transformation group G refers to a collection of mappings from a set X to itself, satisfying: (i) the identity map is a member of G, (ii) composing any two elements of G produces another element of G, (iii) each map in G admits an inverse map, which is also an element of G.

2.1. Lie group action and homogeneous space

As per Cartan's theorem [**30**], if G is a Lie group with a closed subgroup H, then H is also a Lie group. Denoting the set of left cosets of H as $G/H = \{gH : g \in G\}$, a consequential well-known result states that, for a Lie group G with H as a closed subgroup and G/H equipped with the quotient topology, G/H exhibits a unique smooth manifold structure. In this structure, the projection map $p\colon G \to G/H$, defined by mapping $g \in G$ to its equivalence class $[g]$ through the expression $p(g) = gH = [g]$, is a smooth submersion. Moreover, G acts smoothly on G/H, see [**21**].

Definition 2.1 (Section of projection map)**.** A section s of the projection map p is defined as a right inverse of p, that is, a smooth map $s\colon G/H \to G$ satisfying $p(s(x)) = x$ for all $x \in G/H$.

As an example, let $G = \mathbb{R}^2$ and $H = \{(0, y) : y \in \mathbb{R}\}$, a closed Lie subgroup. Then $G/H \cong \mathbb{R}$, and the canonical projection map is $p(x, y) = x$. A section is given by $s(x) = (x, 0)$ since $p \circ s = \mathrm{id}_{\mathbb{R}}$.

Another notable result in this context, to be referenced in our results, is the following theorem:

Theorem 2.2 ([**30**], Homogeneous space as quotient space)**.** *Consider a Lie group G and a homogeneous G-space K. Let $p \in K$, and let H denote the isotropy subgroup of the point p. Then H is a closed subgroup of G. Furthermore, the map $F\colon G/H \to K$ defined as $gH \mapsto g(p)$ is an isomorphism.*

2.2. Matrix Lie group, Lie algebra and adjoint representation

In accordance with the definition presented in [**16**], a matrix Lie group can be characterized as a subgroup G of $\mathrm{GL}(n, \mathbb{C})$ with the property that whenever a sequence of matrices $\{S_m\} \subset G$ converges to a matrix S, the limit satisfies either $S \in G$ or $S \notin \mathrm{GL}(n, \mathbb{C})$. The Lie algebra associated with a matrix Lie group G, denoted as $\mathfrak{g}$, is comprised of all matrices X for which e^{tX} belongs to G for every real number t.

In this context, consider $\mathrm{GL}(\mathfrak{g})$ as the group of all invertible linear transformations of $\mathfrak{g}$. Then, the adjoint representation of G is described by the mapping $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g}),\ g \mapsto \mathrm{Ad}_g$, defined as

$$\mathrm{Ad}_g(X) = gXg^{-1}, \text{ where } g \in G \text{ and for all } X \in \mathfrak{g}.$$

2.3. SL$(2, \mathbb{R})$ action in the EPH cases

As in [**26**], we consider the Möbius action of $\mathrm{SL}(2, \mathbb{R})$ in the elliptic, parabolic and hyperbolic cases, abbreviated as EPH cases, defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw + b}{cw + d}, \tag{1}$$

where $w = u + \iota v$ and $\iota^2 = \sigma = -1, 0, 1$ for EPH cases, respectively.

Proposition 2.3 ([**26**], One-parameter subgroup)**.** *Every one-parameter subgroup within $SL(2,\mathbb{R})$ can be conjugated to one of the following subgroups:*

$$\mathbf{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} : \ a \in \mathbb{R}, \ a > 0 \right\},$$

$$\mathsf{N} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : \ x \in \mathbb{R} \right\},$$

$$\mathsf{K} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} : \ t \in (-\pi, \pi] \right\}.$$

Remark. Any matrix of $\mathbf{A}$ and N is similar to a matrix of A' and N', respectively, where

$$A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : \ t \in \mathbb{R} \right\}, \qquad N' = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : \ t \in \mathbb{R} \right\} \text{ (cf. [26]).}$$

Therefore, as the subgroups A' and N' are conjugate to the subgroups $\mathbf{A}$ and N, respectively, instead of $\mathbf{A}$ and N, we often use subgroups A' and N' in our result (up to conjugacy).

The subsequent result presents a categorization of EPH upper half planes as homogeneous spaces, which will be employed in our main theorems.

Lemma 2.4 ([**26**], EPH upper half plane as homogeneous space)**.** *Consider the Möbius action* (1) *of $SL(2,\mathbb{R})$ in all the EPH cases. In each case, the isotropy subgroup of the point ι can be described as follows:*

1) *In the elliptic case, the isotropy subgroup is the compact subgroup K. Consequently, the elliptic upper half plane is isomorphic to the homogeneous space $SL(2,\mathbb{R})/\mathsf{K}$.*
2) *In the parabolic case, the isotropy subgroup is N'. Hence, the parabolic upper half plane is isomorphic to $SL(2,\mathbb{R})/N'$.*
3) *In the hyperbolic case, the isotropy subgroup is A'. Thus, the hyperbolic upper half plane is isomorphic to $SL(2,\mathbb{R})/A'$.*

2.4. Lines and conics in $\mathbb{RP}^2$

Line L in the projective plane $\mathbb{RP}^2$ is represented by the homogeneous coordinates $L = (a,b,c)^T$ and is defined as $ax + by + cz = 0$, or in matrix notation $p^T L = 0$, where $p = (x,y,z)^T$ is any point in L.

Conic in the projective plane $\mathbb{RP}^2$ is the set of points for which a quadratic form on $\mathbb{R}^3$ vanishes. Thus conic associated to a quadratic form (or matrix) is $\mathbf{C}_{\mathcal{A}} = \{[p] \in \mathbb{RP}^2 : p^T \mathcal{A} p = 0\}$, see [**32**].

Let us consider the equation of conic defined by

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 = 0, \text{ or } p^T \mathcal{A} p = 0,$$

where $p = (x,y,z)^T$ and $\mathcal{A} = \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix}$.

A conic is said to be non degenerate if the associated matrix $\mathcal{A}$ is non singular.

2.5. Two-dimensional homogeneous space $\mathbb{RP}^2$

Let $G = \mathrm{SL}(3,\mathbb{R})$, and let

$$H = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \;\middle|\; a_{22}a_{33} - a_{23}a_{32} = \frac{1}{a_{11}},\ a_{11} \neq 0 \right\}$$

be a closed Lie subgroup of G. We study the quotient space of left cosets $X = G/H$ together with the left action of G given by $g\colon z \mapsto g \cdot z$. With this action, X becomes a homogeneous G-space. For a more intuitive geometric understanding of this action, we parametrize X and represent the G-action using these parameters, as illustrated below.

Since X is a two-dimensional space, each element can be represented by the pair (x, y). With this parametrization, the action on G/H is expressed as

$$g\colon z \mapsto g \cdot z = p(g * s(z)),$$

where p denotes the projection, s a section, and $*$ the group operation in G. Utilizing the maps p and s, we can define an additional map $r\colon G \to H$ such that $r(g) = h$, where $h = s(p(g))^{-1}g$. Thus, g can be uniquely expressed as $g = s(p(g))r(g)$, which yields

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11}/a_{31} & 0 & 1 \\ a_{21}/a_{31} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & \frac{a_{21}}{a_{31}}a_{32} - a_{22} & \frac{a_{21}}{a_{31}}a_{33} - a_{23} \\ 0 & a_{12} - \frac{a_{11}}{a_{31}}a_{32} & a_{13} - \frac{a_{11}}{a_{31}}a_{33} \end{pmatrix}, \qquad a_{31} \neq 0.$$

Accordingly, we define the projection as $p(g) = \left(\dfrac{a_{11}}{a_{31}}, \dfrac{a_{21}}{a_{31}}\right)$, where the matrix $\begin{pmatrix} a_{11}/a_{31} & 0 & 1 \\ a_{21}/a_{31} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ serves as a representative of the equivalence class of $[g]$.

A section $s\colon X \to \mathrm{SL}(3,\mathbb{R})$ is fixed by $s(x,y) = \begin{pmatrix} x & 0 & 1 \\ y & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Within this framework, the $\mathrm{SL}(3,\mathbb{R})$ action then takes the form

$$(x,y) \mapsto \left(\frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}} \right),$$

provided $a_{31}x + a_{32}y + a_{33} \neq 0$ (see [**8**] for the details).

Now, if we allow $a_{31}x + a_{32}y + a_{33} = 0$, then this action indeed gives us a projective transformation of the space $\mathbb{RP}^2$. This $\mathrm{SL}(3,\mathbb{R})$ action on $\mathbb{RP}^2$ is denoted as $g\colon [p] \mapsto [g \cdot p]$. Let ϕ be the action

$$\phi\colon \mathrm{SL}(3,\mathbb{R}) \times \mathbb{RP}^2 \to \mathbb{RP}^2, \qquad \phi(g,[p]) = [g \cdot p].$$

We consider the projective transformation $\phi_g\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ such that $\phi_g([p]) = [g \cdot p]$ for all $g \in \mathrm{SL}(3,\mathbb{R})$.

2.6. Stabilizer of the projective unit circle

Let us consider the equation of conic C as $p^T Ap = 0$, where A be the matrix associated with the conic C. If we transform the conic projectively, then under the action of g, the transformed conic is represented by the matrix $(g^{-1})^T Ag^{-1}$, see [**32**].

Let us now consider the equation of the unit circle C in homogeneous coordinates:

$$x^2 + y^2 = z^2. \tag{2}$$

As the subgroup $\mathrm{SO}(2,1)$ of $\mathrm{SL}(3,\mathbb{R})$ preserves the bilinear form $[\cdot,\cdot]_{2,1}$ on $\mathbb{R}^{2+1}$, defined by

$$[\xi,\eta]_{2,1} = \xi_1\eta_1 + \xi_2\eta_2 - \xi_3\eta_3$$

for $\xi = (\xi_1,\xi_2,\xi_3)^T$, $\eta = (\eta_1,\eta_2,\eta_3)^T \in \mathbb{R}^{2+1}$ (see [**16, 31**] for further details), the following result holds.

Theorem 2.5. *Under the projective action, the stabilizer M of the projective unit circle is the Lie group* $\mathrm{SO}(2,1)$*, the generalized special orthogonal subgroup of* $\mathrm{SL}(3,\mathbb{R})$.

3. RELATIONSHIP BETWEEN THE STABILIZER OF THE PROJECTIVE UNIT CIRCLE AND $\mathrm{SL}(2,\mathbb{R})$

Let M be the stabilizer of the projective unit circle as mentioned in Theorem 2.5.

Lemma 3.1. *The Lie subalgebra $\mathfrak{m}$ of the Lie subgroup M is isomorphic to $\mathfrak{sl}(2)$, the Lie algebra of $SL(2,\mathbb{R})$.*

Proof. In Theorem 2.5, we have seen that the stabilizer M is $\mathrm{SO}(2,1)$. The Lie algebra of $\mathrm{SO}(2,1)$, denoted as $\mathfrak{so}(2,1)$, is given by

$$\mathfrak{so}(2,1) = \left\{ \begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} : a,b,c \in \mathbb{R} \right\}.$$

The Lie algebra of the special linear group $\mathfrak{sl}(2)$ consists of all 2×2 trace less real matrices. It is a standard result that there exists an isomorphism of Lie algebras

$$\phi\colon \mathfrak{sl}(2) \longrightarrow \mathfrak{so}(2,1) \text{ such that } \phi([X,Y]) = [\phi(X),\phi(Y)] \\ \text{for all } X,\ Y \in \mathfrak{sl}(2) \text{ (see, e.g., [\textbf{16}])}, \tag{3}$$

where the Lie bracket is defined as $[X,Y] = XY - YX$, X, $Y \in \mathfrak{sl}(2)$. Hence, the result follows. ☐

The component of $\mathrm{SO}(2,1)$ containing the identity is a subgroup of $\mathrm{SO}(2,1)$ and denoted as $\mathrm{SO}^+(2,1)$, see [**16**]. In connection to this, the next result states the following:

Theorem 3.2. *The identity component of the stabilizer of the projective unit circle is isomorphic to the Lie group* $\mathrm{PSL}(2,\mathbb{R})$.

Proof. To establish the result, we give a rigorous construction of the well-known result that $\mathrm{SO}^+(2,1)$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ through the adjoint representation of $\mathrm{SL}(2,\mathbb{R})$.

We consider the Lie algebra $\mathfrak{sl}(2)$ equipped with the symmetric bilinear form

$$\langle\cdot,\cdot\rangle\colon \mathfrak{sl}(2)\times\mathfrak{sl}(2)\to\mathbb{R},\quad \langle X,Y\rangle=\mathrm{trace}(XY),$$

whose signature is $(2,1)$. Hence, $\mathfrak{sl}(2)$ is a real vector space endowed with a form of signature $(2,1)$.

Next, we consider the adjoint representation of $\mathrm{SL}(2,\mathbb{R})$ (see Subsection 2.2), which acts on $\mathfrak{sl}(2)$ by conjugation $\mathrm{Ad}\colon \mathrm{SL}(2,\mathbb{R})\to\mathrm{GL}(\mathfrak{sl}(2))$ defined by the formula

$$\mathrm{Ad}_g(X)=gXg^{-1},\qquad g\in\mathrm{SL}(2,\mathbb{R}),\quad X\in\mathfrak{sl}(2). \tag{4}$$

Since $\mathfrak{sl}(2)$ is a real vector space of dimension 3, we have $\mathrm{GL}(\mathfrak{sl}(2))\simeq\mathrm{GL}(3,\mathbb{R})$ (see [**34**]). The adjoint map Ad preserves the form $\langle\cdot,\cdot\rangle$ of signature $(2,1)$. Therefore, Ad maps $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SO}(2,1)$, since $\mathrm{SO}(2,1)$ is the subgroup of $\mathrm{SL}(3,\mathbb{R})$ preserving a symmetric bilinear form of signature $(2,1)$ on $\mathbb{R}^{2+1}$. Furthermore, as $\mathrm{SL}(2,\mathbb{R})$ is connected and Ad is a Lie group homomorphism, Ad maps $\mathrm{SL}(2,\mathbb{R})$ into the identity component $\mathrm{SO}^+(2,1)$ of $\mathrm{SO}(2,1)$ (cf. [**16, 17**]).

Now, we explicitly express

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{R})$$

in the form of an element of $\mathrm{SO}(2,1)$ under the adjoint map. The invertible linear map Ad_g of $\mathfrak{sl}(2)$ is determined uniquely (up to similarity) once an ordered basis of $\mathfrak{sl}(2)$ is fixed. Let

$$E_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad E_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad E_3=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

be an orthogonal ordered basis of $\mathfrak{sl}(2)$. Through straightforward computation, we obtain:

$$\mathrm{Ad}_g(E_1)=(ad+bc)E_1+(cd-ab)E_2+(ab+cd)E_3, \tag{5}$$

$$\mathrm{Ad}_g(E_2)=(bd-ac)E_1+\tfrac{1}{2}(a^2+d^2-b^2-c^2)E_2+\tfrac{1}{2}(b^2+d^2-a^2-c^2)E_3, \tag{6}$$

$$\mathrm{Ad}_g(E_3)=(ac+bd)E_1+\tfrac{1}{2}(c^2+d^2-a^2-b^2)E_2+\tfrac{1}{2}(a^2+b^2+c^2+d^2)E_3. \tag{7}$$

Hence, the matrix representation of the linear map Ad_g with respect to the ordered basis $\{E_1,E_2,E_3\}$ is

$$\mathrm{Ad}_g=\begin{pmatrix}ad+bc & bd-ac & ac+bd\\ cd-ab & \tfrac{1}{2}(a^2+d^2-b^2-c^2) & \tfrac{1}{2}(c^2+d^2-a^2-b^2)\\ ab+cd & \tfrac{1}{2}(b^2+d^2-a^2-c^2) & \tfrac{1}{2}(a^2+b^2+c^2+d^2)\end{pmatrix}.$$

We see that Ad_g is the identity map if and only if $g = \pm I_{2\times 2}$. Thus, we obtain the homomorphism

$$\mathrm{Ad}\colon \mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}^+(2,1)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} ad+bc & bd-ac & ac+bd \\ cd-ab & \frac{1}{2}(a^2+d^2-b^2-c^2) & \frac{1}{2}(c^2+d^2-a^2-b^2) \\ ab+cd & \frac{1}{2}(b^2+d^2-a^2-c^2) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix} \tag{8}$$

with kernel $\{\pm I_{2\times 2}\}$. Hence,

$$\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}^+(2,1),$$

i.e., the identity component $\mathrm{SO}^+(2,1)$ of the stabilizer $\mathrm{SO}(2,1)$ of the projective unit circle is isomorphic to the Lie group $\mathrm{PSL}(2,\mathbb{R})$. □

4. ISOTROPY SUBGROUPS AND CORRESPONDING ACTIONS ON THE HOMOGENEOUS SPACES

In the previous section, we demonstrated that the identity component of the stabilizer of the projective unit circle is isomorphic to the Lie group $\mathrm{SL}(2,\mathbb{R})$. Therefore, recalling the Möbius action of $\mathrm{SL}(2,\mathbb{R})$ in the elliptic, parabolic, and hyperbolic (abbreviated as EPH) cases, we get that this action generates three homogeneous spaces that are isomorphic to the elliptic, parabolic, and hyperbolic upper half planes. In the present section, we embed the elliptic, parabolic, and hyperbolic subgroups of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(3,\mathbb{R})$ and construct explicit mappings that translate these EPH upper half plane geometries into their projective analogues in $\mathbb{RP}^2$.

These concrete subgroup actions thereby illustrate the geometric transition from EPH upper half planes to the real projective plane through the induced projective action.

Theorem 4.1. *The correspondence map between the parabolic upper half plane and the boundary of the projective unit circle is*

$$\mathfrak{f}_1(u,v) = \left(\frac{2u}{v}, \frac{1-u^2}{v}, \frac{1+u^2}{v}\right).$$

Proof. According to Lemma 2.4, we see that the parabolic upper half plane is isomorphic to $\mathrm{SL}(2,\mathbb{R})/N'$. Hence, we consider the subgroup $N' = \left\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} : a \in \mathbb{R}\right\}$ of $\mathrm{SL}(2,\mathbb{R})$. Therefore, under the homomorphism Ad (see equation (8)), we have

$$\mathrm{Ad}\,(N') = N^\dagger = \begin{pmatrix} 1 & -a & a \\ a & \frac{2-a^2}{2} & \frac{a^2}{2} \\ a & -\frac{a^2}{2} & \frac{2+a^2}{2} \end{pmatrix}.$$

Hence, the characteristic polynomial of $N^\dagger$ is

$$\lambda^3 - \operatorname{trace}\left(N^\dagger\right)\lambda^2 + \frac{1}{2}\left\{\left(\operatorname{trace}\left(N^\dagger\right)\right)^2 - \operatorname{trace}\left({N^\dagger}^2\right)\right\}\lambda - \det\left(N^\dagger\right) = 0$$
$$\implies (\lambda-1)^3 = 0 \implies \lambda = 1.$$

Therefore, the corresponding fixed vectors[1] $(x_1, x_2, x_3)^T$ are

$$\begin{pmatrix} 0 & -a & a \\ a & -\frac{a^2}{2} & \frac{a^2}{2} \\ a & -\frac{a^2}{2} & \frac{a^2}{2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies -ax_2 + ax_3 = 0, \tag{9}$$

$$\text{and} \quad ax_1 - \frac{a^2}{2}x_2 + \frac{a^2}{2}x_3 = 0. \tag{10}$$

$$\therefore \quad x_1 = 0, \; x_2 = x_3 = \lambda_1 \text{ for some non zero } \lambda_1.$$

Thus, $(0, \lambda_1, \lambda_1)^T \in \mathbb{RP}^2$ is the only fixed vector of the matrix $N^\dagger$.
Conversely, we can verify that the matrices fixing the vector $(0,1,1)^T$ are in $N^\dagger$.
If $\operatorname{Ad}(\mathcal{M}) = \begin{pmatrix} ad+bc & bd-ac & ac+bd \\ cd-ab & \frac{1}{2}(a^2+d^2-b^2-c^2) & \frac{1}{2}(c^2+d^2-a^2-b^2) \\ ab+cd & \frac{1}{2}(b^2+d^2-a^2-c^2) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix}$ fixes the vector $(0,1,1)^T$, where $\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} ad+bc & bd-ac & ac+bd \\ cd-ab & \frac{1}{2}(a^2+d^2-b^2-c^2) & \frac{1}{2}(c^2+d^2-a^2-b^2) \\ ab+cd & \frac{1}{2}(b^2+d^2-a^2-c^2) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ \lambda \end{pmatrix}$$

$$\implies bd - ac + ac + bd = 0, \tag{11}$$

$$\frac{1}{2}(a^2+d^2-b^2-c^2) + \frac{1}{2}(c^2+d^2-a^2-b^2) = \lambda, \tag{12}$$

$$\frac{1}{2}(b^2+d^2-a^2-c^2) + \frac{1}{2}(a^2+b^2+c^2+d^2) = \lambda. \tag{13}$$

Therefore, equations (11), (12) and (13) refer that up to factor $\pm I_{2\times 2}$, the matrices are of the form of the lower triangular matrices $\Gamma = \left\{\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} : \gamma \in \mathbb{R}\right\}$ (up to similarity). Hence, the matrices of $\mathrm{SO}(2,1)$ stabilizing the vector $(0,1,1)^T$ are in $\operatorname{Ad}(\Gamma) = N^\dagger$.

Again, for any point $(x,y,z)^T$ on the boundary of the projective unit circle, it satisfies the equation of the circle, which implies that $x^2 + y^2 = z^2$. Now, there is

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[1] Given that $x \sim \lambda x$ in projective space $\mathbb{RP}^2$, the fixed point $[x] \in \mathbb{RP}^2$ can be regarded as an eigenvector in $\mathbb{R}^3$ because if $[x] \in \mathbb{RP}^2$ is a fixed point of $A \in \mathrm{SL}(3,\mathbb{R})$, then $A[x] = [x] = [\lambda x] = \lambda[x]$, where $\lambda \in \mathbb{R}$ is a non-zero scalar.

$g \in \mathrm{SO}^+(2,1)$ such that

$$g \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} y/z & x(z^2+1)/2z^2 & x(z^2-1)/2z^2 \\ -x/z & y(z^2+1)/2z^2 & y(z^2-1)/2z^2 \\ 0 & z^2-1/2z & z^2+1/2z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which makes the boundary of the projective unit circle a homogeneous space under the $\mathrm{SO}^+(2,1)$ action. Also, we find that the isotropy subgroup of the point $(0,1,1)^T$ on the boundary is the subgroup $N^\dagger$. Therefore, using Theorem 2.2, we identify the boundary of the projective unit circle with the homogeneous space $\mathrm{SO}^+(2,1)/N^\dagger$ under the isomorphism $g \cdot p \mapsto gN^\dagger$, where $p = (0,1,1)^T$.

Again, we have that the parabolic upper half plane is isomorphic to $\mathrm{SL}(2,\mathbb{R})/N'$. In particular, we have a map from the parabolic upper half plane to the boundary of the projective unit circle, both of which are identified with isomorphic homogeneous spaces. Consequently, for any point $u + \iota v$ in the upper half plane, we consider $g = \begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ such that

$$\begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix} \iota = \frac{\sqrt{v}\iota + \frac{u}{\sqrt{v}}}{\frac{1}{\sqrt{v}}} = u + \iota v, \text{ where } \iota^2 = \sigma = 0.$$

Now, under the Ad map (8), we have

$$\begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix} \mapsto \begin{pmatrix} 1 & \frac{u}{v} & \frac{u}{v} \\ -u & \frac{1}{2}\left(v + \frac{1}{v} - \frac{u^2}{v}\right) & \frac{1}{2}\left(\frac{1}{v} - v - \frac{u^2}{v}\right) \\ u & \frac{1}{2}\left(\frac{u^2}{v} + \frac{1}{v} - v\right) & \frac{1}{2}\left(v + \frac{1}{v} + \frac{u^2}{v}\right) \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} 1 & \dfrac{u}{v} & \dfrac{u}{v} \\ -u & \dfrac{1}{2}\left(v + \dfrac{1}{v} - \dfrac{u^2}{v}\right) & \dfrac{1}{2}\left(\dfrac{1}{v} - v - \dfrac{u^2}{v}\right) \\ u & \dfrac{1}{2}\left(\dfrac{u^2}{v} + \dfrac{1}{v} - v\right) & \dfrac{1}{2}\left(v + \dfrac{1}{v} + \dfrac{u^2}{v}\right) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{2u}{v} \\ \dfrac{1-u^2}{v} \\ \dfrac{1+u^2}{v} \end{pmatrix}.$$

Thus, the function $\mathfrak{f}_1 \colon$ parabolic upper half plane $\longrightarrow \mathbb{RP}^2$ given by

$$(u,v) \mapsto \left(\frac{2u}{v}, \frac{1-u^2}{v}, \frac{1+u^2}{v}\right)$$

establishes the correspondence between the parabolic upper half plane and the boundary of the projective unit circle. $\square$

Theorem 4.2. *The function* $\mathfrak{f}_2$ *that establishes a correspondence between the elliptic upper half plane and the interior of the projective unit circle is given by* $\mathfrak{f}_2(u,v) = \left(\dfrac{u}{v}, \dfrac{1-v^2-u^2}{2v}, \dfrac{1+v^2+u^2}{2v}\right)$.

Proof. Here we see that the elliptic upper half plane is isomorphic to $\mathrm{SL}(2,\mathbb{R})/\mathsf{K}$ (cf. Lemma 2.4). Thus, we consider the compact subgroup

$$\mathsf{K} = \left\{ \begin{pmatrix} a & \sqrt{1-a^2} \\ -\sqrt{1-a^2} & a \end{pmatrix} : a \in [0,1] \right\}$$

of $\mathrm{SL}(2,\mathbb{R})$. Under the Ad homomorphism (8), we have

$$\mathrm{Ad}(\mathsf{K}) = K^{\dagger} = \begin{pmatrix} 2a^2-1 & 2a\sqrt{1-a^2} & 0 \\ -2a\sqrt{1-a^2} & 2a^2-1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As in the parabolic case, we observe that $(0,0,\lambda_2)^T \in \mathbb{RP}^2$ is the only real fixed vector of the subgroup $K^{\dagger}$.

Conversely, the matrices stabilizing the vector $(0,0,1)^T$ are of the form $K^{\dagger}$.

Again, for any point $(x,y,z)^T$ in the interior of the projective unit circle with $x^2+y^2<z^2$, we can choose the homogeneous coordinates as $(x,y,z)^T$ such that $x^2+y^2+1=z^2$, since $(x,y,z)^T$ and $(\lambda x,\lambda y,\lambda z)^T$ represent the same point, for all $\lambda \neq 0$ in $\mathbb{R}$. For example, let $p=(x,y,z)^T$ be a point in the interior of the unit circle such that $z^2-(x^2+y^2)\neq 1$. Since $x^2+y^2<z^2$, let $z^2-(x^2+y^2)=\alpha>0$. Now, we select $k=\frac{1}{\sqrt{\alpha}}$ and set $(\frac{x}{\sqrt{\alpha}}, \frac{y}{\sqrt{\alpha}}, \frac{z}{\sqrt{\alpha}})^T$ as the representation of p, so that $(\frac{z}{\sqrt{\alpha}})^2-(\frac{x}{\sqrt{\alpha}})^2-(\frac{y}{\sqrt{\alpha}})^2=1$. Furthermore, there is $g\in \mathrm{SO}^+(2,1)$ such that

$$g\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix} xz/\sqrt{z^2-1} & -y/\sqrt{z^2-1} & x \\ yz/\sqrt{z^2-1} & x/\sqrt{z^2-1} & y \\ \sqrt{z^2-1} & 0 & z \end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix},$$

which makes the interior of the projective unit circle a homogeneous space under the $\mathrm{SO}^+(2,1)$ action. Also, the isotropy subgroup of the point $(0,0,1)^T$ inside the projective unit circle is the subgroup $K^{\dagger}$. Thus, using Theorem 2.2, we identify the space $\mathbb{RP}^2$ with the homogeneous space $\mathrm{SO}^+(2,1)/K^{\dagger}$ under the isomorphism $g\cdot p\mapsto gK^{\dagger}$, where $p=(0,0,1)^T$. Again, the elliptic upper half plane is isomorphic to $\mathrm{SL}(2,\mathbb{R})/\mathsf{K}$.

Similar to the parabolic case, here we have a map from the elliptic upper half plane to the space $\mathbb{RP}^2$, both of which are identified with isomorphic homogeneous spaces. Therefore, for any point $u+\iota v$ in the elliptic upper half plane, we consider $g=\begin{pmatrix}\sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}}\end{pmatrix}\in \mathrm{SL}(2,\mathbb{R})$, so that $\begin{pmatrix}\sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}}\end{pmatrix}\iota = u+\iota v$, where $\iota^2=\sigma=-1$. Under the Ad map (8), we have

$$\begin{pmatrix}\sqrt{v} & \dfrac{u}{\sqrt{v}} \\ 0 & \dfrac{1}{\sqrt{v}}\end{pmatrix} \mapsto \begin{pmatrix} 1 & \frac{u}{v} & \frac{u}{v} \\ -u & \frac{1}{2}\left(v+\frac{1}{v}-\frac{u^2}{v}\right) & \frac{1}{2}\left(\frac{1}{v}-v-\frac{u^2}{v}\right) \\ u & \frac{1}{2}\left(\frac{u^2}{v}+\frac{1}{v}-v\right) & \frac{1}{2}\left(v+\frac{1}{v}+\frac{u^2}{v}\right) \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} 1 & \dfrac{u}{v} & \dfrac{u}{v} \\ -u & \dfrac{1}{2}\left(v+\dfrac{1}{v}-\dfrac{u^2}{v}\right) & \dfrac{1}{2}\left(\dfrac{1}{v}-v-\dfrac{u^2}{v}\right) \\ u & \dfrac{1}{2}\left(\dfrac{u^2}{v}+\dfrac{1}{v}-v\right) & \dfrac{1}{2}\left(v+\dfrac{1}{v}+\dfrac{u^2}{v}\right) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{u}{v} \\ \dfrac{1}{2}\left(\dfrac{1}{v}-v-\dfrac{u^2}{v}\right) \\ \dfrac{1}{2}\left(v+\dfrac{1}{v}+\dfrac{u^2}{v}\right) \end{pmatrix}.$$

Thus, the map $\mathfrak{f}_2$: elliptic upper half plane $\longrightarrow \mathbb{RP}^2$ is defined as

$$(u,v) \mapsto \left(\frac{u}{v}, \frac{1-v^2-u^2}{2v}, \frac{1+v^2+u^2}{2v}\right),$$

establishing the correspondence between the elliptic upper half plane and the interior of the projective unit circle. □

Theorem 4.3. *The function* $\mathfrak{f}_3(u,v) = \left(\dfrac{u}{v}, \dfrac{1+v^2-u^2}{2v}, \dfrac{1-v^2+u^2}{2v}\right)$ *establishes the correspondence between the hyperbolic upper half plane and the exterior of the projective unit circle.*

Proof. Proceeding analogously to the elliptic case, the hyperbolic upper half plane is isomorphic to $\mathrm{SL}(2,\mathbb{R})/A'$, where $A' = \left\{ \begin{pmatrix} a & \sqrt{a^2-1} \\ \sqrt{a^2-1} & a \end{pmatrix} : a \in \mathbb{R} \right\}$ (cf. Lemma 2.4). Under the adjoint homomorphism (8),

$$\mathrm{Ad}(A') = A^\dagger = \begin{pmatrix} 2a^2-1 & 0 & 2a\sqrt{a^2-1} \\ 0 & 1 & 0 \\ 2a\sqrt{a^2-1} & 0 & 2a^2-1 \end{pmatrix},$$

whose fixed vector is $(0,1,0)^T \in \mathbb{RP}^2$. Conversely, the matrices fixing the vector $(0,1,0)^T$ are in $A^\dagger$.

Also, there is $g \in \mathrm{SO}^+(2,1)$ such that

$$\begin{pmatrix} y/\sqrt{z^2+1} & x & xz/\sqrt{z^2+1} \\ -x/\sqrt{z^2+1} & y & yz/\sqrt{z^2+1} \\ 0 & z & \sqrt{z^2+1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Thus, employing a parallel approach to the other two cases, we find that the exterior of the projective unit circle is the homogeneous space $\mathrm{SO}^+(2,1)/A^\dagger$.

Let $g = \begin{pmatrix} \sqrt{v} & \dfrac{u}{\sqrt{v}} \\ 0 & \dfrac{1}{\sqrt{v}} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, $g\iota = u + \iota v$, $\iota^2 = \sigma = 1$.

Using equation (8), we obtain

$$\mathrm{Ad}(g) = \begin{pmatrix} 1 & \dfrac{u}{v} & \dfrac{u}{v} \\ -u & \dfrac{1}{2}\left(v + \dfrac{1}{v} - \dfrac{u^2}{v}\right) & \dfrac{1}{2}\left(\dfrac{1}{v} - v - \dfrac{u^2}{v}\right) \\ u & \dfrac{1}{2}\left(\dfrac{u^2}{v} + \dfrac{1}{v} - v\right) & \dfrac{1}{2}\left(v + \dfrac{1}{v} + \dfrac{u^2}{v}\right) \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} 1 & \frac{u}{v} & \frac{u}{v} \\ -u & \frac{1}{2}\left(v + \frac{1}{v} - \frac{u^2}{v}\right) & \frac{1}{2}\left(\frac{1}{v} - v - \frac{u^2}{v}\right) \\ u & \frac{1}{2}\left(\frac{u^2}{v} + \frac{1}{v} - v\right) & \frac{1}{2}\left(v + \frac{1}{v} + \frac{u^2}{v}\right) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{u}{v} \\ \dfrac{1}{2}\left(v + \frac{1}{v} - \frac{u^2}{v}\right) \\ \dfrac{1}{2}\left(\frac{u^2}{v} + \frac{1}{v} - v\right) \end{pmatrix}.$$

Therefore, $\mathfrak{f}_3(u,v)$ provides the desired correspondence between the hyperbolic upper half plane and the exterior of the projective unit circle. □

Remark. In each of the preceding cases, we observe that the maps that are generated are comparable. Thus, we express the mappings from the upper half planes in the elliptic, parabolic, and hyperbolic cases (EPH cases) to the corresponding domain of the projective plane using a unified formula. This is defined as

$$f(w) = f(u,v) = \left(\frac{u}{v}, \frac{1 + \sigma v^2 - u^2}{2v}, \frac{1 - \sigma v^2 + u^2}{2v}\right), \tag{14}$$

where $w = u + \iota v = (u,v)$ and $\iota^2 = \sigma = -1,\ 0,\ 1$ for EPH cases, respectively. The inverse map of (14) in all the EPH cases is given by

$$f^{-1}(x,y,z) = \left(\frac{x}{y+z}, \frac{1}{y+z}\right) \text{ such that } x^2 + y^2 - \sigma = z^2. \tag{15}$$

5. Images of cycles in EPH cases

Now, we consider the map f defined in equation (14) and the corresponding inverse map f^{-1} given in equation (15) to study the images of Möbius invariant cycle (as described in [**24**]) within the elliptic, parabolic and hyperbolic upper half plane.

Theorem 5.1. *Under the correspondence map f as in equation* (14)*, cycles in the elliptic case are mapped to ellipses and cycles in the hyperbolic case are mapped to hyperbolas.*

Proof. We consider the equation of cycle C as

$$k(u^2-\sigma v^2)-2lu-2nv+m=0, \tag{16}$$

where k,l,n,m are real parameters, with the condition that not all of them are zero. Here $\sigma=-1,\ 1$ in elliptic and hyperbolic cases, respectively, [**26**].

Now, under the isomorphism f, the image of the cycle C is the set of points $P\in\mathbb{RP}^2$ for which $(u,v)=f^{-1}(P)$ is on the cycle. Let $\mathcal{C}$ represents the image of the cycle C. Therefore, we substitute the formula for (u,v) (see the relation in (15)) into the equation of the cycle, and hence we get

$$\begin{aligned}&k\left(\frac{x^2}{(y+z)^2}-\sigma\frac{1}{(y+z)^2}\right)-2l\frac{x}{y+z}-2n\frac{1}{y+z}+m=0\\ &\implies k(z^2-y^2)-2lx(y+z)-2n(y+z)+m(y+z)^2=0\ (\because\ x^2-\sigma=z^2-y^2)\\ &\implies (4l^2-\sigma 4n^2)x^2+((m-k)^2-\sigma 4n^2)y^2+((m+k)^2+\sigma 4n^2)z^2-4l(m-k)xy\\ &\qquad -4l(m+k)xz+2(m^2-k^2)yz=0 \qquad (\text{putting } \sigma=x^2+y^2-z^2,\ \sigma^2=1).\end{aligned}$$

Therefore, the cycle C is transformed to the conic $\mathcal{C}$ given by

$$Ax^2+Cy^2+Fz^2+2Bxy+2Dxz+2Eyz=0, \tag{17}$$

where

$$\begin{aligned}&A=4l^2-\sigma 4n^2, && C=(m-k)^2-\sigma 4n^2, && F=(m+k)^2+\sigma 4n^2,\\ &B=-2l(m-k), && D=-2l(m+k), && E=(m^2-k^2).\end{aligned} \tag{18}$$

We now consider the symmetric matrix $\mathcal{A}$ associated to the conic $\mathcal{C}$, where $\mathcal{A}=\begin{pmatrix}A&B&D\\B&C&E\\D&E&F\end{pmatrix}$. Here, determinant of $\mathcal{A}$ is equal to $64n^4(km+\sigma n^2-l^2)$.

In the elliptic case, $\dfrac{(l^2+n^2-mk)}{k^2}$ measures squared radius of the cycle C, whereas for the hyperbolic case, $\dfrac{4(l^2-n^2-km)}{k^2}$ measures the square length of the transverse axis of the cycle C. Thus, we have $\det(\mathcal{A})\neq 0$. Hence, the conic is non-degenerate.

Let us again consider the intersection of conic $\mathcal{C}$ with the line at infinity $z=0$, then from the equation (17), we have $Ax^2+Cy^2+2Bxy=0$. Let $t=\dfrac{x}{y}$, provided $y\neq 0$. Then,

$$At^2+2Bt+C=0. \tag{19}$$

i) For the elliptic case, this quadratic equation (19) has no real solution as the discriminant $4B^2-4AC=-16n^2(4n^2+(m-k)^2+4l^2)$ is negative. Thus, for the elliptic case, the equation (18) represents the equation of an ellipse, which has no point at infinity.

ii) For the hyperbolic case, this quadratic equation (19) has two real solutions as the discriminant $4B^2-4AC=16n^2(4(l^2-n^2-km)+(m+k)^2)$ is positive (since $\frac{4(l^2-n^2-km)}{k^2}$ measures the square length of the transverse axis). Thus, for the hyperbolic case, the equation (18) represents the equation of a hyperbola, which has two points at infinity, see [**32**].

Hence, under the isomorphism f, respective cycles are mapped to ellipses in the elliptic case and hyperbolas in the hyperbolic case. □

As we have the degeneracy ($\sigma = 0$) in the parabolic case (cf. [**25**]), it requires a special treatment to discuss the images of cycle on parabolic upper half plane.

Proposition 5.2. *In the parabolic case, the isomorphism f maps cycles to projective unit circle.*

Proof. We see that under the map

$$(u, v) \mapsto \left(\frac{u}{v}, \frac{1-u^2}{2v}, \frac{1+u^2}{2v}\right) = (x, y, z)\,,$$

the image of parabolic upper half plane is the unit circle $x^2 + y^2 = z^2$ on $\mathbb{RP}^2$. In particular, we can say that any cycle $ku^2 - 2lu - 2nv + m = 0$ on the parabolic upper half plane maps to the unit circle and this can be seen as follows:
Let us consider the cycle in the parabolic case as

$$ku^2 - 2lu - 2nv + m = 0,$$

where k, l, n, m are real parameters not all of them are zero, see [**26**]. As before, we substitute the formula for (u, v) (see the relation in (15)) into the equation of the cycle, and hence we get

$$k\frac{x^2}{(y+z)^2} - 2l\frac{x}{y+z} - 2n\frac{1}{y+z} + m = 0, \text{ where } x^2 + y^2 = z^2.$$

Thus,

$$kx^2 - 2lx(y+z) - 2n(y+z) + m(y+z)^2 = 0. \tag{20}$$

Therefore, we have

$$\begin{aligned} & k(z^2 - y^2) - 2lx(y+z) - 2n(y+z) + m(y+z)^2 = 0 \\ & \Longrightarrow K(z-y) - 2lx - 2n + m(y+z) = 0 \text{ (as } v = \frac{1}{z+y} \neq 0) \\ & \Longrightarrow 2n = K(z-y) - 2lx + m(y+z). \end{aligned} \tag{21}$$

Now, putting the value of $2n$ from equation (21) into the equation (20), we get

$$\begin{aligned} & kx^2 - 2lx(y+z) - (K(z-y) - 2lx + m(y+z))(y+z) + m(y+z)^2 = 0 \\ & \Longrightarrow kx^2 - k(z^2 - y^2) = 0 \\ & \Longrightarrow x^2 + y^2 = z^2 \end{aligned}$$

Thus, in the parabolic case, the cycles, which are represented by parabolas, are mapped to the unit circle given by the equation $x^2 + y^2 = z^2$. □

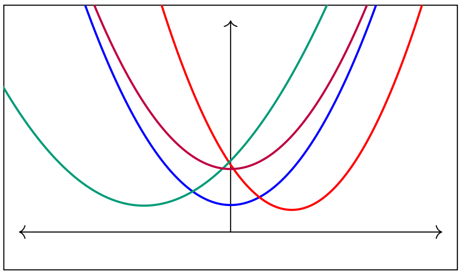

Figure 1. cycles in the parabolic upper half plane

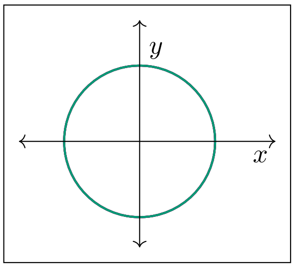

Figure 2. images of cycles under f in the affine plane ($x = \frac{X}{Z}, y = \frac{Y}{Z}$)

Remark. The equivalence map f defined by equations (14) and (15) sends lines in the projective plane to lines in the elliptic case. For example, the projective line $X + 2Y + Z = 0$ is mapped to the curve $(u - 1)^2 + v^2 = 4$, which is a semicircle with centre on the real axis (see Figures 3 and 4). In the Lobachevsky upper half plane model, such semicircles represent geodesics (cf. [**11**]), and in our context, the Lobachevsky upper half plane is referred to as the elliptic upper half plane. In the hyperbolic setting, the same projective line $X + 2Y + Z = 0$ is mapped to the curve $(u - 1)^2 - v^2 = 4$, which describes a hyperbola and represents a geodesic in this case (see Figures 3 and 5).

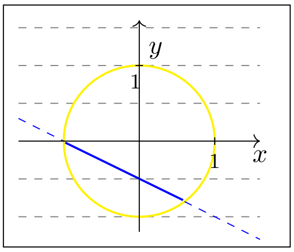

Figure 3. Unit circle and the line projected to affine plane ($x = \frac{X}{Z}, y = \frac{Y}{Z}$)

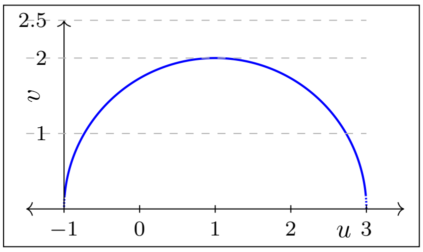

Figure 4. Image of the line under f in elliptic upper half plane

Remark. From equations (16) and (18), it follows that the quadric $\mathbf{C}$ of the form

$$\mathbf{C} : AX^2 + CY^2 \pm 2\sqrt{(C+F)(A-C)} + FZ^2 = 0$$

corresponds to the cycle

$$(F + C)(u^2 - \sigma v^2 + 1) \pm 2\sqrt{(C+F)(A-C)}u - 2\sqrt{-\sigma C(F+C)}v = 0,$$

under the condition $-\sigma C(F + C) \geq 0$; $\sigma^2 = \pm 1$. In the elliptic case ($\sigma^2 = -1$), this correspondence is illustrated in Figures 6 and 7 for some specific choices of parameters A, C and F. The analogous correspondence in the hyperbolic case ($\sigma^2 = 1$) is shown in Figures 8 and 9, for corresponding selections of A, C and F.

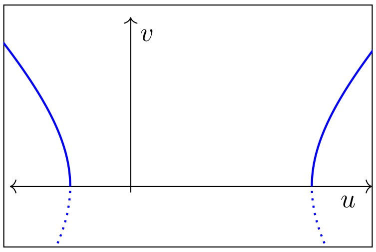

Figure 5. Image of the line under f in hyperbolic upper half plane

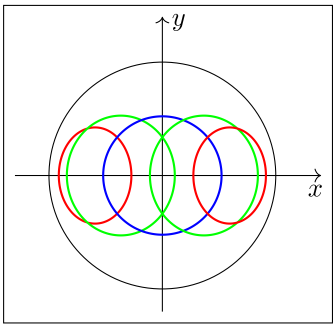

Figure 6. Quadrics **C** in the affine plane

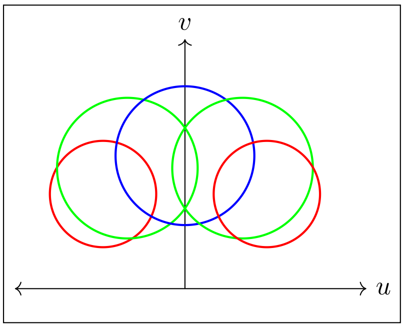

Figure 7. Image of the quadrics **C** under f in elliptic upper half plane

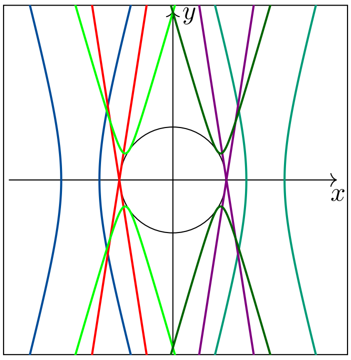

Figure 8. Quadrics **C** in the affine plane

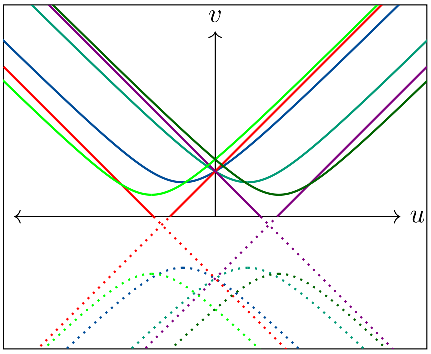

Figure 9. Image of the quadrics **C** under f in hyperbolic upper half plane

6. Applications

In our investigation, we have considered the EPH upper half planes, as well as the interior, exterior, and boundary of the projective unit disk. These geometric models have promising applications in physical models of space-time and various extensions of phase space. However, Poincaré's analysis of geometry in physics highlights that the choice of a geometric model depends on its convenience for a specific application, rather than its inherent truthfulness. In particular, the Poincaré ball model of three-dimensional hyperbolic geometry (referred to as elliptic in our terminology) is gaining significance in the development of hyperbolic browsers within the field of computer graphics. This development, discussed in [**35**], enables us to utilize the obtained isomorphism f (cf. equation (14)) and explore hyperbolic trigonometry within the projective disk model, leading to potential benefits and new perspectives. Moreover, the understanding of the conversion between the Klein model and the Poincaré model is essential in various scenarios. For instance, it plays a pivotal role in unraveling the "Seven Circles Theorem," as evidenced in [**15**]. In this context, we offer analogous conversions from the elliptic upper half plane to the projective disk, which have the potential to reveal valuable insights, such as the orthogonality of quadrics within the Klein disk model.

7. Conclusion

We investigate the projective actions of the subgroups of $\mathrm{SL}(3,\mathbb{R})$ on the non-degenerate conics in $\mathbb{RP}^2$. The identification of the identity component of the stabilizer of the projective unit circle as the isomorphic Lie group $\mathrm{SL}(2,\mathbb{R})$ highlights the fundamental role played by $\mathrm{SL}(2,\mathbb{R})$ in the projective transformations associated with conics. Furthermore, the exploration of the Möbius action of $\mathrm{SL}(2,\mathbb{R})$ in the elliptic, parabolic, and hyperbolic cases enables us to understand how different subgroups within $\mathrm{SL}(3,\mathbb{R})$ yield distinct mappings from the EPH upper half plane to various regions in the projective space. Also, the observation that cycles in the elliptic and hyperbolic cases map to ellipses and hyperbolas, respectively, highlights the correspondence between these conic sections and their associated subgroup actions. Finally, the recognition that cycles in the parabolic case are mapped to the projective unit circle due to their degeneracy provides important insights into the nature of parabolic transformations and their connection to the projective space. This sheds light on the special characteristics and limitations of parabolic subgroups within $\mathrm{SL}(3,\mathbb{R})$. These findings deepens our comprehension of the geometric structures and symmetries present in projective geometry, paving the way for applications in various fields, including computer graphics, and geometric modeling.

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