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AVERAGE OF JORDAN'S FUNCTION OVER SHIFTED SMOOTH NUMBERS OVER SUM OF DIGITS OF CONSECUTIVE INTEGERS

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Abstract. We derive asymptotic estimates for some average values of the Jordan function evaluated over shifted smooth numbers in arithmetic progressions whose sum of digits as well as that of their successors are in arithmetic progression.

1. Introduction

Let $q \geq 2$ be an integer. It is a well-known fact, that every non-negative integer can be written uniquely in base q as

$$n = \sum_{k=0}^{+\infty} n_k \, q^k,$$

where the integers n_k satisfy $0 \leq n_k \leq q-1$ and $n_k \neq 0$ only for finitely many values. The sum of digits function in base q is then defined by

$$S(n) = \sum_{k=0}^{+\infty} n_k.$$

In [**9**], Gelfond proved the following theorem.

Theorem A. *If $q \geq 2$, $m \geq 2$ are integers such that $(m, q-1) = 1$ and $\gamma \in \mathbb{R}$, then we have for $1 \leq h \leq m-1$*

$$\sum_{n=1}^{N} e\left(\gamma \, n + \frac{h}{m} \, S(n)\right) = O_q\left(N^{\lambda}\right) \quad \textit{as } N \to +\infty,$$

where $\lambda = \frac{1}{2\,\log(q)} \, \log\left(\frac{q\,\sin\left(\frac{\pi}{2\,m}\right)}{\sin\left(\frac{\pi}{2\,m\,q}\right)}\right) < 1.$

A similar estimate was made by Aloui–Mauduit–Mkaouar [**3**] under a constraint over the sum of digits of consecutive integers.

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Theorem B. *Let* $(\alpha,\beta,\gamma)\in\mathbb{R}^3$ *and let* $q\geq 2$ *be an integer such that* $(q-1)(\alpha+\beta)\in\mathbb{R}\setminus\mathbb{Z}$. *Then, for any real number* x *sufficiently large, we have*

$$B(x,\alpha,\beta,\gamma)=\sum_{n\leq x} e\left(\alpha S(n)+\beta S(n+1)+\gamma n\right)\ll x^{\lambda}\log x,$$

where λ *is the constant defined in* Theorem A *and the implicit constant depends only on* q.

Recently, the first author [**2**] extended this result by estimating the exponential sum

$$\sum_{n\leq N} e(\alpha_0 S(n)+\cdots+\alpha_k S(n+k)),$$

where $k\in\mathbb{N}$ and $(\alpha_0,\ldots,\alpha_k)\in\mathbb{R}^{k+1}$, making a new proof of the Thue–Morse sequence recursion formula.

Given a positive real number y, an integer $n\geq 1$ is called y-smooth if every prime factor p of n satisfies $p\leq y$. We denote by $S(x,y)$ the set of numbers less than or equal to x that are y-smooth, that is,

$$S(x,y)=\{1\leq n\leq x\mid P^{+}(n)\leq y\},$$

where $P^{+}(n)$ is the greatest prime factor of a non-negative integer n. Besides, in the sequel, let $\Psi(x,y)=|S(x,y)|$ be the counting function for smooth numbers and $u=\dfrac{\log(x)}{\log(y)}$. Obviously, if $x\leq y$ then $\Psi(x,y)=\lfloor x\rfloor$; hence we assume along this article that $x>y\geq 2$. The ratio $\dfrac{\Psi(x,y)}{\lfloor x\rfloor}$ can be interpreted as the probability that a randomly chosen integer from the interval $[1,x]$ has all its prime factors $\leq y$.

Non-trivial estimates for $\Psi(x,y)$ can be obtained by various methods, depending on the relative size of x and y as well as on the nature of the desired result. In fact, several researchers investigated the asymptotic behavior of the function $\Psi(x,y)$, including Dickman [**7**], Hildebrand [**12**] and Tenenbaum [**16**]. For a detailed introduction to smooth numbers, their counting function $\Psi(x,y)$ and their properties, the reader is advised to refer to [**6, 11, 12, 13, 16, 17**]. Throughout this paper, we denote by $\mathbb{N}_0,\mathbb{N},\mathbb{Z},\mathbb{R}_{\geq 0},\mathbb{R}$ and $\mathbb{C}$ the sets of non-negative integers, positive integers, integers, non-negative real numbers, real and complex numbers, respectively. Given a real number x, $\lfloor x\rfloor$ denotes the greatest integer $\leq x$ and $e(x)=\mathrm{e}^{2\mathrm{i}\pi x}$. Besides, if $x>0$, then $\log^{(2)}(x)=\log(\log(x))$ and $\log^{(3)}(x)=\log(\log(\log(x)))$, where log is the natural logarithm. If $z=x+\mathrm{i}y$ is a complex number, we denote its complex conjugate by $\bar{z}=x-\mathrm{i}y$. The greatest common divisor of two non-zero integers a and b will be denoted by (a,b) and if $a\leq b$, $[\![a,b]\!]$ denotes the set of integers $\{a,\ldots,b\}$. If $\mathcal{A}$ is a given set, we denote its cardinality by $|\mathcal{A}|$.

We recall Vinogradov's notation $U\ll V$ equivalent to Bachmann–Landau's notation $U=O(V)$ for complex valued functions U and V, where the implied constants in the symbols "O", "$\ll$" are absolute. If the implied constants depend on certain parameters $\alpha,\beta,\ldots$ (but on no other parameters), then we write $U(N)=O_{\alpha,\beta,\ldots}(V(N))$ or $U(N)\ll_{\alpha,\beta,\ldots}V(N)$.

In fact, we focus on the following functions depending on a positive integer n:

- The Möbius function,

$$\mu\colon n \longmapsto \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ is a product of distinct primes}, \\ 0 & \text{otherwise.} \end{cases}$$

- The Euler totient function that assigns to n, the number of positive integers $\leq n$, and relatively prime to n, $\varphi\colon n \longmapsto \sum\limits_{\substack{k=1 \\ (k,n)=1}}^{n} 1$, which can be also written (see [**5**, Theorem 2.3]) as

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

- The δ^{th} Jordan totient function, denoted as $\mathcal{J}_\delta$, where δ is a positive integer, that assigns to a positive integer n, the number of δ-tuples of positive integers that are less than or equal to n and that together with n form a coprime set of $\delta + 1$ integers. This is a generalization of Euler's totient function. It is easy (see, for instance, [**4**]) to check that $\mathcal{J}_\delta$ satisfies

$$\mathcal{J}_\delta(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^\delta.$$

With respect to the notation used in [**1**], we take n, r, a, a', q, h, m and $m' \in \mathbb{Z}$ such that $n, q, h, m, m' \geq 2$, and we define the following function:

$$\widehat{\mathcal{J}_\delta}(n) = \sum_{\substack{d|n \\ d \equiv r \bmod h \\ S(d) \equiv a \bmod m \\ S(d+1) \equiv a' \bmod m'}} \mu(d) \left(\frac{n}{d}\right)^\delta.$$

We study shifted numbers in arithmetic progressions, a topic central to analytic number theory, particularly in the context of arithmetic functions distribution and exponential sums (see, for instance, [**14, 15**]). In particular, given $s \in \mathbb{N}$, we are interested in average behaviors of a function $f(n - s)$ where n ranges over some residue class. In this study, we derive asymptotic formulae for certain average values of Jordan function taken over shifted smooth numbers in arithmetic progressions with a particular focus on their sum of digits. More specifically, for $s, \delta \in \mathbb{N}$, we define

$$T_{\widehat{\mathcal{J}_\delta}}(x, y) = \sum_{\substack{s<n\leq x \\ n\in S(x,y)}} \frac{\widehat{\mathcal{J}_\delta}(n - s)}{(n - s)^\delta}$$

and

$$V_{\widehat{\mathcal{J}_\delta}}(x, y) = \frac{1}{\Psi(x, y)} \sum_{\substack{s<n\leq x \\ n\in S(x,y)}} \frac{\widehat{\mathcal{J}_\delta}(n - s)}{(n - s)^{\delta-1}}.$$

We ought to set

$$\mathfrak{G}^* = [\![0, h-1]\!] \times [\![0, m-1]\!] \times [\![0, m'-1]\!] \setminus \{(0,0,0)\},$$

and we denote by ζ the Riemann zeta function defined by the equations

$$\zeta(s) = \begin{cases} \displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^s} & \text{if} \quad s > 1, \\ \displaystyle\lim_{x \to +\infty} \left(\sum_{n \le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) & \text{if} \quad 0 < s < 1. \end{cases} \tag{1}$$

We now proceed to the statement of our main results.

Theorem 1.1. *Let q, h, m and m' be integers ≥ 2, let $\delta \in \mathbb{N}$ and $r, a, a' \in \mathbb{Z}$. Then, there exists an absolute constant $C > 0$ such that, uniformly in the domain*

$$x \geq y \geq \exp\left(C\sqrt{\log(x)\log_3(x)}\right),$$

we have

$$T_{\widehat{\mathcal{J}_\delta}}(x,y) = \Psi(x,y)\left(\frac{\frac{1}{\zeta(\delta+1)} + \mathcal{Z}_\delta}{hmm'} + O\left(\frac{\log_2(x)\log_2(y)}{\log(y)}\right)\right),$$

ζ being defined in (1) *and*

$$\begin{aligned} \mathcal{Z}_\delta = (\delta+1) & \sum_{(u,j,k) \in \mathfrak{G}^*} e\left(-\frac{ur}{h} - \frac{ja}{m} - \frac{ka'}{m'}\right) \\ & \times \left(\int_1^{+\infty} \left(\sum_{d \le u} \mu(d) e\left(\frac{u}{h}d + \frac{j}{m}S(d) + \frac{k}{m'}S(d+1)\right)\right) \frac{\mathrm{d}u}{u^{\delta+2}}\right). \end{aligned} \tag{2}$$

In particular,

$$T_{\widehat{\varphi}}(x,y) = \Psi(x,y)\left(\frac{\frac{6}{\pi^2} + \mathcal{Z}_1}{hmm'} + O\left(\frac{\log_2(x)\log_2(y)}{\log(y)}\right)\right),$$

which enables to recover [**14**, Theorem 1].

Theorem 1.2. *Let q, h, m and m' be integers ≥ 2, let $\delta \in \mathbb{N}$ and $r, a, a' \in \mathbb{Z}$. Then, there exists an absolute constant $C > 0$ such that, uniformly in the domain*

$$x \geq y \geq \exp\left(C\sqrt{\log(x)\log_3(x)}\right),$$

we have

$$V_{\widehat{\mathcal{J}_\delta}}(x,y) = \frac{\frac{1}{\zeta(\delta+1)} + \mathcal{Z}_\delta}{2hmm'}x + O\left(\frac{x\log_2(x)\log_2(y)}{\log(y)}\right)$$

where ζ is defined in (1) *and $\mathcal{Z}_\delta$ is defined in* (2).

In particular,

$$V_{\widehat{\varphi}}(x,y) = \frac{\frac{6}{\pi^2} + \mathcal{Z}_1}{2hmm'}x + O\left(\frac{x\log_2(x)\log_2(y)}{\log(y)}\right),$$

which meets the estimate proven in [**14**, Theorem 2].

Despite the complex appearance of $\mathcal{Z}_\delta$, we can easily see that $\mathcal{Z}_\delta = \overline{\mathcal{Z}_\delta}$ (so $\mathcal{Z}_\delta \in \mathbb{R}$). Indeed, we check easily that $(u,j,k) \in \mathfrak{G}^*$ if and only if $(h-u, m-j, m'-k) \in \mathfrak{G}^*$ and

$$\begin{aligned}
\overline{\mathcal{Z}_\delta} &= (\delta+1) \sum_{(u,j,k)\in\mathfrak{G}^*} e\left(\frac{ur}{h} + \frac{ja}{m} + \frac{ka'}{m'}\right) \\
&\quad \times \left(\int_1^{+\infty} \left(\sum_{d \le u} \mu(d) e\left(-\frac{u}{h}d - \frac{j}{m}S(d) - \frac{k}{m'}S(d+1) \right) \right) \frac{\mathrm{d}u}{u^{\delta+2}} \right) \\
&= (\delta+1) \sum_{(u,j,k)\in\mathfrak{G}^*} e\left(\frac{(h-u)r}{h} + \frac{(m-j)a}{m} + \frac{(m'-k)a'}{m'}\right) \\
&\quad \times \left(\int_1^{+\infty} \left(\sum_{d \le u} \mu(d) e\left(-\frac{h-u}{h}d - \frac{m-j}{m}S(d) - \frac{m'-k}{m'}S(d+1) \right) \right) \frac{\mathrm{d}u}{u^{\delta+2}} \right) \\
&= (\delta+1) \sum_{(u,j,k)\in\mathfrak{G}^*} e\left(-\frac{ur}{h} - \frac{ja}{m} - \frac{ka'}{m'}\right) \\
&\quad \times \left(\int_1^{+\infty} \left(\sum_{d \le u} \mu(d) e\left(\frac{u}{h}d + \frac{j}{m}S(d) + \frac{k}{m'}S(d+1) \right) \right) \frac{\mathrm{d}u}{u^{\delta+2}} \right) \\
&= \mathcal{Z}_\delta.
\end{aligned}$$

In the same way, we can prove that all the main terms in the following theorems are real numbers.

The article is structured as follows: in Section 2, we state the main tools of our estimates. Next, in Section 3, we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2.

2. Preliminary results

In this section, we recall some relevant facts and collect some important theorems and lemmas which play an important role in the proof of our main results. First, we shall recall the classic orthogonality relation.

Lemma 2.1. *For any positive integers m and n, we have*

$$\frac{1}{m}\sum_{j=0}^{m-1} e\left(\frac{n}{m}j\right) = \begin{cases} 1 & \text{if } m \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we will recall Abel's summation formula.

Lemma 2.2. *Let $x \ge 0$ be a real number, $a \in \mathbb{N}$ be such that $a < x$ and $f\colon [a,x] \longrightarrow \mathbb{C}$ be a function. Let g be a complex-valued function which is continuous and piece-wise continuously differentiable on the interval $[a,x]$. Then we have*

$$\sum_{a\le n\le x} f(n)g(n) = \sum_{a \le n \le x} f(n)g(x) - \int_a^x g'(t)\Big(\sum_{a\le n\le t} f(n)\Big)\mathrm{d}t.$$

Next, we state a useful lemma, its proof can be found in [**5**, Theorem 3.2].

Lemma 2.3. *As* $x \longrightarrow +\infty$*, we have* $\displaystyle\sum_{n>x} \frac{1}{n^{\alpha}} = O\left(x^{1-\alpha}\right)$ *if* $\alpha > 1$.

Dickman [**7**] showed that for all $t > 0$ fixed, a positive proportion of integers smaller than x is $x^{\frac{1}{t}}$-smooth

$$\lim_{x \longrightarrow +\infty} \frac{\Psi(x, x^{\frac{1}{t}})}{x} = \rho(t),$$

where the function ρ, called Dickman–de Bruijn function, is the unique function, continuous over $\mathbb{R}_{\geq 0}$, differentiable over $(1, +\infty)$, solution of the differential-difference equation

$$t\rho'(t) = \rho(t-1)$$

with the initial condition $\rho(t) = 1$ for $0 \leq t \leq 1$. Henceforth, $\rho(t)$ is the probability that an integer smaller than x is $x^{\frac{1}{t}}$-smooth.

The reader can find an interesting survey on Dickman–de Bruijn function in [**16**, Chapter III.5]. This function decreases rapidly as t goes towards $+\infty$ as shown by Hildebrand and Tenenbaum [**13**, Corollary 2.3]:

$$\rho(t) = \exp\left(-t\left(\log(t) + \log_2(t+2) - 1 + O\left(\frac{\log_2(t+2)}{\log(t+2)}\right)\right)\right).$$

Dickman's function is implemented in several math software programs, such as Sage. We share some of its values extracted from [**11**]: $\rho(2) \approx 3.07 \times 10^{-2}$, $\rho(10) \approx 2.77 \times 10^{-11}$, $\rho(20) \approx 2.46 \times 10^{-29}$.

Hildebrand linked between the Dickman–de Bruijn function and Ψ. He obtained the following lemma whose proof appears in [**16**, Chapter III.5, Corollary 5.19].

Lemma 2.4. *For any* $\varepsilon > 0$*, the estimate*

$$\Psi(x, y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log(y)}\right)\right)$$

holds uniformly in the range

$$\exp\left((\log_2(x))^{\frac{5}{3}+\varepsilon}\right) \leq y \leq x.$$

The following asymptotic estimate on ρ can be crude however useful (see, for instance, [**13**, (1.7)]): for any $t \longrightarrow \infty$, we have

$$\rho(t) = \exp(-(1 + o(1))t\log(t)).$$

Consequently, we get

$$\Psi(x, y) = xt^{-t+o(t)}. \tag{3}$$

The following upper bound on the derivative of ρ is a very weak form of a much more precise result in [**16**, Chapter III.5, Corollary 5.14].

Lemma 2.5. *For any* $t > 0$*, we have*

$$\rho'(t) \ll \rho(t)\log(t+1).$$

For any integers a and d, let

$$\Psi_d(x,y) = |\{n \in S(x,y) : (n,d) = 1\}|$$

and

$$\Psi(x,y;a,d) = |\{n \in S(x,y) : n \equiv a \bmod d\}|.$$

If $\ell = (a,d) > 1$ and is y-smooth, we can derive easily that $\Psi(x,y;a,d) = \Psi\left(\frac{x}{\ell}, y, \frac{a}{\ell}, \frac{d}{\ell}\right)$. So, we can limit ourselves to the case $(a,d) = 1$.

It is interesting to find a formula linking $\Psi_d(x,y)$ and $\Psi(x,y)$. Fouvry and Tenenbaum [**8**] gave the following asymptotic formula for the number of smooth numbers that are coprime to d.

Lemma 2.6. *For any $\varepsilon > 0$, there exists $x_0(\varepsilon)$ such that for $x \geq x_0(\varepsilon)$, the estimate*

$$\Psi_d(x,y) = \frac{\varphi(d)}{d}\Psi(x,y)\left(1 + O\left(\frac{\log_2(dy)\log_2(x)}{\log(y)}\right)\right)$$

holds uniformly in the range

$$\exp\left((\log_2(x))^{\frac{5}{3}+\varepsilon}\right) \leq y \leq x, \quad \log_2(d+2) \leq \left(\frac{\log(y)}{\log(u+1)}\right)^{1-\varepsilon}.$$

In the same spirit, we might wonder about the link between $\Psi_d(x,y)$ and $\Psi(x,y;a,d)$. Granville [**10**] proved the following bounds on the average of smooth numbers in a fixed arithmetic progression.

Lemma 2.7. *Let A be a fixed positive number. Then, there exist positive constants η and β depending only on A, such that for*

$$\Delta = \min\left\{\exp\left(\eta\frac{\log(y)\log_2(y)}{\log_3(y)}\right), \frac{\sqrt{x}}{(\log(x))^{\beta}}\right\}$$

uniformly over $y \geq 100$, we have

$$\sum_{d\leq\Delta}\max_{z\leq x}\max_{(a,d)=1}\left|\Psi(z,y;a,d) - \frac{\Psi_d(z,y)}{\varphi(d)}\right| = O\left(\frac{\Psi(x,y)}{(\log(y))^A}\right),$$

where the implied constant depends only on A.

We can derive the next lemma.

Lemma 2.8. *Let the setup be as in* Lemma 2.7. *Then*

$$\frac{x}{\Delta^{\delta}} \ll \Psi(x,y)\frac{\log_2(x)\log_2(y)}{\log(y)}.$$

Proof. This follows mainly from the definition of Δ. Indeed, thanks to (3), we may write

$$\begin{aligned}\frac{x}{\Delta^{\delta}} &= \max\left\{x\exp\left(-\delta\eta\frac{\log(y)\log_2(y)}{\log_3(y)}\right), x^{1-\frac{\delta}{2}}(\log(x))^{\beta\delta}\right\} \\ &\ll x\exp\left(-2\frac{\log(x)\log_2(x)}{\log(y)}\right) \ll x\exp\left(-2u\log(u) - \log_2(y)\right) \\ &\ll \Psi(x,y)\frac{1}{\log(y)} \ll \Psi(x,y)\frac{\log_2(x)\log_2(y)}{\log(y)}.\end{aligned}$$

□

3. Proof of Theorem 1.1

We write

$$\begin{aligned}
T_{\widehat{\mathcal{J}_\delta}}(x,y) &= \sum_{\substack{s<n\leq x\\ n\in S(x,y)}} \frac{\widehat{\mathcal{J}_\delta}(n-s)}{(n-s)^\delta} = \sum_{\substack{s<n\leq x\\ n\in S(x,y)}} \sum_{\substack{d|n-s\\ d\equiv r \bmod h'\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)}{d^\delta}\\
&= \sum_{\substack{d<x\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)}{d^\delta} \sum_{\substack{s<n\leq x\\ n\in S(x,y)\\ n\equiv s \bmod d}} 1 \qquad (4)\\
&= \sum_{\substack{d<x\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)}{d^\delta}\Psi(x,y;s,d) + O(1)\\
1ex &= S_1 + S_2,
\end{aligned}$$

where

$$S_1 = \sum_{\substack{d\leq\Delta\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)}{d^\delta}\Psi(x,y;s,d),$$

$$S_2 = \sum_{\substack{\Delta<d<x\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)}{d^\delta}\Psi(x,y;s,d),$$

$$\Delta = \min\left\{\exp\left(\eta\frac{\log(y)\log_2(y)}{\log_3(y)}\right), \frac{\sqrt{\frac{x}{s}}}{\left(\log\left(\frac{x}{s}\right)\right)^\beta}\right\},$$

and β,η are chosen to correspond to $A=1$ in Lemma 2.7.

First, we set

$$M = \sum_{\substack{d\leq\Delta\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{1}{d^\delta}\left|\Psi(x,y;s,d) - \frac{\Psi_d(x,y)}{\varphi(d)}\right|,$$

in order to use Lemma 2.6.

Then, we express S_1 as

$$S_1 = \sum_{\substack{d \leq \Delta \\ d \equiv r \bmod h \\ S(d) \equiv a \bmod m \\ S(d+1) \equiv a' \bmod m'}} \frac{\mu(d)\Psi_d(x,y)}{d^\delta \varphi(d)} + O(M). \tag{5}$$

For each divisor $\ell \mid s$, we gather the terms satisfying $(s,d) = \ell$ (in particular $\left(\frac{s}{\ell}, \frac{d}{\ell}\right) = 1$), obtaining

$$M = \sum_{\ell \mid s} M_\ell, \tag{6}$$

where

$$\begin{aligned} M_\ell &= \sum_{\substack{d \leq \Delta \\ (s,d)=\ell \\ d \equiv r \bmod h \\ S(d) \equiv a \bmod m \\ S(d+1) \equiv a' \bmod m'}} \frac{1}{d^\delta} \left| \Psi(x,y;s,d) - \frac{\Psi_d(x,y)}{\varphi(d)} \right| \\ &= \sum_{\substack{d \leq \Delta \\ (s,d)=\ell \\ d \equiv r \bmod h \\ S(d) \equiv a \bmod m \\ S(d+1) \equiv a' \bmod m'}} \frac{1}{d^\delta} \left| \Psi\left(\frac{x}{\ell}, y; \frac{s}{\ell}, \frac{d}{\ell}\right) - \frac{\Psi_d(x,y)}{\varphi(d)} \right|. \end{aligned} \tag{7}$$

Here, we assume that $y > s$ since otherwise the set $\{n \in S(x,y) : n \equiv s \bmod d\}$ will be empty. We note that Lemma 2.6 implies

$$\frac{\Psi_{\frac{d}{\ell}}\left(\frac{x}{\ell}, y\right)}{\varphi\left(\frac{d}{\ell}\right)} = \frac{\ell}{d}\Psi\left(\frac{x}{\ell}, y\right)\left(1 + O\left(\frac{\log_2(dy)\log_2(x)}{\log(y)}\right)\right). \tag{8}$$

Besides, letting

$$u_\ell = \frac{\log\left(\frac{x}{\ell}\right)}{\log(y)} = u + O\left(\frac{1}{\log(y)}\right),$$

we conclude from Lemma 2.5 that

$$\rho(u_\ell) = \rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log(y)}\right)\right).$$

Hence, thanks to Lemma 2.4, we get

$$\Psi\left(\frac{x}{\ell}, y\right) = \frac{1}{\ell}\Psi(x,y)\left(1 + O\left(\frac{\log(u+1)}{\log(y)}\right)\right).$$

Substituting into (8), and noting that $u \ll x$, we find

$$\frac{\Psi_{\frac{d}{\ell}}\left(\frac{x}{\ell}, y\right)}{\varphi\left(\frac{d}{\ell}\right)} = \frac{1}{d}\Psi(x,y)\left(1 + O\left(\frac{\log_2(dy)\log_2(x)}{\log(y)}\right)\right).$$

Using Lemma 2.6 again, we have

$$\frac{\Psi_d(x,y)}{\varphi(d)} = \frac{\Psi_{\frac{d}{\ell}}\left(\frac{x}{\ell},y\right)}{\varphi\left(\frac{d}{\ell}\right)}\left(1+O\left(\frac{\log_2(dy)\log_2(x)}{\log(y)}\right)\right).$$

As the series

$$\sum_{\substack{d\geq 1\\ (s,d)=\ell}} \frac{1}{d^{\delta}\varphi\left(\frac{d}{\ell}\right)}$$

is convergent, then we deduce from (7) that

$$\begin{aligned}M_\ell = & \sum_{\substack{d\leq\Delta\\ (s,d)=\ell\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{1}{d^{\delta}}\left|\Psi\left(\frac{x}{\ell},y;\frac{s}{\ell},\frac{d}{\ell}\right)-\frac{\Psi_{\frac{d}{\ell}}\left(\frac{x}{\ell},y\right)}{\varphi\left(\frac{d}{\ell}\right)}\right|\\ &+O\left(\Psi(x,y)\frac{\log_2(\Delta y)\log_2(x)}{\log(y)}\right)\\ \ll & \sum_{\substack{d\leq\Delta\\ (s,d)=\ell}}\left|\Psi\left(\frac{x}{\ell},y;\frac{s}{\ell},\frac{d}{\ell}\right)-\frac{\Psi_{\frac{d}{\ell}}\left(\frac{x}{\ell},y\right)}{\varphi\left(\frac{d}{\ell}\right)}\right| + \Psi(x,y)\frac{\log_2(\Delta y)\log_2(x)}{\log(y)}.\end{aligned}$$

Furthermore, in the given range of x and y and for x sufficiently large, we have $y\leq\Delta^3$, so

$$\log_2(\Delta y)\leq\log_2(\Delta^4)\leq\log\left(\frac{4\eta\log(y)\log_2(y)}{\log_3(y)}\right)=O(\log_2(y)).$$

As η and β are chosen to correspond to $A=1$ in the definition of Δ in Lemma 2.7, we derive

$$M_\ell \ll \Psi(x,y)\frac{\log_2(x)\log_2(y)}{\log(y)},$$

which implies by substitution into (6)

$$M = O\left(\Psi(x,y)\frac{\log_2(x)\log_2(y)}{\log(y)}\right). \tag{9}$$

Next, note that for $d\leq\Delta$, it is easy to check that the bounding $\log_2(d+2)\leq\left(\frac{\log(y)}{\log(u+1)}\right)^{1-\varepsilon}$ holds true (as long as x is large enough) so Lemmas 2.6 and 2.1

imply

$$
\begin{aligned}
\sum_{\substack{d\le\Delta\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)\Psi_d(x,y)}{d^\delta\varphi(d)} &= \Psi(x,y) \sum_{\substack{d\le\Delta\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)}{d^{\delta+1}}\left(1+O\left(\frac{\log_2(dy)\log_2(x)}{\log(y)}\right)\right)\\
&= \frac{\Psi(x,y)}{hmm'}\sum_{u=0}^{h-1}\sum_{j=0}^{m-1}\sum_{k=0}^{m'-1} e\left(-\frac{ur}{h}-\frac{ka'}{m'}-\frac{ja}{m}\right)\\
&\quad\times\sum_{d\le\Delta}\mu(d)\frac{e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)}{d^{\delta+1}}\left(1+O\left(\frac{\log_2(dy)\log_2(x)}{\log(y)}\right)\right)\\
&= \frac{\Psi(x,y)}{hmm'}\sum_{d\le\Delta}\frac{\mu(d)}{d^{\delta+1}} + \frac{\Psi(x,y)}{hmm'}\sum_{(u,j,k)\in\mathfrak{G}^*} e\left(-\frac{ur}{h}-\frac{ja}{m}-\frac{ka'}{m'}\right)\\
&\quad\times\sum_{d\le\Delta}\mu(d)\frac{e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)}{d^{\delta+1}}\\
&\quad+O\left(\Psi(x,y)\frac{\log_2(x)}{\log(y)}\sum_{d\le\Delta}\frac{\log_2(dy)}{d^{\delta+1}}\right).
\end{aligned}
\tag{10}
$$

As the series $\left(\sum_{d\ge1}\frac{1}{d^{\delta+1}}\right)$ and $\left(\sum_{d\ge1}\frac{\mu(d)}{d^{\delta+1}}\right)$ converge absolutely and satisfy

$$
\left(\sum_{d=1}^{+\infty}\frac{1}{d^{\delta+1}}\right)\left(\sum_{m=1}^{+\infty}\frac{\mu(m)}{m^{\delta+1}}\right) = \sum_{k=1}^{+\infty}\frac{1}{k^{\delta+1}}\left(\sum_{m\mid k}\mu(m)\right) = 1,
$$

we get

$$
\sum_{d=1}^{+\infty}\frac{\mu(d)}{d^{\delta+1}} = \frac{1}{\zeta(\delta+1)},
$$

where ζ is defined in (1). Hence, we may write thanks to Lemma 2.3,

$$
\begin{aligned}
\sum_{d\le\Delta}\frac{\mu(d)}{d^{\delta+1}} &= \sum_{d=1}^{+\infty}\frac{\mu(d)}{d^{\delta+1}} + O\left(\sum_{d>\Delta}\frac{1}{d^{\delta+1}}\right)\\
&= \frac{1}{\zeta(\delta+1)} + O\left(\frac{1}{\Delta^\delta}\right).
\end{aligned}
\tag{11}
$$

Recalling the definition of Δ, we get an estimation of the error term in (10) as follows

$$
\begin{aligned}
\Psi(x,y)\frac{\log_2(x)}{\log(y)}\sum_{d\le\Delta}\frac{\log_2(dy)}{d^{\delta+1}} &\le \Psi(x,y)\frac{\log_2(x)}{\log(y)}\sum_{d\le\Delta}\frac{\log_2(\Delta y)}{d^{\delta+1}}\\
&\ll \Psi(x,y)\frac{\log_2(x)}{\log(y)}\log_2(\Delta y)\\
&\ll \Psi(x,y)\frac{\log_2(x)\log_2(y)}{\log(y)}.
\end{aligned}
\tag{12}
$$

Gathering (10), (11), (12) and using Lemma 2.8 (as $\Psi(x,y)\le x$), it follows that

$$\sum_{\substack{d\le\Delta\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)\Psi_d(x,y)}{d^\delta\varphi(d)} = \Psi(x,y)\frac{1}{hmm'\zeta(\delta+1)}+\Lambda_\delta + O\left(\Psi(x,y)\frac{\log_2(x)\log_2(y)}{\log(y)}\right) \tag{13}$$

with

$$\Lambda_\delta = \frac{\Psi(x,y)}{hmm'}\sum_{(u,j,k)\in\mathfrak{G}^*} e\left(-\frac{ur}{h}d-\frac{ja}{m}-\frac{ka'}{m'}\right) \times\sum_{d\le\Delta}\mu(d)\frac{e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)}{d^{\delta+1}}. \tag{14}$$

Next, applying Lemma 2.2, we derive for $(u,j,k)\in\mathfrak{G}^*$,

$$\begin{aligned}&\sum_{d\le\Delta}\mu(d)\frac{e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)}{d^{\delta+1}}\\ &=\frac{1}{\Delta^{\delta+1}}\sum_{d\le\Delta}\mu(d)e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)\\ &\quad+(\delta+1)\int_1^\Delta\left(\sum_{d\le u}\mu(d)e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)\right)\frac{\mathrm{d}u}{u^{\delta+2}}.\end{aligned}$$

Trivially, we have

$$\frac{1}{\Delta^{\delta+1}}\sum_{d\le\Delta}\mu(d)e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)=O\left(\frac{1}{\Delta^\delta}\right), \tag{15}$$

and, as the integral is convergent, we obtain

$$\begin{aligned}&\int_1^\Delta\left(\sum_{d\le u}\mu(d)e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)\right)\frac{\mathrm{d}u}{u^{\delta+2}}\\ &=\int_1^{+\infty}\left(\sum_{d\le u}\mu(d)e\left(\frac{u}{h}d+\frac{j}{m}S(d)+\frac{k}{m'}S(d+1)\right)\right)\frac{\mathrm{d}u}{u^{\delta+2}}+O\left(\frac{1}{\Delta^\delta}\right)\end{aligned} \tag{16}$$

for each $(u,j,k)\in\mathfrak{G}^*$. Taking into account (16) and (15) jointly in (14), we find

$$\Lambda_\delta=\frac{\Psi(x,y)}{hmm'}\mathcal{Z}_\delta+O\left(\frac{\Psi(x,y)}{\Delta^\delta}\right) \tag{17}$$

with

$$\mathcal{Z}_\delta = (\delta+1) \sum_{(u,j,k)\in\mathfrak{G}^*} e\left(-\frac{ur}{h} - \frac{ka'}{m'} - \frac{ja}{m}\right) \times \int_1^{+\infty} \left(\sum_{d\le u} \mu(d) e\left(\frac{u}{h}d + \frac{j}{m}S(d) + \frac{k}{m'}S(d+1)\right)\right) \frac{\mathrm{d}u}{u^{\delta+2}}.$$

Substituting (17) into (13) then in (5) and taking (9) into account, we get

$$S_1 = \Psi(x,y)\frac{\frac{1}{\zeta(\delta+1)} + \mathcal{Z}_\delta}{hmm'} + O\left(\Psi(x,y)\frac{\log_2(x)\log_2(y)}{\log(y)}\right). \tag{18}$$

Returning back to S_2, we write

$$\begin{aligned}
S_2 &= \sum_{\substack{\Delta<d<x \\ d\equiv r \bmod h \\ S(d)\equiv a \bmod m \\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)}{d^\delta}\Psi(x,y;s,d) \\
&= \frac{1}{hmm'} \sum_{u=0}^{h-1}\sum_{j=0}^{m-1}\sum_{k=0}^{m'-1} e\left(-\frac{ur}{h} - \frac{ja}{m} - \frac{ka'}{m'}\right) \\
&\quad \times \sum_{\Delta<d<x} e\left(\frac{u}{h}d + \frac{j}{m}S(d) + \frac{k}{m'}S(d+1)\right)\frac{\mu(d)}{d^\delta}\Psi(x,y;s,d) \\
&\ll \sum_{\Delta<d<x} \frac{1}{d^\delta} \sum_{\substack{1\le n\le x \\ n\equiv s \bmod d}} 1 \\
&\ll \sum_{\Delta<d<x} \frac{1}{d^\delta}\left\lfloor \frac{x}{d} \right\rfloor \\
&\ll x\sum_{\Delta<d} \frac{1}{d^{\delta+1}} \\
&\ll \frac{x}{\Delta^\delta}.
\end{aligned} \tag{19}$$

The last bound follows from Lemma 2.3.

By substituting (18) and (19) in (4), and recalling Lemma 2.8, we obtain

$$T_{\widehat{\mathcal{J}_\delta}}(x,y) = \Psi(x,y)\left(\frac{\frac{1}{\zeta(\delta+1)} + \mathcal{Z}_\delta}{hmm'} + O\left(\frac{\log_2(x)\log_2(y)}{\log(y)}\right)\right).$$

Accordingly, the desired result is established. ☐

4. Proof of Theorem 1.2

Recall that

$$\widehat{\mathcal{J}_\delta}(n) = \sum_{\substack{d\mid n\\ d\equiv r \bmod h\\ S(d)\equiv a \bmod m\\ S(d+1)\equiv a' \bmod m'}} \frac{\mu(d)}{d^\delta} n^\delta.$$

Now, using Lemma 2.2, we get

$$\begin{aligned} V_{\widehat{\mathcal{J}_\delta}}(x,y) &= \frac{1}{\Psi(x,y)} \sum_{\substack{s<n\le x\\ n\in S(x,y)\\ n\equiv s \bmod d}} \frac{(n-s)\widehat{\mathcal{J}_\delta}(n-s)}{(n-s)^\delta}\\ &= \frac{1}{\Psi(x,y)}\left(T_{\widehat{\mathcal{J}_\delta}}(x,y)(x-s) - \int_1^x T_{\widehat{\mathcal{J}_\delta}}(t,y)\mathrm{d}t\right). \end{aligned}$$

Taking $t\le x$ and $y \ge \exp\left(C\sqrt{\log(x)\log_3(x)}\right)$, we deduce from Theorem 1.1 that

$$T_{\widehat{\mathcal{J}_\delta}}(t,y) = \Psi(t,y)\left(\frac{\frac{1}{\zeta(\delta+1)} + \mathcal{Z}_\delta}{hmm'} + O\left(\frac{\log_2(t)\log_2(y)}{\log(y)}\right)\right)$$

henceforth, as $\Psi(t,y)\le\Psi(x,y)$, we obtain

$$\begin{aligned} &V_{\widehat{\mathcal{J}_\delta}}(x,y)\\ &= \frac{1}{\Psi(x,y)}\left(x\, T_{\widehat{\mathcal{J}_\delta}}(x,y) - \left(\frac{\frac{1}{\zeta(\delta+1)} + \mathcal{Z}_\delta}{hmm'} + O\left(\frac{\log_2(x)\log_2(y)}{\log(y)}\right)\right)\int_1^x \Psi(t,y)\mathrm{d}t\right)\\ &= \frac{1}{\Psi(x,y)}\left(x\, T_{\widehat{\mathcal{J}_\delta}}(x,y) - \left(\frac{\frac{1}{\zeta(\delta+1)} + \mathcal{Z}_\delta}{hmm'}\right)\int_1^x \Psi(t,y)\mathrm{d}t\right) + O\left(\frac{x\log_2(x)\log_2(y)}{\log(y)}\right). \end{aligned} \tag{20}$$

Applying Lemma 2.4, we get

$$\Psi(t,y) = t\rho\left(\frac{\log(t)}{\log(y)}\right) + O\left(\Psi(x,y)\frac{\log(u+1)}{\log(y)}\right).$$

Therefore, (20) rewrites as

$$\begin{aligned} V_{\widehat{\mathcal{J}_\delta}}(x,y) = \frac{1}{\Psi(x,y)}&\left(x\, T_{\widehat{\mathcal{J}_\delta}}(x,y) - \left(\frac{\frac{1}{\zeta(\delta+1)} + \mathcal{Z}_\delta}{hmm'}\right)\mathcal{H}(x,y)\right)\\ &+ O\left(\frac{x\log_2(x)\log_2(y)}{\log(y)}\right) \end{aligned} \tag{21}$$

with

$$\mathcal{H}(x,y) = \int_1^x t\rho\left(\frac{\log(t)}{\log(y)}\right)\mathrm{d}t.$$

Now, applying integration by parts to $\mathcal{H}$ yields,

$$\mathcal{H}(x,y) = \frac{x^2}{2}\rho(u) - \frac{1}{2\log(y)}\int_1^x t\rho'\left(\frac{\log(t)}{\log(y)}\right)\mathrm{d}t + O(1).$$

Thanks to Lemma 2.5, we get

$$\begin{aligned}\frac{1}{2\log(y)}\int_1^x t\rho'\left(\frac{\log(t)}{\log(y)}\right)\mathrm{d}t &\ll \frac{\log(u+1)}{2\log(y)}\int_1^x t\rho\left(\frac{\log(t)}{\log(y)}\right)\mathrm{d}t \\ &\ll \mathcal{H}(x,y)\frac{\log(u+1)}{\log(y)}.\end{aligned}$$

Hence,

$$\mathcal{H}(x,y) = \frac{x^2}{2}\rho(u) + O\left(1 + \mathcal{H}(x,y)\frac{\log(u+1)}{\log(y)}\right).$$

Applying Lemma 2.4, it follows that

$$\begin{aligned}\mathcal{H}(x,y) &= \frac{x^2}{2}\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log(y)}\right)\right) \\ &= \frac{1}{2}x\Psi(x,y)\left(1 + O\left(\frac{\log(u+1)}{\log(y)}\right)\right).\end{aligned} \tag{22}$$

Consequently, by substituting (22) into (21) and using Theorem 1.1, we reach the desired result. □

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