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RICCI SOLITONS ON A FAMILY OF FOUR-DIMENSIONAL WALKER MANIFOLDS

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Abstract. In this paper, we investigate the curvature properties of a family of four-dimensional Walker manifolds. We establish the existence of non-trivial Ricci solitons on this family of manifolds and provide explicit examples to illustrate the results.

1. Introduction

A four-dimensional pseudo-Riemannian manifold (M, g) of signature $(2, 2)$ is said to be a Walker manifold if it admits a two-dimensional null distribution $\mathcal{D}$, which is parallel with respect to the Levi-Civita connection of g [**4**]. This type of manifolds was introduced by Walker [**19**] who showed that the metric depends on three smooth functions. In local coordinates (x_1, x_2, x_3, x_4), the Walker metric has the following form:

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{1}$$

where a, b and c are functions of (x_1, x_2, x_3, x_4), and the two-dimensional null distribution $\mathcal{D}$ is locally generated by $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$. Curvature properties of four-dimensional Walker metrics were studied in [**9**], where examples with interesting geometric properties were given. Examples of Walker Osserman metrics of signature $(3, 3)$ which admits a field of parallel null 3-planes were given in [**12, 13**]. In [**10**], the authors studied the existence of Killing vector fields on a family of four-dimensional Walker manifolds. Also, in [**17**], conditions for a four-dimensional strict Walker manifold to be locally conformally flat and locally conformally symmetric were established. Numerous examples of Walker structures have appeared, which proved to be important in differential geometry and general relativity as

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well. Several results suggest that Walker structures are responsible for many of the basic differences between the strictly Riemannian and the pseudo-Riemannian cases.

A Ricci soliton is a natural generalization of an Einstein metric. It is defined on a pseudo-Riemannian manifold (M, g) by

$$\mathcal{L}_X g + \rho = \lambda \cdot g, \tag{2}$$

where X is a smooth vector field on M, $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, ρ is the Ricci tensor and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Moreover, we say that a Ricci soliton is a gradient Ricci soliton if it admits a vector field X satisfying $X = \operatorname{grad} h$ for some potential function h.

Although Ricci solitons were first investigated in Riemannian and Lorentzian signature, they have recently been investigated in a more general context by several authors. In [**1**], the authors investigated the Ricci soliton equation on Lorentzian manifolds whose isotropy group has dimension at least $\frac{1}{2}n(n-1)+1$, where n is the dimension of the underlying manifold. Brozos-Vázquez et al. [**2**] analyzed the existence of three-dimensional Lorentzian homogeneous Ricci solitons, showing the existence of shrinking, expanding and steady Ricci solitons in that setting. In [**3**], the authors studied and investigated locally conformally flat gradient Ricci solitons in the Lorentzian setting. Calvaruso and De Leo [**5**] gave some examples of Ricci solitons within the family of three-dimensional Walker manifolds. In [**6**], the authors obtained a full classification of homogeneous Ricci solitons on four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds. Non-trivial examples were obtained both in the Lorentzian case and for metrics of neutral signature $(2, 2)$. In particular, the authors showed the existence of shrinking, expanding and steady examples. In [**7**], the authors obtained a full classification of homogeneous pseudo-Riemannian Ricci solitons on 4-dimensional homogeneous manifolds with non-trivial isotropy. In [**18**], the authors showed that four-dimensional Lorentzian Walker manifolds admit different vector fields resulting in expanding, steady and shrinking Ricci solitons. Also, it was proved that those Ricci solitons are gradient only in the steady case.

Motivated by the above works, we consider the family of Walker metrics g_a on $\mathcal{O} \subset \mathbb{R}^4$ given by:

$$\begin{aligned} g_a = {}& 2(\mathrm{d}x_1 \circ \mathrm{d}x_3 + \mathrm{d}x_2 \circ \mathrm{d}x_4) + a(x_1, x_2, x_3, x_4)\mathrm{d}x_3 \circ \mathrm{d}x_3 \\ & + a(x_1, x_2, x_3, x_4)\mathrm{d}x_4 \circ \mathrm{d}x_4, \end{aligned} \tag{3}$$

where $a(x_1, x_2, x_3, x_4) = x_1 b(x_3, x_4) + x_2 c(x_3, x_4) + d(x_3, x_4)$ and we study the existence of Ricci solitons on the four-dimensional Walker metric (3). We organize the paper as follow: in Section 2, we describe the curvature of the metric considered. In Section 3, we study and characterize Ricci solitons on the four-dimensional Walker metric (3). Finally, in Section 4, we express the result in terms of Riemannian extensions.

2. Description of the metric

Henceforth, we denote $\partial_i := \frac{\partial}{\partial x_i}$ and $a_i := \frac{\partial a(x_1,x_2,x_3,x_4)}{\partial x_i}$. A straightforward calculation shows that the non-zero components of the Levi-Civita connection of the metric (3) are given by:

$$
\begin{aligned}
&\nabla_{\partial_1}\partial_3 = \frac{1}{2}b\partial_1, \quad \nabla_{\partial_1}\partial_4 = \frac{1}{2}b\partial_2, \quad \nabla_{\partial_2}\partial_3 = \frac{1}{2}c\partial_1, \quad \nabla_{\partial_2}\partial_4 = \frac{1}{2}c\partial_2,\\
&\nabla_{\partial_3}\partial_3 = \frac{1}{2}\Big[(x_1b+x_2c+d)b+(x_1b_3+x_2c_3+d_3)\Big]\partial_1\\
&\qquad + \frac{1}{2}\Big[(x_1b+x_2c+d)c-(x_1b_4+x_2c_4+d_4)\Big]\partial_2\\
&\qquad - \frac{1}{2}b\partial_3 - \frac{1}{2}c\partial_4,\\
&\nabla_{\partial_3}\partial_4 = \frac{1}{2}(x_1b_4+x_2c_4+d_4)\partial_1 + \frac{1}{2}(x_1b_3+x_2c_3+d_3)\partial_2,\\
&\nabla_{\partial_4}\partial_4 = \frac{1}{2}\Big[(x_1b+x_2c+d)b-(x_1b_3+x_2c_3+d_3)\Big]\partial_1\\
&\qquad + \frac{1}{2}\Big[(x_1b+x_2c+d)c+(x_1b_4+x_2c_4+d_4)\Big]\partial_2\\
&\qquad - \frac{1}{2}b\partial_3 - \frac{1}{2}c\partial_4.
\end{aligned} \tag{4}
$$

Using (4), we can completely determine the curvature tensor of the metric (3). Then, taking into account (3), we can determine all components of the $(0,4)$-curvature tensor $R_{ijkl} = g_a(\mathcal{R}(\partial_i,\partial_j)\partial_k,\partial_l)$. We obtain that, the non-zero components of the $(0,4)$-curvature tensor of the metric (3) are given as follows:

$$
\begin{aligned}
&R_{1334} = \frac{1}{4}(bc-2b_4), \quad R_{1434} = \frac{1}{4}(2b_3-b^2),\\
&R_{2334} = \frac{1}{4}(c^2-2c_4), \quad R_{2434} = \frac{1}{4}(2c_3-bc),\\
&R_{3434} = \frac{1}{4}\Big[x_1(2b_{33}+2b_{44}-b(b^2+c^2))+x_2(2c_{33}+2c_{44}-c(b^2+c^2))\Big]\\
&\qquad + \frac{1}{4}\Big[2d_{33}+2d_{44}-d(b^2-c^2)\Big].
\end{aligned} \tag{5}
$$

By (5), we can calculate the compoments ρ_{ij} with respect to ∂_i of the Ricci tensor of the metric (3). We find that, the non-zero component of the Ricci tensor are given by:

$$
\begin{aligned}
\rho_{33} &= \frac{1}{2}(c^2-2c_4),\\
\rho_{34} &= \frac{1}{2}(-bc+b_4+c_3),\\
\rho_{44} &= \frac{1}{2}(b^2-2b_3).
\end{aligned} \tag{6}
$$

The scalar curvature defined by $\tau = \operatorname{trace}\rho$ of the metric (3) is zero.

3. Ricci solitons on a family of four-dimensional Walker manifolds

In this section, we analyze the conditions for the existence of Ricci soliton on the four-dimensional Walker manifold (3). We start by giving some basic definitions. The Lie derivative of a tensor field T, with respect to a vector field X, is given by

$$(\mathcal{L}_X T) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\big((\varphi_t)^* T\big)_p,$$

where $\varphi\colon I \times M \subset \mathbb{R} \times M \to M$ is the local flow induced by X and $(\varphi_t)^*$ is the pullback along the diffeomorphism φ_t for all $t \in I$. Then:

- the Lie derivative of a function is the directional derivative of the function, so if f is a real function in M, we have that $\mathcal{L}_X f = X(f) = \nabla_X f$;
- the Lie derivative of a vector field is the Lie bracket, therefore, if Y is a vector field, $\mathcal{L}_X Y = [X, Y]$.

On the other hand, the Lie derivative of the metric is defined as a usual derivation on a $(0, 2)$-tensor by

$$(\mathcal{L}_X g)(Y, Z) = X \cdot g(Y, Z) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z),$$

which can also be written as

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y).$$

A vector field X on a pseudo-Riemannian manifold (M, g) is said to be Killing if the Lie derivative of the metric g with respect to X vanishes identically: $\mathcal{L}_X g = 0$; this is, the local flow of X is performed by isometries. In [**10**], the authors study question of the existence of Killing vector fields on a family of Walker metrics.

A Ricci soliton is a pseudo-Riemannian manifold (M, g) which admits a smooth vector field X on M such that

$$\mathcal{L}_X g + \rho = \lambda \cdot g, \tag{7}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, ρ is the Ricci tensor and λ is a real number. The Ricci soliton equation may also give some insight into the general study of Einstein field equations.

The description of Ricci solitons can be regarded as a first step in understanding the Ricci flow, since they are the fixed points of the flow. Moreover, they are important in understanding singularities of the Ricci flow [**2**].

Let $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$ be an arbitrary vector field on (M, g_a), where X^i, $i = 1, 2, 3, 4$ are smooth functions of the variables x_1, x_2, x_3, x_4. Locally, the components of the Lie derivative $(\mathcal{L}_X g_a)$ of the metric (3) with respect to X are given by

$$(\mathcal{L}_X g)(\partial_i, \partial_j) = \sum_{k=1}^{4} \Big[X^k \partial_k g_{ij} + (\partial_i X^k) g_{kj} + (\partial_j X^k) g_{ik}\Big]. \tag{8}$$

From the formula (8), the non-zero components of the Lie derivative of the Walker metric (3) in the direction of $X = \sum_{k=1}^{4} X^k \partial_k$ are as follows:

$$
(9)\quad
\begin{aligned}
&(\mathcal{L}_X g_a)(\partial_1, \partial_1) = 2\partial_1 X^3,\\
&(\mathcal{L}_X g_a)(\partial_1, \partial_2) = (\mathcal{L}_X g_a)(\partial_2, \partial_1) = \partial_1 X^4 + \partial_2 X^3,\\
&(\mathcal{L}_X g_a)(\partial_1, \partial_3) = (\mathcal{L}_X g_a)(\partial_3, \partial_1) = \partial_1 X^1 + a(\partial_1 X^3) + \partial_3 X^3,\\
&(\mathcal{L}_X g_a)(\partial_1, \partial_4) = (\mathcal{L}_X g_a)(\partial_4, \partial_1) = \partial_1 X^2 + a(\partial_1 X^4) + \partial_4 X^3,\\
&(\mathcal{L}_X g_a)(\partial_2, \partial_2) = 2\partial_2 X^4,\\
&(\mathcal{L}_X g_a)(\partial_2, \partial_3) = (\mathcal{L}_X g_a)(\partial_3, \partial_2) = \partial_2 X^1 + a(\partial_2 X^3) + \partial_3 X^4,\\
&(\mathcal{L}_X g_a)(\partial_2, \partial_4) = (\mathcal{L}_X g_a)(\partial_4, \partial_2) = \partial_2 X^2 + a(\partial_2 X^4) + \partial_4 X^4,\\
&(\mathcal{L}_X g_a)(\partial_3, \partial_3) = X^k a_k + 2\partial_3 X^1 + 2a\partial_3 X^3,\\
&(\mathcal{L}_X g_a)(\partial_3, \partial_4) = (\mathcal{L}_X g_a)(\partial_4, \partial_3) = \partial_3 X^2 + a\partial_3 X^4 + \partial_4 X^1 + a\partial_4 X^3,\\
&(\mathcal{L}_X g_a)(\partial_4, \partial_4) = X^k a_k + 2\partial_4 X^2 + 2a\partial_4 X^4.
\end{aligned}
$$

Thus, using (7) and (9), a standard calculation shows that the four-dimensional Walker manifold (3) is a Ricci soliton if and only if the following system holds:

(C1) $2\partial_1 X^3 = 0$,
(C2) $\partial_1 X^4 + \partial_2 X^3 = 0$,
(C3) $\partial_1 X^1 + a(\partial_1 X^3) + \partial_3 X^3 = \lambda$,
(C4) $\partial_1 X^2 + a(\partial_1 X^4) + \partial_4 X^3 = 0$,
(C5) $2\partial_2 X^4 = 0$,
(C6) $\partial_2 X^1 + a(\partial_2 X^3) + \partial_3 X^4 = 0$,
(C7) $\partial_2 X^2 + a(\partial_2 X^4) + \partial_4 X^4 = \lambda$,
(C8) $X^k a_k + 2\partial_3 X^1 + 2a\partial_3 X^3 + \frac{1}{2}\left(a_2^2 - 2a_{24}\right) = \lambda a$,
(C9) $\partial_3 X^2 + a\partial_3 X^4 + \partial_4 X^1 + a\partial_4 X^3 + \frac{1}{2}\left(a_{14} + a_{23} - a_1 a_2\right) = 0$,
(C10) $X^k a_k + 2\partial_4 X^2 + 2a\partial_4 X^4 + \frac{1}{2}\left(a_1^2 - 2a_{13}\right) = \lambda a$.

We can now prove the following.

Theorem 3.1. *The four-dimensional Walker manifold (M, g_a), where g_a is the metric given by (3) is a Ricci soliton if and only if there exists a function A satisfying the relation*

$$A_4(x_3, x_4) = A_3(x_3, x_4) = \frac{1}{2} b(x_3, x_4) A(x_3, x_4) = \frac{1}{2} c(x_3, x_4) A(x_3, x_4).$$

In this case, the Ricci soliton equation (7) is satisfied by $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$, where the functions X^i, $i = 1, 2, 3, 4$ are given as follows:

$$
\begin{aligned}
X^1 = &\frac{1}{2}(x_2^2 + x_1 x_2) b(x_3, x_4) A(x_3, x_4) + x_1 (\lambda - C_3)\\
&+ x_2 (d(x_3, x_4) A(x_3, x_4) - B_3(x_3, x_4)) + G(x_3, x_4),
\end{aligned}
$$

$$\begin{aligned} X^2 &= -\frac{1}{2}(x_1^2 + x_1x_2)b(x_3,x_4)A(x_3,x_4) + x_2\,(\lambda - B_4) \\ &\quad - x_2(d(x_3,x_4)A(x_3,x_4) - C_4(x_3,x_4)) + K(x_3,x_4), \\ X^3 &= -x_2A(x_3,x_4) + C(x_3,x_4), \\ X^4 &= x_1A(x_3,x_4) + B(x_3,x_4). \end{aligned}$$

Also, note that the functions A, B, C, K, G must satisfy the following system of partial differential equations:

$$\begin{cases} b_3A = b_4A = \frac{1}{2}b^2A = bA_3 = bA_4, \\ d_3A + 2C_{34} - b(B_3 + C_4) = 0, \\ d_4A - 2B_{34} - c(B_3 + C_4) = 0, \\ (B_3 + C_4)d + K_3 + G_4 + \frac{1}{2}(b_4 + c_3 - bc) = 0, \\ d_4A + C_{44} - C_{33} + (C_3 - B_4)b = 0, \\ d_3A + B_{44} - B_{33} + (C_3 - B_4)c = 0, \\ G_3 + K_4 + (C_3 - B_4)d + \frac{1}{4}(c^2 - 2c_4 - b^2 + 2b_3) = 0. \end{cases}$$

Proof. Let $a(x_1,x_2,x_3,x_4) = x_1b(x_3,x_4) + x_2c(x_3,x_4) + d(x_3,x_4)$ and $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$, on (M, g_a).

1. step: we establish the form of X^3 and X^4. From conditions (C1) and (C5), we have

$$X^3 = f(x_2,x_3,x_4) \quad \text{and} \quad X^4 = f(x_1,x_3,x_4). \tag{10}$$

Using condition (C2), we find that the functions X^3 and X^4 have the following forms:

$$X^3 = -x_2A(x_3,x_4) + C(x_3,x_4), \tag{11}$$

and

$$X^4 = x_1A(x_3,x_4) + B(x_3,x_4), \tag{12}$$

where A, B, C are some functions.

2. step: we give the form of X^1. From the condition (C3), we get

$$X^1 = x_1\,(\lambda - C_3) + x_1x_2A_3(x_3,x_4) + D(x_2,x_3,x_4), \tag{13}$$

where D is a function and $A_3 = \partial_3 A$, $C_3 = \partial_3 C$.

3. step: we give the form of X^2. From the condition (C7), we get

$$X^2 = x_2(\lambda - B_4) - x_1x_2A_4(x_3,x_4) + E(x_1,x_3,x_4), \tag{14}$$

where E is a function, $A_4 = \partial_4 A$ and $B_4 = \partial_4 B$.

The functions X^3, X^4, X^1, X^2 found in (11), (12), (13), (14) must satisfy the conditions (C4), (C6), (C8), (C9) and (C10).

- Firstly, computing (C4) + (C6), we find

$$2x_2A_4 - 2x_1A_3 + D_2(x_2,x_3,x_4) + E_1(x_1,x_3,x_4) + C_4 + B_3 = 0. \tag{15}$$

By differentiation of (15) with respect ∂_1 and ∂_2, respectively, we get $D_{22} = A_4$ and $E_{11} = -A_3$. So, we have

$$D = x_2^2 A_4(x_3, x_4) + x_2 F(x_3, x_4) + G(x_3, x_4), \tag{16}$$

and

$$E = -x_1^2 A_3(x_3, x_4) + x_1 H(x_3, x_4) + K(x_3, x_4), \tag{17}$$

where F, G, H and K are functions depending on the variables x_3 and x_4. Injecting (16) in (13) and (17) in (14), respectively, we obtain the following:

$$\begin{aligned} X^1 &= x_1\,(\lambda - C_3) + x_1 x_2 A_3(x_3, x_4) \\ &\quad + x_2^2 A_4(x_3, x_4) + x_2 F(x_3, x_4) + G(x_3, x_4) \end{aligned} \tag{18}$$

and

$$\begin{aligned} X^2 &= x_2(\lambda - B_4) - x_1 x_2 A_4(x_3, x_4) \\ &\quad - x_1^2 A_3(x_3, x_4) + x_1 H(x_3, x_4) + K(x_3, x_4), \end{aligned} \tag{19}$$

Equations (18) and (19) in (C4) and (C6), imply

$$F = dA - B_3 \quad \text{and} \quad H = -dA - C_4. \tag{20}$$

Relations (20) in (18) and (19) give

$$\begin{aligned} X^1 &= x_1\,(\lambda - C_3) + x_1 x_2 A_3(x_3, x_4) \\ &\quad + x_2^2 A_4(x_3, x_4) + x_2(dA - B_3) + G(x_3, x_4) \end{aligned} \tag{21}$$

and

$$\begin{aligned} X^2 &= x_2(\lambda - B_4) - x_1 x_2 A_4(x_3, x_4) \\ &\quad - x_1^2 A_3(x_3, x_4) + x_1(-dA - C)_+ K(x_3, x_4). \end{aligned} \tag{22}$$

The expression (11), (12), (21) and (22) must satisfy (C8), (C9), (C10).

- By (C9), we get the following partial differential equations system:

$$\begin{cases} b_3 A = b_4 A = \dfrac{1}{2} b^2 A, \\ d_3 A + 2C_{34} - b(B_3 + C_4) = 0, \\ d_4 A - 2B_{34} - c(B_3 + C_4) = 0, \\ (B_3 + C_4)d + K_3 + G_4 + \frac{1}{2}(b_4 + c_3 - bc) = 0. \end{cases} \tag{23}$$

- By (C8) and (C10), we get the following partial differential equations system:

$$\begin{cases} d_4 A + C_{44} - C_{33} + (C_3 - B_4)b = 0, \\ d_3 A + B_{44} - B_{33} + (C_3 - B_4)c = 0, \\ G_3 + K_4 + (C_3 - B_4)d + \frac{1}{4}(c^2 - 2c_4 - b^2 + 2b_3) = 0. \end{cases} \tag{24}$$

This ends the proof of the theorem. □

From $A_4(x_3, x_4) = A_3(x_3, x_4) = \frac{1}{2} b(x_3, x_4) A(x_3, x_4) = \frac{1}{2} c(x_3, x_4) A(x_3, x_4)$, we have the two following consequences.

Corollary 3.2. *Assume $A \in \mathbb{R}^*$. Then the four-dimensional Walker manifold (M, g_a), where g_a is the metric given by (3) is strict Walker. In this case, it is Ricci soliton if the equation (7) is satisfied by $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$, where the functions $X^i, i = 1, 2, 3, 4$ are given by:*

$$\begin{aligned} X^1 &= x_1 (\lambda - C_3) - x_2 B_3(x_3, x_4) + G(x_3, x_4), \\ X^2 &= x_2 (\lambda - B_4) - x_1 C_4(x_3, x_4) + K(x_3, x_4), \\ X^3 &= -Ax_2 + C(x_3, x_4), \\ X^4 &= Ax_1 + B(x_3, x_4), \end{aligned}$$

where the functions B, C, G and K satisfy the following system:

$$\begin{cases} 2C_{34} + B_{33} - B_{44} = 0, \\ 2B_{34} + C_{44} - C_{33} = 0, \\ (B_3 + C_4)d + K_3 + G_4 = 0, \\ (C_3 - B_4)d + G_3 + K_4 = 0. \end{cases} \tag{25}$$

Example 3.3. The four-dimensional strict Walker manifold (M, g_a), where g_a is the metric (3), is a Ricci soliton for $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$, with

$$\begin{aligned} X^1 &= (\lambda - \alpha) x_1 - \alpha x_2 - \cos(x_3 + x_4), \\ X^2 &= (\lambda - \alpha) x_2 + \alpha x_1 + \cos(x_3 + x_4), \\ X^3 &= -Ax_2 + \alpha(x_3 - x_4 + 1), \\ X^4 &= Ax_1 + \alpha(x_3 + x_4 + 3), \end{aligned}$$

where $\alpha \in \mathbb{R}$.

Corollary 3.4. *Assume $A = 0$. Then the four-dimensional Walker (M, g_a), where g_a is the metric (3), is a Ricci soliton if the equation (7) is satisfied by $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$, where the functions $X^i, i = 1, 2, 3, 4$ are given by:*

$$\begin{aligned} X^1 &= x_1 (\lambda - C_3) - x_2 B_3(x_3, x_4) + G(x_3, x_4), \\ X^2 &= x_2 (\lambda - B_4) - x_1 C_4(x_3, x_4) + K(x_3, x_4), \\ X^3 &= B(x_3, x_4), \\ X^4 &= C(x_3, x_4), \end{aligned}$$

where the functions B, C, G and K satisfy the following system:

$$\begin{cases} 2C_{34} - b(B_3 + C_4) = 0, \\ -2B_{34} - c(B_3 + C_4) = 0, \\ (B_3 + C_4)d + K_3 + G_4 + \dfrac{1}{2}(b_4 + c_3 - bc) = 0, \\ C_{44} - C_{33} + (C_3 - B_4)b = 0, \\ B_{44} - B_{33} + (C_3 - B_4)c = 0, \\ G_3 + K_4 + (C_3 - B_4)d + \dfrac{1}{4}(2b_3 - b^2 + c^2 - 2c_4) = 0. \end{cases} \tag{26}$$

Example 3.5. Let g_a be the Walker metric (3), where the functions $b(x_3, x_4)$ and $c(x_3, x_4)$ are given by: $b(x_3, x_4) = \frac{-2\beta}{\alpha+\beta x_3+\gamma x_4}$ and $c(x_3, x_4) = \frac{-2\gamma}{\alpha+\beta x_3+\gamma x_4}$. Consider the vector field $X = X^1\,\partial_1 + X^2\,\partial_2 + X^3\,\partial_3 + X^4\,\partial_4$, where $\alpha, \beta, \gamma \in \mathbb{R}$ and the components of X are given by:

$$\begin{aligned} X^1 &= (\lambda - \gamma)x_1 - \gamma x_2 + \mathrm{e}^{\beta(x_3+x_4)}, \\ X^2 &= (\lambda - \gamma)x_2 + \gamma x_1 - \mathrm{e}^{\beta(x_3+x_4)}, \\ X^3 &= \gamma(x_3 - x_4 + 1), \\ X^4 &= \gamma(x_3 + x_4 + 2). \end{aligned}$$

Then X defines a *R*icci soliton on the Walker manifold (M, g_a),

Ricci solitons can be viewed as generalizations of Einstein manifolds. One of the aims of the different geometric evolution equations is to produce (or deduce the existence of) manifolds with an optimal behavior with respect to the given invariants: the Ricci flow makes it possible to construct Einstein metrics under certain conditions. However, there are conditions under which the initial structure does not evolve under the flow but remains as a fixed point of it. Ricci solitons are the geometric fixed points (modulo homotheties and diffeomorphisms) of the Ricci flow. Moreover, since they appear as singular models for the flow, analyzing their geometry is an important step towards an understanding of the Ricci flow itself.

4. Riemannian extensions

The simplest examples of non-Lorentzian Walker metrics are provided by the so-called Riemannian extensions. This construction, which relates affine and pseudo-Riemannian geometry, associates to any torsion-free connection D on a manifold M, a metric of neutral signature on the cotangent bundle T^*M. Riemannian extensions have been widely used both to investigate problems in affine geometry and to address curvature questions (see, for example, [**11, 16**]).

It is a remarkable fact that Walker metrics satisfying certain natural curvature conditions are locally realized as Riemannian extensions. Consequently, the corresponding classification problem can be reduced to a problem in affine geometry, as is shown in [**8, 14**].

In [**15**], the authors generalized the Riemannian extension to the so-called deformed Riemannian extensions. Let D be a torsion-free affine connection on an affine manifold M. The deformed Riemannian extension $g_{D,\Phi}$ is the pseudo-Riemannian metric on T^*M of neutral signature (n, n) given by

$$g_D = 2\mathrm{d}x^i \circ \mathrm{d}x^{i'} + (-2x_{k'}\Gamma^k_{ij} + \Phi_{ij})\mathrm{d}x^i \circ \mathrm{d}x^j,$$

where Γ^k_{ij} are the coefficients of the affine connection D, Φ is a symmetric $(0, 2)$-tensor on M and $i' = i + n$, $k' = k + n$ with $i, j = 1, \ldots, n$.

It is easy to show that the Walker metric (3) is the deformed Riemannian extension of a torsion-free affine connection D whose nonzero components are given by:

$$\begin{aligned} D_{\partial_3}\partial_3 &= \frac{1}{2}b(x_3,x_4)\partial_3 + \frac{1}{2}c(x_3,x_4)\partial_4, \\ D_{\partial_4}\partial_4 &= \frac{1}{2}b(x_3,x_4)\partial_3 + \frac{1}{2}c(x_3,x_4)\partial_4, \end{aligned}$$

where $\Phi(x_3,x_4) = d(x_3,x_4)$. When $\Phi(x_3,x_4) = 0$, the Walker metric (3) reduces to the (classical) Riemannian extension of the torsion-free affine connection D.

We now state the main results of Section 3 in terms of Riemannian extensions.

Theorem 4.1. *The four-dimensional Walker manifold (M, g_a), where g_a is the metric given by* (3)*, is a Ricci soliton if and only if there exists a function A satisfying the relation*

$$A_4(x_3,x_4) = A_3(x_3,x_4) = \frac{1}{2}b(x_3,x_4)A(x_3,x_4) = \frac{1}{2}c(x_3,x_4)A(x_3,x_4).$$

In this case, the Ricci soliton equation (7) *is satisfied by $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$, where the functions X^i, $i = 1,2,3,4$ are given by:*

$$\begin{aligned} X^1 &= \frac{1}{2}(x_2^2 + x_1x_2)b(x_3,x_4)A(x_3,x_4) + x_1\,(\lambda - C_3) \\ &\quad - x_2B_3(x_3,x_4) + G(x_3,x_4), \\ X^2 &= -\frac{1}{2}(x_1^2 + x_1x_2)b(x_3,x_4)A(x_3,x_4) + x_2\,(\lambda - B_4) \\ &\quad - x_2C_4(x_3,x_4)) + K(x_3,x_4), \\ X^3 &= -x_2A(x_3,x_4) + C(x_3,x_4), \\ X^4 &= x_1A(x_3,x_4) + B(x_3,x_4). \end{aligned}$$

Also, note that the functions A, B, C, K, G must satisfy the following system of partial differential equations:

$$\begin{cases} b_3A = b_4A = \frac{1}{2}b^2A = bA_3 = bA_4, \\ 2C_{34} - b(B_3 + C_4) = 0, \\ -2B_{34} - c(B_3 + C_4) = 0, \\ K_3 + G_4 + \tfrac{1}{2}(b_4 + c_3 - bc) = 0, \\ C_{44} - C_{33} + (C_3 - B_4)b = 0, \\ B_{44} - B_{33} + (C_3 - B_4)c = 0, \\ G_3 + K_4 + \tfrac{1}{4}(c^2 - 2c_4 - b^2 + 2b_3) = 0. \end{cases}$$

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