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YAMABE SOLITONS AND QUOTIENT ALMOST YAMABE SOLITONS ON RIEMANNIAN MANIFOLDS

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Abstract. The object of the present article is to characterize Yamabe solitons admitting torse-forming vector fields with supporting non-trivial examples. Quotient Yamabe almost solitons with potential vector fields as infinitesimal harmonic transformations are studied. In addition, a triviality result of quotient almost Yamabe soliton is established using an integral inequality.

1. Introduction

Historically, the word 'soliton' first came off in the literature of fluid mechanics to represent a solitary wave. In the theory of partial differential equations, solitons are self similar solutions of PDEs representing certain evolution equations. In the context of geometric analysis, a soliton is a fixed point, upto diffeomorphisms and scaling, of a geometric flow. For example, a Ricci soliton is a self similar solution of a Ricci flow introduced by Hamilton [**15**] in order to solve the long standing open problem 'Poincaré Conjecture'. The root of Yamabe flow was originaly implied within the work of Hamilton. A self similar solution of Yamabe flow is known as Yamabe soliton. Chow [**8**] proved that the scalar curvature of a compact Yamabe soliton is constant. Daskalopoulos and Sesum [**9**] proved that complete locally conformally flat Yamabe solitons with positive sectional curvature are rotationally symmetric and must belong to the conformal class of Euclidean space. During the last decade, the theories of Ricci solitons and Yamabe solitons have been intensively studied by several authors from various perspectives [**3, 4, 10, 11, 12, 21, 22, 25, 26, 27, 28, 29**].

Riemannian manifolds associated with torqued vector fields and Ricci solitons were studied by Chen in [**6, 7**]. In [**16**], the author studied Ricci solitons on Riemannian manifolds with certain vector fields. In [**17**], Yamabe solitons and τ-quasi

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Yamabe gradient solitons were studied on a Riemannian manifold admitting concurrent recurrent vector fields. Among all vector fields, torse-forming vector fields are more general. As particular cases we have concircular, concurrent, recurrent and geodesic vector fields. In [**22**], the first author of the present paper studied torse-forming vector fields and its applications on Riemannian manifolds. Thus, the study of Yamabe solitons and their generalizations admitting torse-forming vector fields is an interesting topic of investigation. Motivated by this, we intend to study Yamabe solitons with torse-forming vector fields in Section 3 after compiling the brief required terminologies in Section 2. In Section 3, we also construct non-trivial examples to support the established results.

Almost Yamabe solitons were introduced by Barbosa and Ribeiro [**1**] as generalizations of self similar solutions of the Yamabe flow. Essentially they replaced the soliton constant by a smooth function. Guan and Guofang [**14**] introduced a generalization of Yamabe flow by replacing the scalar curvature constraint by the constraint of logarithmic function of scalar curvature. Self similar solutions of such a flow has been provided with the terminology 'quotient Yamabe soliton' by Bo et al. [**5**]. Such solitons where the soliton constants are substituted by smooth functions are known as quotient almost Yamabe solitons or quotient Yamabe almost solitons.

It is well known that every 1-parameter group of transformations generates a vector field on a manifold. A 1-parameter group of harmonic diffeomorphisms induces a vector field known as an infinitesimal harmonic transformation [**23**]. Ghosh [**13**] studied Ricci almost soliton with the potential vector field as an infinitesimal harmonic transformation and proved that in such a case the almost Ricci soliton reduces to Ricci soliton. Barros and Ribeiro [**3**] established an integral inequality for triviality of Ricci solitons by generalizing the results of Petersen and Wylie [**18**]. Ghosh [**13**] considered such integral inequality to study triviality of Ricci almost soliton. Now it is a natural question that what will we obtain if we study quotient almost Yamabe soliton with potential vector field as infinitesimal harmonic transformation? Also what will be the scenario if we study quotient almost Yamabe soliton with the integral inequality established by Barros and Ribeiro? The answers to these questions are provided in Sections 4 and 5, respectively.

2. Basic terminology

A unit vector field ξ on a Riemannian manifold (M, g) is said to be a torse-forming vector field if it satisfies

$$\nabla_X \xi = \rho X + \eta(X)\xi, \tag{1}$$

where $\rho\colon M \to R$ is a smooth function and η is a 1-from.

In particular, if $\rho = 0$, then the vector field is called recurrent vector field.

The vector field is called concurrent recurrent if $\nabla_X \xi = \rho(X - \eta(X)\xi)$ for a constant ρ.

A vector field V on a Riemannian manifold (M, g) is said to be conformal vector field if it satisfies $\pounds_V g = 2qg$, where $q \in C^\infty(M)$ is called the potential function.

Suppose (M, g) be a compact Riemannian manifold having s as its scalar curvature. Then the Yamabe problem is associated with the existence of a Riemannian metric g' conformal to g for which the scalar curvature s' of the metric g' is constant. In order to solve the Yamabe problem, Hamilton in [**15**] introduced the concept of Yamabe flow. Given a conformal class of Riemannian metrics, Yamabe flow is used for obtaining metrics whose scalar curvature s is constant. The Yamabe flow is an evolving metric family $g(t)$ given by

$$\frac{\partial}{\partial t} g(t) = -s(t) g(t) \tag{2}$$

with the initial data $g(0) = g$. Geometric flows such as the Ricci flow, the mean curvature flow, the inverse mean curvature flow, and the KählerRicci flow have applications in a wide variety of topological, geometrical, and physical problems. It should be noted that, in dimension two, the Yamabe flow and the Ricci flow are equivalent, whereas in higher dimensions they are not.

The Yamabe solitons are self-similar solutions of the Yamabe flow, that is, there are scalars $\sigma(t)$ and the diffeomorphisms ϕ_t such that $g(t) = \sigma(t)\phi_t^*(g_0)$ is the solutions of the Yamabe flow (2) with $\sigma(0) = 1$ and $\phi_0 = I$. Thus, a Riemannian metric g is called Yamabe soliton if there is a smooth vector field $V \in \mathfrak{X}(M)$ (called the potential vector field) and a scalar $\lambda \in \mathbb{R}$ satisfying

$$\frac{1}{2} \pounds_V g = (s - \lambda) g, \tag{3}$$

where $\pounds_V$ is the Lie derivative operator along V. When V is Killing, we say that the Yamabe soliton is trivial.

If λ is a smooth soliton function on M, then the Riemannian metric g is called an almost Yamabe soliton.

The Riemann curvature tensor R of (M, g) admits the following decomposition:

$$R = W_g + \Sigma_g \owedge g,$$

where W_g and Σ_g are the tensors of Weyl and Schouten, respectively, and $\owedge$ is the Kulkarni–Nomizu product of (M, g). It is known that the Schouten tensor is given by

$$\Sigma_g = \frac{1}{n-2}\Big(S - \frac{s}{2n-1} g\Big).$$

The σ_k-curvature of g is defined as the k-th elementary symmetric function of the eigenvalues $\lambda_1, \dots, \lambda_n$ of the endomorphism $g^{-1}\Sigma_g$, that is

$$\sigma_k(g) = \sigma_k(g^{-1}\Sigma_g) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \le k \le n.$$

Now, we consider $\sigma_0(g) = 1$ for convenience. A simple calculation shows that $\sigma_1(g) = \frac{s}{2(n-1)}$ which hints that the σ_k-curvature is a suitable substitution of the scalar curvature of the Yamabe flow. Guan and Guofang [**14**] introduced the following nonlinear flow

$$\frac{\partial g}{\partial t} = -\left(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)\right) g, \quad g(0) = g_0, \tag{4}$$

where

$$\log r_{k,l}(g) = \frac{\int_M \sigma_l(g)\log\frac{\sigma_k(g)}{\sigma_l(g)}\mathrm{d}v_g}{\int_M \sigma_l(g)\mathrm{d}v_g}$$

was defined as to make the flow preserve the quantities

$$\mathcal{E}_l(g) = \begin{cases} \displaystyle\int_M \sigma_l(g)\mathrm{d}v_g, & l \neq \dfrac{n}{2}, \\ \displaystyle -\int_0^1 \mathrm{d}t \int_M u\sigma_{\frac{n}{2}}(g)\mathrm{d}v_g, & l = \dfrac{n}{2}, \end{cases}$$

where $u \in C^\infty(M)$, $g(t) = \mathrm{e}^{-2tu}g_0$. The convergence of the above stated flow was then approved under certain conditions to be shifted by the eigenvalues of the Schouten tensor. The authors [**14**] also constructed geometric inequalities such as the Sobolev-type inequality in case $0 \leq l < k < \frac{n}{2}$, the conformal quasi-mass-integral-type inequality for $\frac{n}{2} \leq k \leq n$, $1 \leq l < k$ and the Moser–Trudinger-type inequality for $k = \frac{n}{2}$.

Bo et al. [**5**] deduced the self-similar solution of (4) namely quotient almost Yamabe soliton as

$$\frac{1}{2}\pounds_V g = \left(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \lambda\right)g, \quad \sigma_k \cdot \sigma_l > 0. \tag{5}$$

The soliton is called expanding if $\lambda < 0$, steady if $\lambda = 0$, shrinking if $\lambda > 0$, and indefinite if λ change signs on M. For details, see [**2**].

3. Yamabe solitons admitting torse-forming vector fields

In this section, we first establish some results and lemmas to prove the main results.

Lemma 3.1. *In a Riemannian manifold with a torse-forming vector field ξ,*

$$\xi(s) = 2\{-(n-1)\Delta\rho + 2(n-1)(\xi\rho) + n(n-1)\rho^2 - \rho s\}$$

holds.

Proof. Using (1) in the definition of Riemann curvature tensor, we obtain

$$R(X,Y)\xi = (X\rho)Y - (Y\rho)X + \rho[\eta(Y)X - \eta(X)Y]. \tag{6}$$

Contracting the above equation, we have

$$S(X,\xi) = -(n-1)(X\rho) + \rho(n-1)\eta(X), \tag{7}$$

which yields

$$Q\xi = -(n-1)D\rho + \rho(n-1)\xi, \tag{8}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

Differentiating both sides of (8) along X implies

$$(\nabla_X Q)\xi = -(n-1)\nabla_X D\rho + (n-1)(X\rho)\xi + \rho^2(n-1)Xr - \rho QX + (n-1)\eta(X)D\rho. \tag{9}$$

Taking the g-trace of the above equation, one obtains

$$\xi(s) = 2\{-(n-1)\Delta\rho + 2(n-1)(\xi\rho) + n(n-1)\rho^2 - \rho s\}. \qquad \square$$

Lemma 3.2. *A conformal vector field V on a Riemannian n-manifold (M^n, g) satisfies*

$$(\pounds_V S)(X,Y) = -(n-2)g(\nabla_X Dq, Y) - (\Delta q)g(Y,X),$$
$$\pounds_V s = -2qs - 2(n-1)\Delta q,$$

where $\Delta = \operatorname{div} D$ is the Laplacian operator of g.

Proof. In [**30**], Yano established the following:

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = g((\pounds_V\nabla)(X,Z),Y) - g((\pounds_V\nabla)(X,Y),Z).$$

Since ∇ is Levi-Civita connection, it follows that

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V\nabla)(X,Y),Z) + g((\pounds_V\nabla)(X,Z),Y).$$

Utilizing the symmetric property of $\pounds_V\nabla$, the foregoing equation yields

$$2g((\pounds_V\nabla)(X,Y),Z) = (\nabla_Y \pounds_V g)(Z,X) + (\nabla_X \pounds_V g)(Y,Z) - (\nabla_Z \pounds_V g)(X,Y). \tag{10}$$

Since V is conformal,

$$(\nabla_X \pounds_V g)(Y,Z) = 2(Xq)g(Y,Z).$$

Thus from (10), we get

$$(\pounds_V\nabla)(X,Y) = (Xq)Y + (Yq)X - g(X,Y)Dq. \tag{11}$$

From (11)

$$\begin{aligned}(\nabla_X \pounds_V \nabla)(Y,Z) = {} & (X(Yq))Z + (Yq)\nabla_X Z + (X(Zq))Y \\ & + (Zq)\nabla_X Y - (Xg(Y,Z))Dq - g(Y,Z)\nabla_X Dq \\ & - [(\nabla_X Y)qZ + (Zq)\nabla_X Y - g(\nabla_X Y, Z)Dq] \\ & - [(Yq)\nabla_X Z + (\nabla_X Z)qY - g(Y, \nabla_X Z)Dq].\end{aligned}$$

So

$$\begin{aligned}\nabla_X \pounds_V \nabla(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z) = {} & X(Zq)Y - Y(Zq)X \\ & - g(Y,Z)\nabla_X Dq + g(X,Z)\nabla_Y Dq \\ & - ((\nabla_X Z)q)Y + ((\nabla_Y Z)q)X.\end{aligned} \tag{12}$$

Also, from Yano [**30**], we get

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z).$$

Using equation (12), we get

$$(\pounds_V R)(X,Y)Z = g(\nabla_X Dq, Z)Y - g(\nabla_Y Dq, Z)X \\ - g(Y,Z)\nabla_X Dq + g(X,Z)\nabla_Y Dq.$$

Contracting the forgoing equation with respect to X,

$$(\pounds_V S)(Y,Z) = -(n-2)g(\nabla_Y Dq, Y) - (\Delta q)g(Y,Z).$$

Consequently, $\pounds_V s = 2qs - 2(n-1)\Delta q$. This completes the proof. □

Lemma 3.3. *If a Riemannian manifold is equipped with a torse-forming vector field and the metric of the manifold is a Yamabe soliton, then ρ is constant.*

Proof. First we take Lie derivative to $g(\xi,\xi) = 1$ along V, and employ the equations (3) and (1) to get

$$(\pounds_V \eta)\xi = -\eta(\pounds_V \xi) = (s-\lambda). \tag{13}$$

Now taking $n = 3$ and $q = s - \lambda$ in Lemma 3.2, we find

$$(\pounds_V S)(X,Y) = -g(\nabla_X Ds, Y) - (\Delta s)g(X,Y), \tag{14}$$

$$\pounds_V s = -2s(s-\lambda) - 4\Delta s. \tag{15}$$

In Riemannian three manifolds, the curvature tensor is given by

$$R(X,Y)Z = g(Z,Y)QX - g(Z,X)QY + g(Z,QY)X \\ - g(Z,QX)Y - \frac{s}{2}[g(Z,Y)X - g(Z,X)Y]. \tag{16}$$

Taking $Z = \xi$ in the above equation and using (6), we easily deduce

$$S(X,Y) = \Big(\frac{s}{2} + \xi\rho - \rho\Big)g(X,Y) + \Big(3\rho - \frac{s}{2}\Big)\eta(X)\eta(Y) \\ - (X\rho)\eta(Y) - 2(Y\rho)\eta(X). \tag{17}$$

Since $S(X,Y)$ is symmetric, we get $(X\rho) = 0$, and $(Y\rho) = 0$, i.e., ρ is constant. This completes the proof. □

Let us now prove the following results.

Theorem 3.1. *If the metric of a Riemannian three manifold M equipped with a torse-forming vector field is a Yamabe soliton, then the scalar curvature is harmonic if and only if the function ρ vanishes. If the function ρ is non-zero constant, the scalar curvature is solution of the Poisson equation $\Delta s + As = B$, where $A = \frac{1}{2}\rho(\rho+1)$, $B = 3\rho^2(\rho+1)$. In that case the soliton is shrinking if s is positive.*

Proof. Using Lemma 3.3, we get from (17)

$$S(X,Y) = \Big(\frac{s}{2} - \rho\Big)g(X,Y) + \Big(3\rho - \frac{s}{2}\Big)\eta(X)\eta(Y). \tag{18}$$

Taking Lie derivative of the previous equation along V and employing the equations (15) and (3), we find

$$(\pounds_V S)(X,Y) = -2[\rho(s-\lambda)+\Delta s]g(X,Y) + [s(s-\lambda)+2\Delta s]\eta(X)\eta(Y) + \Big(3\rho - \frac{s}{2}\Big)[(\pounds_V \eta)X\eta(Y) + \eta(X)(\pounds_V \eta)Y].$$

Comparing the above equation with (14), we obtain

$$g(\nabla_X Ds, Y) = (2\rho(s-\lambda)+\Delta s)g(X,Y) - (s(s-\lambda)+2\Delta s)\eta(X)\eta(Y) - \Big(3\rho - \frac{s}{2}\Big)\big[(\pounds_V \eta)X\eta(Y) + \eta(X)(\pounds_V \eta)Y\big]. \tag{19}$$

Replacing X and Y in the above equation by ξ and utilizing (13), we see that

$$\xi(\xi s) = -4\rho(s-\lambda) - \Delta s.$$

Now using Lemmas 3.1 and 3.3, we get

$$\Delta s = 24\rho^3 - 4\rho^2 s - 4\rho s + 4\rho\lambda. \tag{20}$$

We replace Y by ξ in (19) and use the equations (13) and (20) to deduce,

$$g(\nabla_X Ds, \xi) = (-(s-\lambda)\Big(\rho + \frac{s}{2}\Big) - 24\rho^3 + 4\rho^2 s + 4\rho s - 4\rho\lambda)\eta(X) - \Big(3\rho - \frac{s}{2}\Big)(\pounds_V \eta)X.$$

On the other hand differentiating the equation stated in Lemma 3.1 along X and using Lemma 3.3, we obtain

$$g(\nabla_X Ds, \xi) = -3\rho(Xs) - (12\rho^2 - 2\rho s)\eta(X).$$

Using the preceding equation in (19), we get

$$\Big(3\rho - \frac{s}{2}\Big)(\pounds_V \eta)(X) = (-(s-\lambda)\Big(\rho + \frac{s}{2}\Big) - 24\rho^3 + 12\rho^2 + 4\rho^2 s + 2\rho s - 4\rho\lambda)\eta(X) - 3\rho(Xs).$$

Substituting the above equation in (19) and (20), we find

$$\begin{aligned} g(\nabla_X Ds, Y) = &-2\rho(s - \lambda + 24\rho^3 - 4\rho^2 s)g(X,Y) \\ &- 2\rho(\lambda - 3s + 12\rho)\eta(X)\eta(Y) \\ &- 3\rho(Xs)\eta(Y) - 3\rho(Ys)\eta(X), \end{aligned}$$

which further leads to

$$\begin{aligned} \nabla_X Ds = &-2\rho(s - \lambda - 12\rho^2 + 2\rho s)X - 2\rho(\lambda - 3s + 12\rho)\eta(X)\xi \\ &- 3\rho(Xs)\xi - 3\rho\eta(X)Ds. \end{aligned} \tag{21}$$

Replacing X in the previous equation by ξ and applying Lemma 3.1, we get

$$\nabla_\xi Ds = 2\rho((2+\rho)s - 6\rho^2 - 12\rho)\xi - 3\rho Ds.$$

In the above equation, taking covariant derivative along X, we have

$$\nabla_X \nabla_\xi Ds = 2\rho(2+\rho)(Xs)\xi + 2\rho((2+\rho)s - 6\rho^2 - 12\rho)\nabla_X \xi - 3\rho \nabla_X Ds. \tag{22}$$

On the other hand, differentiating (21) along ξ, we get

$$\begin{aligned}\nabla_\xi\nabla_X Ds = &-2\rho(1+2\rho)(\xi s)X - 2\rho((1+2\rho)s-\lambda-12\rho^2)\nabla_\xi X + 6\rho(\xi s)\eta(X)\xi\\ &+2\rho(3s-\lambda-12\rho)\eta(\nabla_\xi X)\xi + 2\rho(3s-\lambda-12\rho)\eta(X)(\rho+1)\xi\\ &-3\rho\xi(Xs)\xi - 3\rho(Xs)(\rho+1)\xi - 3\rho\eta(\nabla_\xi X)Ds - 3\rho\eta(X)\nabla_\xi Ds.\end{aligned}$$

Again from (21), we immediately get

$$\begin{aligned}\nabla_{[X,\xi]}Ds = &-2\rho((1+2\rho)s-\lambda-12\rho^2)(\nabla_X\xi - \nabla_\xi X)\\ &+2\rho(3s-\lambda-12\rho)\eta(\nabla_X\xi-\nabla_\xi X)\xi\\ &-3\rho g(\nabla_X\xi-\nabla_\xi X, Ds)\xi - 3\rho\eta(\nabla_X\xi-\nabla_\xi X)Ds.\end{aligned}\tag{23}$$

Now employing the equations (22) and (23) one can easily get,

$$\begin{aligned}R(X,\xi)Ds = &\,\rho(7+5\rho)(Xs)\xi - 3\rho\nabla_X Ds + 2\rho((2+\rho)s-6\rho^2-12\rho)\nabla_X\xi\\ &+2\rho(1+2\rho)(\xi s)X + 2\rho((1+2\rho)s-\lambda-12\rho^2)\nabla_\xi X - 6\rho(\xi s)\eta(X)\xi\\ &+3\rho\eta(X)\nabla_\xi Ds - 2\rho(3s-\lambda-12\rho)\eta(\nabla_\xi X)\xi + 3\rho\xi(Xs)\xi + 3\rho\eta(\nabla_\xi X)Ds\\ &+2\rho((1+2\rho)s-\lambda-12\rho^2)(\nabla_X\xi-\nabla_\xi X) + 3\rho g(\nabla_X\xi-\nabla_\xi X, Ds)\xi\\ &+3\rho\eta(\nabla_X\xi-\nabla_\xi X)Ds - 2\rho(3s-\lambda-12\rho)\eta(\nabla_X\xi-\nabla_\xi X)\xi\\ &-2\rho(3s-\lambda-12\rho)(\rho+1)\eta(X)\xi.\end{aligned}$$

Contracting the above equation, we find

$$\begin{aligned}S(\xi, Ds) = &\,10\rho(1+2\rho)(\xi s) + 6\rho\xi(\xi s) - 3\rho\Delta s\\ &+2\rho(3\rho+1)(3s+3\rho s-18\rho^2-\lambda-12\rho) - 2\rho(\rho+1)(3s-\lambda-12\rho).\end{aligned}$$

Making use of Lemma 3.1 together with (20) in the preceding equation, we obtain

$$S(\xi, Ds) = 36\rho^3 - 84\rho^4 + 10\rho^2 s + 14\rho^3 s - 16\rho^2\lambda.$$

Now we employ Lemma 3.1 and equation (7) in the above equation to deduce the value of soliton constant as follows:

$$\lambda = \Big(\frac{3}{4} - \frac{21}{4}\rho\Big)\rho + \frac{7}{8}(1+\rho)s.$$

Using the value of λ, from (20) we get

$$\Delta s = 3\rho^3 + 3\rho^2 - \frac{1}{2}\rho^2 s - \frac{1}{2}\rho s,$$

$$\Delta s = -\frac{1}{2}\rho(\rho+1)s + 3\rho^2(\rho+1),$$

$$\Delta s + As = B,$$

where $A = \frac{1}{2}\rho(\rho+1)$ and $B = 3\rho^2(\rho+1)$. This completes the proof. □

Theorem 3.2. *Let (M, g) be a Riemannian three manifold admitting torse-forming vector field. If the metric g is a Yamabe soliton then the scalar curvature s satisfies the following results:*

$$\begin{aligned} s = 0 &\quad \text{iff} \quad \rho = 0, \\ s > 0 &\quad \text{if} \quad \rho > 0,\ \rho < -2, \\ s < 0 &\quad \text{if} \quad \rho > -2, \end{aligned}$$

provided $s - 6\rho \neq 0$ and $g(Ds, e_1) \neq 0$. If $g(Ds, e_1) = 0$ then either $s = 0$ or $s = 6\rho$.

Proof. First differentiate (3) along Z to achieve

$$(\nabla_Z \pounds_V g) = 2(Zs)g(X, Y). \tag{24}$$

In [**24**], Yano reveals the following:

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y, Z) = -g((\pounds_V \nabla)(X, Z), Y) - g((\pounds_V \nabla)(X, Y), Z).$$

Since ∇ is Levi-Civita connection, it appears from the preceding equation that

$$(\nabla_X \pounds_V g)(Y, Z) = g((\pounds_V \nabla)(X, Z), Y) + g((\pounds_V \nabla)(X, Y), Z).$$

Utilizing the symmetric property of $\pounds_V \nabla$, the foregoing equation brings into

$$2g((\pounds_V \nabla)(X, Y), Z) = (\nabla_Y \pounds_V g)(Z, X) + (\nabla_X \pounds_V g)(Y, Z) - (\nabla_Z \pounds_V g)(X, Y).$$

Utilizing (24) in the previous equation, we find

$$(\pounds_V \nabla)(X, Y) = (Xs)Y + (Ys)X - g(Y, X)Ds. \tag{25}$$

Differentiating (25) covariantly along Z yields

$$(\nabla_Z \pounds_V \nabla)(X, Y) = g(\nabla_Z Ds, X)Y + g(\nabla_Z Ds, Y)X - g(Y, X)\nabla_Z Ds.$$

Applying the above relation on the following well-known formula

$$(\pounds_V R)(X, Y)Z = (\nabla_X \pounds_V \nabla)(Y, Z) - (\nabla_Y \pounds_V \nabla)(X, Z),$$

we get

$$(\pounds_V R)(X, Y)Z = g(\nabla_X Ds, Z)Y - g(Z, Y)\nabla_X Ds - g(\nabla_Y Ds, Z)X + g(X, Z)\nabla_Y Ds.$$

Replacing Z in the previous equation by ξ and using (21), we get

$$\begin{aligned} (\pounds_V R)(X, Y)\xi = &\Big(4\rho\Big(-\frac{15}{16}s + \frac{1}{16}\rho s - \frac{3}{8}\rho^2 + \frac{45}{8}\rho\Big)\eta(Y) + 3\rho(Ys)\Big)X \\ &- 3\rho(Ys)\eta(X)\xi \\ &- \Big(4\rho\Big(-\frac{15}{16}s + \frac{1}{16}\rho s - \frac{3}{8}\rho^2 + \frac{45}{8}\rho\Big)\eta(X) + 3\rho(Xs)\Big)Y \\ &+ 3\rho(Xs)\eta(Y)\xi. \end{aligned} \tag{26}$$

We differentiate (6), use Lemma 3.3 and apply (3) to obtain

$$\begin{aligned}(\pounds_V R)(X,Y)\xi &= \rho\Big(g(Y,\pounds_V\xi)+2\Big(s-\frac{3}{4}\rho+\frac{21}{4}\rho^2-\frac{7}{8}s-\frac{7}{8}\rho s\Big)\eta(Y)\Big)X\\ &\quad-\rho\Big(g(X,\pounds_V\xi)+2\Big(s-\frac{3}{4}\rho+\frac{21}{4}\rho^2-\frac{7}{8}s-\frac{7}{8}\rho s\Big)\eta(X)\Big)Y\\ &\quad-R(X,Y)\pounds_V\xi.\end{aligned} \tag{27}$$

Subtracting (26) from (27), we see that

$$\begin{aligned}R(X,Y)\pounds_V\xi &= \Big(4\rho\Big(-s+\frac{1}{2}\rho s-3\rho^2+6\rho\Big)\eta(X)+3\rho(Xs)-\rho g(\pounds_V\xi,X)\Big)Y\\ &\quad+3\rho(Ys)\eta(X)\xi\\ &\quad-\Big(4\rho\Big(-s+\frac{1}{2}\rho s-3\rho^2+6\rho\Big)\eta(Y)+3\rho(Ys)-\rho g(\pounds_V\xi,Y)\Big)X\\ &\quad-3\rho(Xs)\eta(Y)\xi.\end{aligned}$$

Contracting the above equation, we obtain that

$$S(Y,\pounds_V\xi)=(8\rho s+2\rho^2 s-12\rho^3-48\rho^2)\eta(Y)-3\rho(Ys)+2\rho g(Y,\pounds_V\xi).$$

By the support of (18) and (13) the above equation yields

$$(s-6\rho)\pounds_V\xi=(s-6\rho)\Big(\frac{67}{4}\rho-\frac{5}{4}\rho^2-\frac{1}{8}s+\frac{7}{8}\rho s\Big)\xi-6\rho Ds. \tag{28}$$

We suppose that on an open subset O of M there holds $s\neq 6\rho$. Then it appears from equation (28) that

$$\pounds_V\xi=\Big(\frac{67}{4}\rho-\frac{5}{4}\rho^2-\frac{1}{8}s+\frac{7}{8}\rho s\Big)\xi-\Big(\frac{6\rho}{s-6\rho}\Big)Ds. \tag{29}$$

Replacing Y with ξ in the well-known formula,

$$(\pounds_V\nabla)(X,Y)=\pounds_V\nabla_X Y-\nabla_X\pounds_V Y-\nabla_{[V,X]}Y,$$

and then using (1) and (29), we obtain the following equality:

$$\begin{aligned}(\pounds_V\nabla)(X,\xi) &= \Big(\frac{1}{8}s-\frac{7}{8}\rho s+\frac{61}{4}\rho+\frac{139}{4}\rho^2-\frac{21}{2}\rho^3\Big)\eta(X)\xi\\ &\quad+\frac{s-7\rho s-54\rho-102\rho^2}{8(s-6\rho)}(Xs)\xi-\frac{18\rho^2+6\rho}{s-6\rho}\eta(X)Ds\\ &\quad-\Big(\frac{67}{4}\rho^2-\frac{5}{4}\rho^3-\frac{1}{8}\rho s+\frac{7}{8}\rho^2 s+\frac{3}{2}\rho^2+\frac{27}{2}\rho^3\Big)X\\ &\quad-\frac{6\rho}{s-6\rho}(Xs)Ds.\end{aligned} \tag{30}$$

On the other hand, replacing Y in (25) by ξ and using Lemma 3.1, we have

$$(\pounds_V\nabla)(X,\xi)=(Xs)\xi+(12\rho^2-2\rho s)X-\eta(X)Ds. \tag{31}$$

Comparing (30) and (31), one obtains

$$
\begin{aligned}
&\frac{7(1+\rho)s+6\rho(1+17\rho)}{8(s-6\rho)}(Xs)\xi+\Big(\frac{121}{4}\rho^2+\frac{49}{4}\rho^3-\frac{17}{8}\rho s+\frac{7}{8}\rho^2 s\Big)X\\
&+\frac{18\rho^2+12\rho-s}{s-6\rho}\eta(X)Ds-\Big(\frac{1}{8}s-\frac{7}{8}\rho s+\frac{61}{4}\rho+\frac{139}{4}\rho^2-\frac{21}{2}\rho^3\Big)\eta(X)\xi\\
&+\frac{6\rho}{(s-6\rho)^2}(Xs)Ds=0.
\end{aligned}
\tag{32}
$$

Choosing an orthonormal basis (ξ, e_1, e_2), putting $X=\xi$ in (32) and taking inner product with e_1 in both sides, we have

$$\Big(\frac{6\rho^2+12\rho-s}{s-6\rho}\Big)g(Ds,e_1)=0.$$

If $g(Ds,e_1)\neq 0$, then $6\rho^2+12\rho-s=0$ imply

$$s=6\rho^2+12\rho, \tag{33}$$

provided $s-6\rho\neq 0$. From (33) we get the following results:

$$
\begin{aligned}
&s=0 \quad \text{iff } \rho=0,\\
&s>0 \quad \text{if } \rho>0,\ \rho<-2,\\
&s<0 \quad \text{if } \rho>-2.
\end{aligned}
$$

If $g(Ds,e_1)=0$, following cases arise:
<u>Case 1</u>: Either $Ds=0$ which implies, $g(Ds,\xi)=0$. So, $\rho=0$, or, $s=6\rho$.
<u>Case 2</u>: If Ds is parallel to e_2 and orthogonal to e_1 then, either Ds is parallel to e_2, or, Ds is parallel to ξ. If Ds is parallel to e_2 orthogonal to ξ, then $g(Ds,\xi)=0$, which gives $\rho=0$ or $s=6\rho$.
<u>Case 3</u>: If Ds is parallel to ξ, then $g(Ds,Ds)=kg(\xi,\xi)$. So, Ds is a constant, consequently $\pounds_X(Ds)=0$, Thus, $\rho=0$. □

In the following, we construct two illustrative examples to support the obtained results.

Example 3.1. Consider a manifold $M=\big((u,v,w)\in\mathbb{R}: v>0, w\neq 0\big)$ with a coordinate system (u,v,w). Let us define a Riemannian metric on M as

$$g=2w^2(\mathrm{d}u)^2+\frac{w^2}{2v}(\mathrm{d}u\otimes\mathrm{d}v+\mathrm{d}v\otimes\mathrm{d}u)+\frac{w^2}{4v^2}(\mathrm{d}v)^2+\frac{1}{w^2}(\mathrm{d}w)^2.$$

From Koszul's formula, one can easily compute

$$
\begin{aligned}
&\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u}=-2w^3\frac{\partial}{\partial w}, &&\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v}=-\frac{w^3}{2v}\frac{\partial}{\partial w}, &&\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial w}=\frac{1}{w}\frac{\partial}{\partial u},\\
&\nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial u}=-\frac{w^3}{2v}\frac{\partial}{\partial w}, &&\nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v}=-\frac{1}{v}\frac{\partial}{\partial v}-\frac{w^3}{4v^2}\frac{\partial}{\partial w}, &&\nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial w}=\frac{1}{w}\frac{\partial}{\partial v},\\
&\nabla_{\frac{\partial}{\partial w}}\frac{\partial}{\partial u}=\frac{1}{w}\frac{\partial}{\partial u}, &&\nabla_{\frac{\partial}{\partial w}}\frac{\partial}{\partial v}=\frac{1}{w}\frac{\partial}{\partial v}, &&\nabla_{\frac{\partial}{\partial w}}\frac{\partial}{\partial w}=-\frac{1}{w}\frac{\partial}{\partial w}.
\end{aligned}
$$

Let us take $\xi = -w\frac{\partial}{\partial w}$. Then we can verify that

$$\nabla_{X_i}\xi = \rho X_i + \eta(X_i)\xi,$$

for $\rho = -1$ and for all $1 \leq i \leq 3$, where $X_1 = \frac{\partial}{\partial u}$, $X_2 = \frac{\partial}{\partial v}$, $X_3 = \frac{\partial}{\partial w}$. Thus, the vector field $\xi = -w\frac{\partial}{\partial w}$ is a torse-forming vector field. Now we use Levi-Civita connection to find non-zero components of curvature tensor as given below

$$R\Big(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\Big)\frac{\partial}{\partial u} = -\frac{w^2}{2v}\frac{\partial}{\partial u} + 2w^2\frac{\partial}{\partial v}, \qquad R\Big(\frac{\partial}{\partial v}, \frac{\partial}{\partial w}\Big)\frac{\partial}{\partial v} = \frac{w^2}{2v}\frac{\partial}{\partial w},$$
$$R\Big(\frac{\partial}{\partial u}, \frac{\partial}{\partial w}\Big)\frac{\partial}{\partial w} = -\frac{1}{w^2}\frac{\partial}{\partial u}, \qquad R\Big(\frac{\partial}{\partial u}, \frac{\partial}{\partial w}\Big)\frac{\partial}{\partial v} = \frac{w^2}{2v}\frac{\partial}{\partial w},$$
$$R\Big(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\Big)\frac{\partial}{\partial v} = \frac{w^2}{2v}\frac{\partial}{\partial v} - \frac{w^2}{4v^2}\frac{\partial}{\partial u}, \qquad R\Big(\frac{\partial}{\partial v}, \frac{\partial}{\partial w}\Big)\frac{\partial}{\partial w} = -\frac{1}{w^2}\frac{\partial}{\partial v},$$
$$R\Big(\frac{\partial}{\partial u}, \frac{\partial}{\partial w}\Big)\frac{\partial}{\partial u} = 2w^2\frac{\partial}{\partial w}, \qquad R\Big(\frac{\partial}{\partial v}, \frac{\partial}{\partial w}\Big)\frac{\partial}{\partial u} = \frac{w^2}{2v}\frac{\partial}{\partial w}.$$

From the curvature tensor we find the scalar curvature as $s = -6$.

Now we show that the metric g is a Yamabe soliton on M. Let

$$V = \frac{\ln(v)}{2}\frac{\partial}{\partial u} - 2v(\ln(v) + 2u)\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}$$

be a vector field on M. It is not hard to see that $\pounds_V g = 0$, and so V is Killing. Thus, we see that

$$\pounds_V g = (s - \lambda)g$$

for $\lambda = -6$. Hence g is a Yamabe soliton having the potential vector field $V = \frac{\ln(v)}{2}\frac{\partial}{\partial u} - 2v(\ln(v) + 2u)\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}$ and the soliton constant $\lambda = -6$. we see that this example verifies our Theorem 3.1 and Theorem 3.2.

Example 3.2. Let $M = \big((x, y, z); z > 0\big)$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let

$$g = \frac{1}{z^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z)$$

and consider the linearly independent vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z}.$$

Using Koszul's formula, we get

$$\nabla_{e_1}e_1 = -e_3, \qquad \nabla_{e_1}e2 = 0, \qquad \nabla_{e_1}e_3 = e_1,$$
$$\nabla_{e_2}e_1 = 0, \qquad \nabla_{e_2}e_2 = -e_3, \qquad \nabla_{e_2}e_3 = e_2,$$
$$\nabla_{e_3}e_1 = 0, \qquad \nabla_{e_3}e_2 = 0, \qquad \nabla_{e_3}e_3 = 0.$$

Then the Riemann curvature and Ricci tensor are given by

$$R(e_1, e_2)e_2 = -e_1, \qquad R(e_1, e_3)e_3 = e_1, \qquad R(e_2, e_1)e_1 = -e_2,$$
$$R(e_2, e_3)e_3 = -e_2, \qquad R(e_3, e_1)e_1 = -e_3, \qquad R(e_3, e_2)e_2 = -e_3.$$

and

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2.$$

Hence the scalar curvature $s = -6$. Now we shall show that the metric g is a Yamabe soliton on M. Let

$$V = e_1 + e_2$$

be a vector field on M. Now

$$\begin{aligned}(\pounds_V g)(e_1, e_1) &= 2g(\nabla_{e_1} V, e_1) = 0,\\ (\pounds_V g)(e_2, e_2) &= 2g(\nabla_{e_2} V, e_2) = 0,\\ (\pounds_V g)(e_3, e_3) &= 2g(\nabla_{e_3} V, e_3) = 0.\end{aligned}$$

Therefore, for $\lambda = -6$ and $V = e_1 + e_2$,

$$(\pounds_V g)(X, Y) = (s - \lambda)g(X, Y)$$

is satisfied. So, g is a Yamabe soliton having potential vector field $V = e_1 + e_2$ and soliton constant $\lambda = -6$.

Let $\xi = -e_3$. Then we can easily verify that

$$\nabla_{X_i}\xi = \rho X_i + \eta(X_i)\xi,$$

for $\rho = -1$ and for all $1 \le i \le 3$, where $X_1 = e_1$, $X_2 = e_2$, $X_3 = e_3$. Thus the vector field $\xi = -e_3$ is torse-forming vector field. We see that this example also verifies our Theorem 3.1 and Theorem 3.2.

4. Infinitesimal harmonic transformation and quotient almost Yamabe solitons

In [**24**], the authors proved the existence of the potential vector field V of a Ricci soliton on a Riemannian manifold (M, g) as an infinitesimal harmonic transformation on M. Ghosh [**13**] studied Ricci almost soliton with potential vector field as infinitesimal harmonic transformation. We consider here the quotient almost Yamabe soliton whose potential vector field is an infinitesimal harmonic transformation. Let us prove the following:

Theorem 4.1. *If $(M^n, g, V), n \ge 3$, is a quotient Yamabe almost soliton with potential vector field as an infinitesimal harmonic transformation, then g is a Yamabe soliton if and only if $\log\frac{\sigma_k}{\sigma_l}$ is constant.*

The proof of Theorem 4.1 relies on the following lemma.

Lemma 4.1. *For a quotient Yamabe almost soliton (M^n, g, V), we have $\bar{\Delta}V - QV = (n-2)D\theta - (n-2)D\lambda$, where D denotes the gradient operator.*

Proof. Suppose that g is a quotient Yamabe almost soliton. By covariant differentiation of (5) along X and renaming $\log\frac{\sigma_k}{\sigma_l} = \theta$, one obtains

$$(\nabla_X \pounds_V g)(Y, Z) = 2(\nabla_X \theta)(Y, Z) - 2(X\lambda)g(Y, Z). \tag{34}$$

The commutation formula [**30**],

$$\begin{aligned}(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y, Z) = &-g((\pounds_V \nabla)(X, Y), Z)\\ &- g((\pounds_V \nabla)(X, Z), Y),\end{aligned}$$

yields

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{35}$$

By virtue of (34) and (35), one has

$$g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y) = 2(\nabla_X \theta)g(Y,Z) - 2(X\lambda)g(Y,Z). \tag{36}$$

Applying cyclic permutation twice over $\{X,Y,Z\}$ in (36), and then taking their difference, we conclude

$$\begin{aligned} g((\pounds_V \nabla)(Y,X),Z) - g((\pounds_V \nabla)(Z,X),Y) &= 2(\nabla_Y \theta)g(Z,X) - 2(\nabla_Z \theta)(X,Y) \\ &\quad - 2(Y\lambda)g(Z,X) + 2(Z\lambda)g(X,Y). \end{aligned}$$

Combining the foregoing equation with (36) and noting that $(\pounds_V \nabla)(X,Y) = (\pounds_V \nabla)(Y,X)$, we establish

$$\begin{aligned} g((\pounds_V \nabla)(X,Y),Z) &= (\nabla_X \theta)g(Y,Z) + (\nabla_Y \theta)g(Z,X) - (\pounds_Z \theta)g(X,Y) \\ &\quad - (X\lambda)g(Y,Z) - (Y\lambda)g(Z,X) + (Z\lambda)g(X,Y). \end{aligned} \tag{37}$$

By virtue of the formula [**30**]

$$g((\pounds_V \nabla)(X,Y),Z) = g(\nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V - R(X,V)Y, Z),$$

from the above equation, one obtains

$$\begin{aligned} &g(\nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V - R(X,V)Y, Z) \\ &= (\nabla_X \theta)(Y,Z) + (\nabla_Y \theta)g(Z,X) - (\nabla_Z \theta)g(X,Y) \\ &\quad - (X\lambda)g(Y,Z) - (Y\lambda)g(Z,X) + (Z\lambda)g(X,Y). \end{aligned} \tag{38}$$

Let $\{e_i : i = 1,2,\dots,n\}$ be an orthonormal frame of the tangent space at each point of M. Putting $X = Y = e_i$ and considering the definition of the rough Laplacian, we have

$$-\bar{\Delta} V + QV = -(n-2)D\theta + (n-2)D\lambda. \tag{39}$$

This completes the proof. □

Proof of Theorem 4.1. Let V be an infinitesimal harmonic transformation. Then it is known that [**23, 24**] $\Delta V = 2QV$. So, in view of Weitzenbock's formula

$$\Delta V = \bar{\Delta} V + QV, \tag{40}$$

we get $\bar{\Delta} V = QV$. Utilizing this in (39), we see that λ is constant if and inly if θ is so. □

5. Triviality results on quotient almost Yamabe solitons under an integral inequality

In [**3**], the authors proved that a compact Ricci almost soliton (M^n, g, V), $n > 2$, is trivial if it satisfies $\int_M [S(V,V) + (n-2)g(D\lambda, V)]\mathrm{d}M \leq 0$. This result extends the corresponding result of Petersen–Wylie [**18**] for compact Ricci solitons. Ghosh [**13**] studied almost Ricci solitons with the integral inequality that was first established in [**3**] in order to find triviality results. Here we will do so for quotient almost Yamabe soliton. Let us prove the following:

Theorem 5.1. *Let (M^n, g, V), $n \geq 2$, be a compact quotient Yamabe almost soliton with non zero potential vector field V. If it satisfies $\int_M \{(2S(V,V) - (n-2)g(D\lambda, V)\}\mathrm{d}M \leq 0$, then the 1-form associated to V is harmonic and M is Ricci flat.*

Proof. Let us recall the following integral formula (see Yano [**30**, p. 41],)

$$\int_M \Big\{g(\Delta V, V) - \frac{1}{2}|\mathrm{d}V^b|^2 - (\operatorname{div} V)^2\Big\}\mathrm{d}M = 0,$$

where V^b is the 1-form associated with the potential vector field V. Now, using (39) and (40), we have

$$\Delta V + (n-2)D\lambda = 2QV + (n-2)D\theta. \tag{41}$$

By virtue of (41) the above integral formula takes the form

$$\begin{aligned}&\int_M \{2S(V,V) - (n-2)g(D\lambda, V)\}\mathrm{d}M \\ &= \int_M \Big\{\frac{1}{2}|\mathrm{d}V^b|^2 + (\operatorname{div} V)^2\Big\}\mathrm{d}M + (n-2)g(D\theta, V).\end{aligned} \tag{42}$$

By our assumption, left hand side integral of (42) is less or equal to zero, while the integrand on the right side integral of (42) is greater or equal to zero. Thus, one must have

$$\int_M \Big\{\frac{1}{2}|dV^b|^2 + (\operatorname{div} V)^2\Big\}\mathrm{d}M + (n-2)g(D\theta, V) = 0.$$

This yields $\mathrm{d}V^b = 0$, $\operatorname{div} V = 0$ and $V(\theta) = 0$. As a result, V^b is harmonic. So, we can integrate the Bochner formula (which holds for any harmonnic 1-form) to obtain

$$\int_M \{S(V,V) + |\nabla V^b|^2\}\mathrm{d}M = \int_M \frac{1}{2}\Delta|V^b|^2\mathrm{d}M = 0 \tag{43}$$

Since V is divergence free, $\operatorname{div}(D\lambda, V) + \lambda(\operatorname{div} V) = g(D\lambda, V)$. Integrating this over M we achive $\int_M g(D\lambda, V)\mathrm{d}M = 0$. On the other hand, from the harmonicity of V^b and (42), it follows

$$\int_M \{2S(V,V) - (n-2)g(D\lambda, V)\}\mathrm{d}M = 0.$$

Therefore $\int_M S(V,V)\mathrm{d}M = 0$ and using this information into (43) proves that V is parallel. Thus, equation (3) shows that M is Einstein (i.e. $S = \lambda g$) and hence λ is constant. Since $\nabla V = 0$, we have $R(X,Y)V = 0$, and hence $S(X,V) = 0$. From which it follows that $\lambda |V|^2 = 0$. Since $|V|$ is a non-zero constant, we have $\lambda = 0$. Hence, M is Ricci flat. This completes the proof. $\square$

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