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SEMI-POSITONE STURM-LIOUVILLE DIFFERENTIAL SYSTEMS ON UNBOUNDED INTERVALS

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Abstract. This work is devoted to proving existence of nontrivial positive solutions for a system of n second-order differential equations subject to integral boundary conditions of Riemann-Stieltjes type and posed on the positive half-line. The novelty of the results is that the nonlinearity involved in the system is sign-changing and depends on both the solution and its derivative. Existence, multiplicity, and nonexistence results of nontrivial positive solutions are obtained using some fixed point theorems on suitable cones of a weighted Banach space. A numerical example is included to illustrate the applicability of our results.

1. Introduction

In this paper, we are concerned with the following Sturm-Liouville boundary value problem for a system of n second-order equations associated with integral boundary conditions of Riemann-Stieltjes type and posed on the positive half-line:

$$(1.1)\quad \begin{cases} -(p(t)y'(t))' = f(t,y(t),y'(t)) - h(t), \qquad \text{a.e. } t>0\\ \alpha y(0) - \beta p(0)y'(0) = \int_0^{+\infty} \mu(s)y(s)\mathrm{d}\xi(s),\\ \gamma \lim\limits_{t\to+\infty} y(t) + \delta \lim\limits_{t\to+\infty} p(t)y'(t) = \int_0^{+\infty} \nu(s)y(s)\mathrm{d}\eta(s),\end{cases}$$

where $\alpha,\gamma \geq 0$, $\beta,\delta \geq 1$, $p \in C[0,+\infty)$ is a real function with $p \geq 1$ on $\mathbb{R}_+$, $\int_0^{+\infty} 1/p(s)\mathrm{d}s < +\infty$, and $\rho = \gamma\beta + \alpha\delta + \alpha\gamma\int_0^{+\infty} 1/p(s)\mathrm{d}s > 0$. By $I = (0,+\infty)$, we denote the set of positive real numbers, $\mathbb{R}_+ = [0,+\infty)$, and $\mathbb{R}^n_+ = (\mathbb{R}_+)^n$; also we use the vector notations:

$$y = \begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix},\quad h = \begin{pmatrix} h_1\\ h_2\\ \vdots\\ h_n\end{pmatrix},\quad f(t,y,y') = \begin{pmatrix} f_1(t,y_1,\dots,y_n,y'_1,\dots,y'_n)\\ f_2(t,y_1,\dots,y_n,y'_1,\dots,y'_n)\\ \vdots\\ f_n(t,y_1,\dots,y_n,y'_1,\dots,y'_n)\end{pmatrix},$$

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$$\int_0^{+\infty} \mu(s)y(s)\mathrm{d}\xi(s) = \begin{pmatrix} \int_0^{+\infty} \mu_1(s)y_1(s)\mathrm{d}\xi_1(s) \\ \int_0^{+\infty} \mu_2(s)y_2(s)\mathrm{d}\xi_2(s) \\ \vdots \\ \int_0^{+\infty} \mu_n(s)y_n(s)\mathrm{d}\xi_n(s) \end{pmatrix},$$

$$\int_0^{+\infty} \nu(s)y(s)\mathrm{d}\eta(s) = \begin{pmatrix} \int_0^{+\infty} \nu_1(s)y_1(s)\mathrm{d}\eta_1(s) \\ \int_0^{+\infty} \nu_2(s)y_2(s)\mathrm{d}\eta_2(s) \\ \vdots \\ \int_0^{+\infty} \nu_n(s)y_n(s)\mathrm{d}\eta_n(s) \end{pmatrix}.$$

For $i \in \{1,\dots,n\}$, the functions $f_i = f_i(t,y,z)\colon \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R}_+$ are L^1-Carathéodory and the functions $h_i\colon \mathbb{R}_+ \to I$ are Lebesgue integrable. For $i \in \{1,\dots,n\}$, the functions $\xi_i, \eta_i\colon \mathbb{R} \to \mathbb{R}$ are nondecreasing and of bounded variation. $\int_0^{+\infty} \mu_i(t)y_i(t)\mathrm{d}\xi_i(t)$ and $\int_0^{+\infty} \nu_i(t)y_i(t)\mathrm{d}\eta_i(t)$ denote the Riemann--Stieltjes integrals of y_i with respect to ξ_i, η_i, respectively, and the functions $\mu_i, \nu_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfy

$$(\mathcal{C}_0) \qquad 0 < \int_0^{+\infty} \mu_i(s)\mathrm{d}\xi_i(s) < \infty \qquad \text{and} \qquad 0 < \int_0^{+\infty} \nu_i(s)\mathrm{d}\eta_i(s) < \infty.$$

Finally, $\mu(t) = \mathrm{diag}(\mu_1(t),\dots,\mu_n(t))$ and $\nu(t) = \mathrm{diag}(\nu_1(t),\dots,\nu_n(t))$. Let $AC(\mathbb{R}_+,\mathbb{R}^n_+)$ be the space of all absolute continuous vector-valued functions on $\mathbb{R}_+$. Throughout this paper, by a positive solution, we understand a function $y = (y_1,\dots,y_n) \in C^1(\mathbb{R}_+,\mathbb{R}^n_+)$ such that $py' \in AC(\mathbb{R}_+,\mathbb{R}^n_+)$ and y satisfies (1.1) with $y > 0_{\mathbb{R}^n}$ a.e. on $[0,+\infty)$, where $0_{\mathbb{R}^n}$ is the null vector of $\mathbb{R}^n$. For $u = (u_1,\dots,u_n)$, $v = (v_1,\dots,v_n) \in \mathbb{R}^n_+$, $u \geq v$ means that $u_i \geq v_i$ for all $i = 1,2,\dots,n$ and $u > v$ means that $u_i > v_i$, $i = 1,2,\dots,n$, i.e., component-wise inequalities.

Semi-positone boundary value problems (BVPs for short) arise in modeling steady-state reaction-diffusion systems [**1**], where the unknown may stand for a density, a temperature,.... The interest in the existence of positive solutions of such BVPs has been ongoing for many years; we refer the reader to the works [**2, 3, 6, 14**] and the references therein. In [**16**], a nonlocal BVP with integral conditions is considered whilst a semi-positone BVP on the half-line is investigated in [**17**].

In 2009, Zhang [**23**] investigated the existence of positive solutions of a singular multi-point boundary value problem for a system of second-order differential equations on infinite intervals in some Banach space. He used the Mönch fixed point theorem and a monotone iterative technique; the system considered reads

$$\begin{cases} x''(t) + f(t, x(t), x'(t), y(t), y'(t)) = 0, & t > 0, \\ y''(t) + g(t, x(t), x'(t), y(t), y'(t)) = 0, & t > 0, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(+\infty) = x_\infty, \\ y(0) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \quad y'(+\infty) = y_\infty. \end{cases}$$

The same year, Xi, Jia, and Ji [**20**] studied the existence of positive solutions to the following boundary value problem

$$\begin{cases} y_1''(t) + f_1(t, y_1(t), y_2(t)) = 0, & t > 0, \\ y_2''(t) + f_2(t, y_1(t), y_2(t)) = 0, & t > 0, \\ y_1(0) = 0, \quad y_1'(+\infty) = \int_0^{+\infty} g_1(s) y_1(s) \mathrm{d}s, \\ y_2(0) = 0, \quad y_2'(+\infty) = \int_0^{+\infty} g_2(s) y_2(s) \mathrm{d}s. \end{cases}$$

They used the Krasnosel'skii fixed point theorem. Some of the obtained results were improved by the same authors in [**21**] where they employed the three-functional fixed point theorem in a cone (due to Avery-Henderson) and a fixed point theorem (due to Avery-Peterson) in order to establish the existence of multiple positive solutions for a system of n equations.

In 2008, 2010, Webb and Infante [**18, 19**] studied some semi-positone BVPs which can be transformed into nonlinear integral equations; the existence of multiple positive solutions is then established for a Hammerstein equation of the form

$$y(t) = \int_0^1 k(t,s) g(s) f(s, y(s)) \mathrm{d}s,$$

where the kernel k may correspond to a Green's function, the function $g \in L^1([0,1], \mathbb{R}_+)$ may have pointwise singularities, and the nonlinearity $f\,[0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is a Carathéodory function satisfying $f(t,y) \geq -A$ for some $A > 0$.

More recently, in 2011, Feng and Bai [**8**] investigated the existence of positive solutions of the following second order m-point BVP

$$\begin{cases} y''(t) = \lambda f(t, y(t)), & t \in (0,1), \\ y'(0) = \sum_{i=1}^{m-2} \alpha_i y'(\xi_i), \quad y(1) = \sum_{i=1}^{m-2} \beta_i y'(\xi_i), \end{cases}$$

when $0 < \sum_{i=1}^{m-2} \alpha_i y'(\xi_i) < 1$, $0 < \sum_{i=1}^{m-2} \beta_i y'(\xi_i) < 1$, the function $f\colon [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is continuous, and there exists $A > 0$ such that $f(t,y) \geq -A$ for $(t,y) \in [0,1] \times [0, +\infty)$.

In 2013, the authors of [**7**] using the fixed point index theory on a cone of a Banach space established the existence and the multiplicity of nontrivial positive solutions for a boundary value problem consisting of a system posed on the positive half-line with second-order differential equations and integral boundary conditions.

The aim of this work is to extend these works to the case of a system in which the nonlinearities also depend on the first derivative and are allowed to change sign; moreover these nonlinearities satisfy general growth conditions including the classical polynomial growth. New existence, nonexistence, and multiplicity results of nontrivial positive solutions in a suitable cone of some weighted Banach space are proved using recent fixed point theorems on cones.

Regarding the nonlocalities, we point out that the signed measures are not considered in this paper. Semi-positone problems with nonlocal BCs involving signed measures can be found, e.g., in [**11**] (with linear BCs) and in [**9**] (with nonlinear BCs). However, the integral boundary conditions in problem (1.1) cover special cases with multi-point boundary conditions as it will be shown in the illustrative example in Section 5. Moreover, if $\xi_i = \eta_i = \mathrm{Id}$, the identity operator, then we get the usual integral boundary conditions.

The organization of this paper is as follows. Some background material is presented in Section 2. In particular, the corresponding Green's function and some useful inequalities are derived. Then problem (1.1) is formulated as a fixed point problem for a mapping denoted F in Section 3. The main results are then stated and proved in Section 4. The paper ends with an example of application.

2. Problem setting

2.1. Spaces of solutions

Definition 2.1. A function $g\colon [0,+\infty) \times \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R}^+$ is called L^1-Carathéodory if

(i) for each $u = (u_1, \dots, u_n) \in \mathbb{R}^n_+$ and $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, the map $t \mapsto g(t,u,v)$ is measurable on $[0,+\infty)$,

(ii) for a.e. $t \in \mathbb{R}_+$, the map $(u,v) \mapsto g(t,u,v)$ is continuous on $\mathbb{R}^n_+ \times \mathbb{R}^n$,

(iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1[0,+\infty)$ with $\varphi_r(t) > 0$ on $(0,+\infty)$ such that for all $i \in \{1,2,\dots,n\}$,

$$\max\{|u_i|, |v_i|\} \le r \implies g(t,u,v) \le \varphi_r(t) \qquad \text{for a.e. } t \in \mathbb{R}_+.$$

Definition 2.2. A nonempty subset $\mathcal{P}$ of a Banach space E is called a cone if $\mathcal{P}$ is convex, closed, and satisfies the conditions:

(i) $\alpha x \in \mathcal{P}$ for all $x \in \mathcal{P}$ and any real positive number α,

(ii) $x, -x \in \mathcal{P}$ imply $x = 0$.

Every cone $\mathcal{P} \subset E$ induces in E a partial ordering denoted $\preceq$ and given by

$$x \preceq y \ \text{ if and only if } \ y - x \in \mathcal{P}.$$

More details on cones and their properties can be found in [**5, 10, 12, 13, 22**]. If we let

$$C_l([0,+\infty),\mathbb{R}) = \{y \in C([0,+\infty),\mathbb{R}) \mid \lim_{t\to+\infty} y(t) \text{ exists}\},$$

then C_l endowed with the sup-norm $\|y\|_l = \sup_{t\in[0,+\infty)} |y(t)|$ is a Banach space. In this work, we rather consider the weighted space

$$\mathbb{X} = \big\{y = (y_1, \dots, y_n) \mid y_i \in C^1(\mathbb{R}_+, \mathbb{R}),\ \lim_{t\to+\infty} y_i(t) \text{ exists} \\ \text{and } \lim_{t\to+\infty} p(t)y_i'(t) \text{ exists for } i \in \{1, \dots, n\}\big\},$$

where the function p is defined as in (1.1). It is easy to prove that $\mathbb{X}$ is a Banach space normed by

$$\|y\| = \sum_{i=1}^{n} \|y_i\| \quad \text{with} \quad \|y_i\| = \max\{\|y_i\|_1, \|y_i\|_2\},$$

where

$$\|y_i\|_1 = \sup_{t\in\mathbb{R}_+} |y_i(t)| \quad \text{and} \quad \|y_i\|_2 = \sup_{t\in\mathbb{R}_+} |p(t)y_i'(t)|.$$

Let us consider the constants

$$A = \int_0^{+\infty} \frac{1}{p(s)} \mathrm{d}s, \quad \Lambda = \min\Big\{1, \frac{1}{\max\{\alpha, \gamma(\beta+\alpha A)\}}\Big\}, \quad \sigma = \frac{\rho}{(\beta+\alpha A)(\delta+\gamma A)},$$

and the function

$$\mathcal{G}(t) = \frac{1}{\rho}\Big(\beta + \alpha \int_0^t \frac{1}{p(s)}\mathrm{d}s\Big)\Big(\delta + \gamma \int_t^{+\infty} \frac{1}{p(s)}\mathrm{d}s\Big) \qquad t \in \mathbb{R}_+.$$

$\mathcal{P}$ denotes the positive cone defined in $\mathbb{X}$ by

$$\mathcal{P} = \Big\{y = (y_1, \dots, y_n) \in \mathbb{X} \ \mid y_i(t) \geq \sigma\mathcal{G}(t)\|y_i\|_1 \text{ for all } t \in \mathbb{R}_+ \\ \text{and } y_i(0) \geq \Lambda\|y_i\|_2,\ i \in \{1, \dots, n\}\Big\}.$$

Lemma 2.1. *Assume that $y \in \mathcal{P}$. Then for all positive t,*

$$y_i(t) \geq \sigma\Lambda\, \mathcal{G}(t)\|y_i\| \qquad \text{for all } i \in \{1, \dots, n\},$$

and thus $\sum_{i=1}^{n} y_i(t) \geq \sigma\Lambda\, \mathcal{G}(t)\|y\|$.

Proof. For $y = (y_1, \dots, y_n) \in \mathcal{P}$ and $i \in \{1, \dots, n\}$, we have

$$\|y_i\|_1 = \sup_{t\in\mathbb{R}_+} |y_i(t)| \geq y_i(0) \geq \Lambda\|y_i\|_2,$$

that is,

$$\|y_i\| = \max\{\|y_i\|_1, \|y_i\|_2\} \leq \max\{\|y_i\|_1, \Lambda^{-1}\|y_i\|_1\} = \Lambda^{-1}\|y_i\|_1.$$

Therefore, for all $t \geq 0$,

$$y_i(t) \geq \sigma\mathcal{G}(t)\|y_i\|_1 \geq \sigma\Lambda\, \mathcal{G}(t)\|y_i\|,$$

and so

$$\sum_{i=1}^{n} y_i(t) \geq \sigma\Lambda\mathcal{G}(t) \sum_{i=1}^{n} \|y_i\| = \sigma\Lambda\, \mathcal{G}(t)\|y\|.$$

□

2.2. Green's function

We first study the linear problem associated with (1.1). Denote

$$\phi_1(t)=\beta+\alpha\int_0^t\frac{1}{p(\tau)}\mathrm{d}\tau \qquad \text{and} \qquad \phi_2(t)=\delta+\gamma\int_t^{+\infty}\frac{1}{p(\tau)}\mathrm{d}\tau$$

so that $\mathcal{G}(t)=\frac{1}{\rho}\phi_1(t)\phi_2(t)$. Let

$$k_1^{(i)}=1-\frac{1}{\rho}\int_0^{+\infty}\mu_i(t)\phi_2(t)\mathrm{d}\xi_i(t), \qquad k_2^{(i)}=\frac{1}{\rho}\int_0^{+\infty}\mu_i(t)\phi_1(t)\mathrm{d}\xi_i(t),$$
$$k_3^{(i)}=1-\frac{1}{\rho}\int_0^{+\infty}\nu_i(t)\phi_1(t)\mathrm{d}\eta_i(t), \qquad k_4^{(i)}=\frac{1}{\rho}\int_0^{+\infty}\nu_i(t)\phi_2(t)\mathrm{d}\eta_i(t),$$

and

$$k^{(i)}=k_1^{(i)}k_3^{(i)}-k_2^{(i)}k_4^{(i)}, \qquad i\in\{1,\dots,n\}.$$

We have

Lemma 2.2. *Assume that $k^{(i)}\neq 0$ for all $i\in\{1,\dots,n\}$. Then, for any $v=(v_1,\dots,v_n)\in L^1(I,\mathbb{R}^n)$, the problem*

$$\text{(2.1)}\qquad \begin{cases} -(p(t)y'(t))'=v(t), \quad a.e.\ \ t\in I\\ \alpha y(0)-\beta p(0)y'(0)=\int_0^{+\infty}\mu(s)y(s)\mathrm{d}\xi(s),\\ \gamma\lim\limits_{t\to+\infty}y(t)+\delta\lim\limits_{t\to+\infty}p(t)y'(t)=\int_0^{+\infty}\nu(s)y(s)\mathrm{d}\eta(s)\end{cases}$$

has the unique solution

$$\text{(2.2)}\qquad y(t)=\int_0^{+\infty}H(t,s)\,v(s)\mathrm{d}s, \qquad t\in\mathbb{R}_+,$$

where $H(t,s)=\mathrm{diag}(H_1(t,s),\cdots,H_n(t,s))$ and the functions H_i, $i\in\{1,\dots,n\}$, are defined on $\mathbb{R}_+\times\mathbb{R}_+$ by

$$\text{(2.3)}\qquad \begin{aligned} H_i(t,s)=G(t,s)&+\frac{k_4^{(i)}\phi_1(t)+\ k_3^{(i)}\phi_2(t)}{k^{(i)}\rho}\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau)\\ &+\ \frac{k_1^{(i)}\phi_1(t)+k_2^{(i)}\phi_2(t)}{k^{(i)}\rho}\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau),\end{aligned}$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}\phi_1(s)\phi_2(t), & 0\le s\le t<+\infty,\\ \phi_1(t)\phi_2(s), & 0\le t\le s<+\infty.\end{cases}$$

Proof. By integrating twice the equation in problem (2.1) over $[0,t]$ $(t>0)$, we get

$$y_i(t)=y(0)+p(0)y'0)\int_0^t\frac{1}{p(s)}\mathrm{d}s-\int_0^t\frac{1}{p(s)}\Big(\int_0^s v_i(\tau)\mathrm{d}\tau\Big)\mathrm{d}s, \quad i\in\{1,\dots,n\}.$$

Integrating by parts yields

$$\text{(2.4)}\qquad y_i(t)=V+W\int_0^t\frac{1}{p(s)}\mathrm{d}s+\int_0^t\Big(\int_0^s\frac{1}{p(\tau)}\mathrm{d}\tau\Big)v_i(s)\mathrm{d}s-\int_0^t\frac{1}{p(\tau)}\mathrm{d}\tau\int_0^t v_i(\tau)\mathrm{d}\tau,$$

where V and W are constants to be determined. Differentiating (2.4) gives

$$y_i'(t) = \frac{1}{p(t)}\Big(W - \int_0^t v_i(s)\mathrm{d}s\Big). \tag{2.5}$$

From (2.4), (2.5), and the boundary conditions, we find the values

$$\begin{aligned} V = &\Big(\frac{1}{\alpha} - \frac{\beta\gamma}{\alpha\rho}\Big)\int_0^{+\infty} \mu_i(s)y_i(s)\mathrm{d}\xi_i(s) + \frac{\beta}{\rho}\int_0^{+\infty} \nu_i(t)y_i(s)\mathrm{d}\eta_i(s) \\ &+ \frac{\beta\gamma}{\rho}\int_0^{+\infty}\Big(\int_0^{+\infty}\frac{1}{p(\tau)}\mathrm{d}\tau - \int_0^s \frac{1}{p(\tau)}\mathrm{d}\tau + \frac{\delta}{\gamma}\Big)v_i(s)\mathrm{d}s, \\ W = &-\frac{\gamma}{\rho}\int_0^{+\infty} \mu_i(s)y_i(s)\mathrm{d}\xi_i(s) + \frac{\alpha}{\rho}\int_0^{+\infty} \nu_i(s)y_i(s)\mathrm{d}\eta_i(s) \\ &+ \frac{\alpha\gamma}{\rho}\int_0^{+\infty}\Big(\int_0^{+\infty}\frac{1}{p(\tau)}\mathrm{d}\tau - \int_0^s \frac{1}{p(\tau)}\mathrm{d}\tau + \frac{\delta}{\gamma}\Big)v_i(s)\mathrm{d}s. \end{aligned}$$

By substitution in (2.4), we get

$$\begin{aligned} y_i(t) = &\Big(\frac{1}{\alpha} - \frac{\beta\gamma}{\alpha\rho} - \frac{\gamma}{\rho}\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau\Big)\int_0^{+\infty} \mu_i(s)y_i(s)\mathrm{d}\xi_i(s) \\ &+ \Big(\frac{\beta}{\rho} + \frac{\alpha}{\rho}\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau\Big)\int_0^{+\infty} \nu_i(s)y_i(s)\mathrm{d}\eta_i(s) \\ &+ \Big(\frac{\beta\gamma}{\rho} + \frac{\alpha\gamma}{\rho}\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau\Big)\int_0^{+\infty}\Big(\int_0^{+\infty}\frac{1}{p(\tau)}\mathrm{d}\tau - \int_0^s \frac{1}{p(\tau)}\mathrm{d}\tau + \frac{\delta}{\gamma}\Big)v_i(s)\mathrm{d}s \\ &- \int_0^t \Big(\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau - \int_0^s \frac{1}{p(\tau)}\mathrm{d}\tau\Big)v_i(s)\mathrm{d}s, \end{aligned}$$

i.e.,

$$\begin{aligned} y_i(t) = &\Big(\frac{\rho - \beta\gamma - \alpha\gamma\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau}{\alpha\rho}\Big)\int_0^{+\infty} \mu_i(s)y_i(s)\mathrm{d}\xi_i(s) \\ &+ \Big(\frac{\beta + \alpha\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau}{\rho}\Big)\int_0^{+\infty} \nu_i(s)y_i(s)\mathrm{d}\eta_i(s) \\ &+ \frac{\gamma}{\rho}\phi_1(t)\int_t^{+\infty}\Big(\int_s^{+\infty}\frac{1}{p(\tau)}\mathrm{d}\tau + \frac{\delta}{\gamma}\Big)v_i(s)\mathrm{d}s \\ &+ \int_0^t \Big[\Big(\frac{\beta\gamma}{\rho} + \frac{\alpha\gamma}{\rho}\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau - 1\Big)\Big(\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau - \int_0^s \frac{1}{p(\tau)}\mathrm{d}\tau\Big) \\ &\times \Big(\frac{\beta\gamma}{\rho} + \frac{\alpha\gamma}{\rho}\int_0^t \frac{1}{p(\tau)}\mathrm{d}\tau\Big)\Big(\int_t^{+\infty}\frac{1}{p(\tau)}\mathrm{d}\tau + \frac{\delta}{\gamma}\Big)\Big]v_i(s)\mathrm{d}s. \end{aligned}$$

Hence

$$
\begin{aligned}
y_i(t) &= \frac{1}{\rho}\phi_2(t)\int_0^{+\infty}\mu_i(s)y_i(s)\mathrm{d}\xi_i(s) + \frac{1}{\rho}\phi_1(t)\int_0^{+\infty}\nu_i(s)y_i(s)\mathrm{d}\eta_i(s)\\
&\quad + \frac{1}{\rho}\int_0^{t}\phi_1(s)\phi_2(t)\,v_i(s)\mathrm{d}s + \frac{1}{\rho}\int_t^{+\infty}\phi_1(t)\phi_2(s)v_i(s)\mathrm{d}s\\
&= \int_0^{+\infty}G(t,s)\,v_i(s)\mathrm{d}s + \frac{1}{\rho}\phi_2(t)\int_0^{+\infty}\mu_i(s)y_i(s)\mathrm{d}\xi_i(s)\\
&\quad + \frac{1}{\rho}\phi_1(t)\int_0^{+\infty}\nu_i(s)y_i(s)\mathrm{d}\eta_i(s).
\end{aligned}
\tag{2.6}
$$

In addition,

$$
\begin{aligned}
\int_0^{+\infty}\mu_i(t)y_i(t)\mathrm{d}\xi_i(t) &= \frac{1}{\rho}\int_0^{+\infty}\mu_i(t)y_i(t)\mathrm{d}\xi_i(t)\int_0^{+\infty}\mu_i(t)\phi_2(t)\mathrm{d}\xi_i(t)\\
&\quad + \frac{1}{\rho}\int_0^{+\infty}\nu_i(t)y_i(t)\mathrm{d}\eta_i(t)\int_0^{+\infty}\mu_i(t)\phi_1(t)\mathrm{d}\xi_i(t)\\
&\quad + \int_0^{+\infty}\mu_i(t)\Big(\int_0^{+\infty}G(t,s)\,v_i(s)\mathrm{d}s\Big)\mathrm{d}\xi_i(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{+\infty}\nu_i(t)y_i(t)\mathrm{d}\eta_i(t) &= \frac{1}{\rho}\int_0^{+\infty}\mu_i(t)y_i(t)\mathrm{d}\xi_i(t)\int_0^{+\infty}\nu_i(t)\phi_2(t)\mathrm{d}\eta_i(t)\\
&\quad + \frac{1}{\rho}\int_0^{+\infty}\nu_i(t)y_i(t)\mathrm{d}\eta_i(t)\int_0^{+\infty}\nu_i(t)\phi_1(t)\mathrm{d}\eta_i(t)\\
&\quad + \int_0^{+\infty}\nu_i(t)\Big(\int_0^{+\infty}G(t,s)\,v_i(s)\mathrm{d}s\Big)\mathrm{d}\eta_i(t),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&k_1^{(i)}\int_0^{+\infty}\mu_i(t)y_i(t)\mathrm{d}\xi_i(t) - k_2^{(i)}\int_0^{+\infty}\nu_i(t)y_i(t)\mathrm{d}\eta_i(t)\\
&\qquad = \int_0^{+\infty}\Big(\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau)\Big)v_i(s)\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
&k_4^{(i)}\int_0^{+\infty}\mu_i(t)y_i(t)\mathrm{d}\xi_i(t) - k_3^{(i)}\int_0^{+\infty}\nu_i(t)y_i(t)\mathrm{d}\eta_i(t)\\
&\qquad = -\int_0^{+\infty}\Big(\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau)\Big)v_i(s)\mathrm{d}s.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^{+\infty}\mu_i(t)y_i(t)\mathrm{d}\xi_i(t) &= \frac{1}{k^{(i)}}\Big(k_3^{(i)}\int_0^{+\infty}\Big(\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau)\Big)\,v_i(s)\mathrm{d}s\\
&\quad + k_2^{(i)}\int_0^{+\infty}(\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau))\,v_i(s)\mathrm{d}s\Big)
\end{aligned}
$$

and

$$\int_0^{+\infty} \nu_i(t)y_i(t)\mathrm{d}\eta_i(t) = \frac{1}{k^{(i)}}\Big(k_1^{(i)}\int_0^{+\infty}\Big(\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau)\Big)\,v_i(s)\mathrm{d}s + k_4^{(i)}\int_0^{+\infty}\Big(\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau)\Big)\,v_i(s)\mathrm{d}s.$$

By substitution in (2.6), we finally arrive at the formula

$$y_i(t) = \int_0^{+\infty} H_i(t,s)\,v_i(s)\mathrm{d}s, \qquad i \in \{1,2,\dots,n\},$$

where H_i is given by (2.3). Therefore,

$$y(t) = \int_0^{\infty} H(t,s)v(s)\mathrm{d}s, \qquad t \in \mathbb{R}_+,$$

where $H(t,s) = \mathrm{diag}(H_1(t,s),\cdots,H_n(t,s))$. □

Some fundamental properties of the functions G and H are summarized in the following lemmas.

Lemma 2.3. *The function G satisfies*

(a) $G(t,s) \geq 0$ *for all* $t,s \in \mathbb{R}_+$; $\quad G(t,t) = \mathcal{G}(t) \leq \sigma^{-1}$, *for all* $t \in \mathbb{R}_+$.

(b) $\sigma\mathcal{G}(t)\mathcal{G}(s) \leq G(t,s) \leq \mathcal{G}(s)$, *for all* $t,s \in \mathbb{R}_+$.

(c) $\lim\limits_{t\to+\infty} G(t,s) = \dfrac{\delta}{\rho}\phi_1(s) < \infty$, *for all* $s \in \mathbb{R}_+$.

The proof is omitted. Also we have the following lemma.

Lemma 2.4. *The partial derivative* $\frac{\partial G}{\partial t}$ *satisfies*

(a) $\Big|p(t)\dfrac{\partial G}{\partial t}(t,s)\Big| \leq \dfrac{1}{\rho}\Big(\alpha\phi_2(s) + \gamma\phi_1(s)\Big)$, *for all* $t,s \in \mathbb{R}_+$.

(b) *If* $\beta,\delta \geq 1$, $\Big|p(t)\dfrac{\partial G}{\partial t}(t,s)\Big| \leq \max\{\alpha, \gamma(\beta+\alpha A)\}G(0,s)$, *for all* $t,s \in \mathbb{R}_+$.

Proof. Clearly

$$\frac{\partial G}{\partial t}(t,s) = \frac{1}{\rho}\begin{cases} \dfrac{\alpha}{p(t)}\,\phi_2(s), & 0 \leq t < s < +\infty,\\ -\dfrac{\gamma}{p(t)}\,\phi_1(s), & 0 \leq s < t < +\infty.\end{cases}$$

(a) For all positive t,s, we have:

1. if $0 \leq t < s < +\infty$, then $\Big|p(t)\dfrac{\partial G}{\partial t}(t,s)\Big| = \dfrac{\alpha}{\rho}\phi_2(s) \leq \dfrac{1}{\rho}\Big(\alpha\phi_2(s)+\gamma\phi_1(s)\Big)$,
2. if $0 \leq s < t < +\infty$, then $\Big|p(t)\dfrac{\partial G}{\partial t}(t,s)\Big| = \frac{\gamma}{\rho}\phi_1(s) \leq \dfrac{1}{\rho}\Big(\alpha\phi_2(s)+\gamma\phi_1(s)\Big)$.

(b) Note that for all positive s, we have

$$G(0,s) = \frac{1}{\rho}\phi_1(0)\phi_2(s).$$

Since $\beta,\delta \geq 1$, we consider two cases:

1. if $0 \le t < s < +\infty$, then
$$\left|p(t)\frac{\partial G}{\partial t}(t,s)\right| = \frac{\alpha}{\rho}\,\phi_2(s) \le \frac{\alpha\beta}{\rho}\,\phi_2(s) = \frac{\alpha}{\rho}\,\phi_1(0)\phi_2(s) = \alpha G(0,s),$$
2. if $0 \le s < t < +\infty$, then
$$\begin{aligned}\left|p(t)\frac{\partial G}{\partial t}(t,s)\right| &= \frac{\gamma}{\rho}\,\phi_1(s) \le \frac{\gamma\beta\delta}{\rho}\phi_1(s)\\ &\le \frac{\gamma}{\rho}\,\phi_1(s)\phi_1(0)\phi_2(s) \le \gamma(\beta+\alpha A)G(0,s).\end{aligned}$$

□

Let
$$\Lambda_1^{(i)} = \frac{k_4^{(i)}(\beta+\alpha A)+k_3^{(i)}(\delta+\gamma A)}{k^{(i)}\rho} \quad \text{and} \quad \Lambda_2^{(i)} = \frac{k_1^{(i)}(\beta+\alpha A)+k_2^{(i)}(\delta+\gamma A)}{k^{(i)}\rho}.$$
From Lemma 2.3, property (b) and Lemma 2.4, properties (a), (b), we derive the following properties of the function $H = \operatorname{diag}(H_1,\dots,H_n)$.

Lemma 2.5. *Assume that*
$$k_1^{(i)} > 0, \quad k_3^{(i)} > 0, \quad k^{(i)} > 0, \qquad \textit{for all } i \in \{1,\dots,n\}. \tag{2.7}$$
Then

(a) $\sigma\mathcal{G}(t)\mathcal{H}_i(s) \le H_i(t,s) \le \mathcal{H}_i(s) \le \mathcal{A}_i$ *for all, $t,s\in\mathbb{R}_+$, where*
$$\mathcal{H}_i(s) = \mathcal{G}(s) + \Lambda_1^{(i)}\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau) + \Lambda_2^{(i)}\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau),$$
$$\mathcal{A}_i = \Big(1+\Lambda_1^{(i)}\int_0^{+\infty}\mu_i(\tau)\mathrm{d}\xi_i(\tau) + \Lambda_2^{(i)}\int_0^{+\infty}\nu_i(\tau)\mathrm{d}\eta_i(\tau)\Big)\sigma^{-1}.$$

(b) $\left|p(t)\frac{\partial H_i}{\partial t}(t,s)\right| \le \mathcal{K}_i(s) \le \mathcal{B}_i,$ *for all $t,s\in\mathbb{R}_+$, where*
$$\begin{aligned}\mathcal{K}_i(s) = {}& \frac{1}{\rho}\Big(\alpha\phi_2(s)+\gamma\phi_1(s)\Big) + \frac{|\alpha k_4^{(i)}-\gamma k_3^{(i)}|}{k^{(i)}\rho}\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau)\\ &+\frac{|\alpha k_1^{(i)}-\gamma k_2^{(i)}|}{k^{(i)}\rho}\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau),\\ \mathcal{B}_i = {}& \frac{1}{\rho}\Big(\alpha(\delta+\gamma A)+\gamma(\beta+\alpha A)\Big)\\ &+\Big(\frac{|\alpha k_4^{(i)}-\gamma k_3^{(i)}|}{k^{(i)}\rho}\int_0^{+\infty}\mu_i(\tau)\mathrm{d}\xi_i(\tau)+\frac{|\alpha k_1^{(i)}-\gamma k_2^{(i)}|}{k^{(i)}\rho}\int_0^{+\infty}\nu_i(\tau)\mathrm{d}\eta_i(\tau)\Big)\sigma^{-1}.\end{aligned}$$

(c) *If $\beta,\delta\ge 1$, then* $\left|p(t)\frac{\partial H_i}{\partial t}(t,s)\right| \le \max(\alpha,\gamma(\beta+\alpha A))H_i(0,s)$, *for all $t,s\in\mathbb{R}_+$.*

Proof. (a) By Lemma 2.3, property (b) and the monotonicity of ϕ_1, ϕ_2, we have $H_i(t,s) \le \mathcal{H}_i(s)$. Moreover, (2.3) and again the monotonicity of ϕ_1, ϕ_2 guarantee that for all positive t,

$$\phi_1(t) = \frac{\rho\mathcal{G}(t)}{\phi_2(t)} = \frac{\rho\mathcal{G}(t)}{\delta + \gamma\int_t^{+\infty}\frac{1}{p(s)}\mathrm{d}s} \ge \frac{\rho\mathcal{G}(t)}{\delta+\gamma A}, \tag{2.8}$$

$$\phi_2(t) = \frac{\rho\mathcal{G}(t)}{\phi_1(t)} = \frac{\rho\mathcal{G}(t)}{\beta + \alpha\int_0^{t}\frac{1}{p(s)}\mathrm{d}s} \ge \frac{\rho\mathcal{G}(t)}{\beta+\alpha A}. \tag{2.9}$$

By (2.3), (2.8) and (2.9) for all $t, s \in \mathbb{R}_+$, we have

$$\begin{aligned} H_i(t,s) &\ge \sigma\mathcal{G}(t)\mathcal{G}(s) \\ &\quad + \Big(\Lambda_1^{(i)}\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau) + \Lambda_2^{(i)}\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau)\Big)\sigma\mathcal{G}(t) \\ &= \sigma\mathcal{G}(t)\mathcal{H}_i(s). \end{aligned}$$

(b) This property immediately follows from Lemma 2.4, property (a).

(c) Assume that $\beta, \delta \ge 1$. By Lemma 2.4, property (b), for all $t, s \in \mathbb{R}_+$, we have the estimates

$$\begin{aligned} \Big|p(t)\frac{\partial H_i}{\partial t}(t,s)\Big| &\le \max\{\alpha, \gamma(\beta+\alpha A)\}G(0,s) + \frac{\alpha k_4^{(i)} + \gamma k_3^{(i)}}{k^{(i)}\rho}\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau) \\ &\quad + \frac{\alpha k_1^{(i)} + \gamma k_2^{(i)}}{k^{(i)}\rho}\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau) \\ &\le \max\{\alpha, \gamma(\beta+\alpha A)\}G(0,s) + \frac{\alpha\beta k_4^{(i)} + \gamma\delta k_3^{(i)}}{k^{(i)}\rho}\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau) \\ &\quad + \frac{\alpha\beta k_1^{(i)} + \gamma\delta k_2^{(i)}}{k^{(i)}\rho}\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau) \\ &\le \max\{\alpha, \gamma(\beta+\alpha A)\}G(0,s) \\ &\quad + \max\{\alpha,\gamma\}\frac{k_4^{(i)}\phi_1(0) + k_3^{(i)}\phi_2(0)}{k^{(i)}\rho}\int_0^{+\infty}\mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau) \\ &\quad + \max\{\alpha,\gamma\}\frac{k_1^{(i)}\phi_1(0) + k_2^{(i)}\phi_2(0)}{k^{(i)}\rho}\int_0^{+\infty}\nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau) \\ &= \max\{\alpha, \gamma(\beta+\alpha A)\}H_i(0,s). \end{aligned}$$

□

3. Integral formulation

3.1. A fixed point operator

Let H_i be given by (2.3). For $i \in \{1,\dots,n\}$, set

$$\omega_i(t) = \int_0^{+\infty} H_i(t,s)h_i(s)\mathrm{d}s. \tag{3.1}$$

Remark 3.1. Lemma 2.2 and Lemma 2.5 guarantee that ω_i is well defined and $\omega_i > 0$ for all $i \in \{1, \dots, n\}$. Moreover, $\omega = (\omega_1, \dots, \omega_n)$ is the unique solution of (2.1) when $v \equiv h$.

Consider the auxiliary boundary value problem

$$\left\{ \begin{aligned} -(p(t)y'(t))' &= g(t, y(t), y'(t)) - h(t), \quad \text{a.e. } t \in I, \\ \alpha y(0) - \beta p(0)\, y'(0) &= \textstyle\int_0^{+\infty} \mu(s) y(s) \mathrm{d}\xi(s), \\ \gamma \lim_{t \to +\infty} y(t) + \delta \lim_{t \to +\infty} p(t) y'(t) &= \textstyle\int_0^{+\infty} \nu(s) y(s) \mathrm{d}\eta(s), \end{aligned} \right. \tag{3.2}$$

where the modified function g is defined in $[0, +\infty) \times \mathbb{R}^n_+ \times \mathbb{R}^n$ by

$$g(t, y, z) = \begin{cases} f(t, y, z), & y \geq \omega(t), \\ f(t, \omega(t), z), & \text{otherwise}. \end{cases} \tag{3.3}$$

Let $\Omega \subset \mathbb{X}$ be a bounded subset. Then, there exists $r_0 > 0$ such that $\|y\| \leq r_0$ for all $y = (y_1, \dots, y_n) \in \Omega$, that is, for all $t \geq 0$ and $i \in \{1, \dots, n\}$, $|y_i(t)| \leq \|y\| \leq r_0 \leq r_1 = \max\{r_0, \|\omega\|\}$ and $|y_i'(t)| \leq |p(t) y_i'(t)| \leq \|y\| \leq r_1$. Since f_i are L^1-Carathéodory functions for $i \in \{1, \dots, n\}$, there exist functions $\varphi_{r_1} \in L^1[0, +\infty)$ such that

$$\begin{aligned} &\int_0^{+\infty} \Big[g_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) - h_i(s) \Big] \mathrm{d}s \\ &\leq \int_0^{+\infty} \Big[\varphi_{r_1}(s) - h_i(s) \Big] \mathrm{d}s < \infty, \end{aligned}$$

where $g = (g_1, \dots, g_n)$. Lemma 2.2 implies that the boundary value problem (3.2) is equivalent to the nonlinear integral equation

$$y(t) = \int_0^{+\infty} H(t, s) \Big[g\Big(s, y(s), y'(s)\Big) - h(s) \Big] \mathrm{d}s.$$

Thus, for $i \in \{1, \dots, n\}$, we can define the integral operators $F_i \colon \overline{\Omega} \cap \mathcal{P} \to C^1(\mathbb{R}_+, \mathbb{R})$ by

$$(F_i y)(t) = \int_0^{+\infty} H_i(t, s) \Big[g_i\Big(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)\Big) - h_i(s) \Big] \mathrm{d}s.$$

Let $(Fy)(t) = \Big(F_1 y(t), \dots, F_n y(t)\Big)^T$, i.e.,

$$\begin{aligned} &F \colon \overline{\Omega} \cap \mathcal{P} \to C^1(\mathbb{R}_+, \mathbb{R}^n), \\ &\qquad y \mapsto (Fy)(t) = \int_0^{+\infty} H(t, s) \Big[g\Big(s, y(s), y'(s)\Big) - h(s) \Big] \mathrm{d}s. \end{aligned} \tag{3.4}$$

Remark 3.2. By Lemma 2.2, if y is a fixed point of F in $\mathbb{X}$, then y is a solution of problem (3.2). If further $y \geq \omega$, then y is a solution of problem (1.1).

3.2. Compactness of the fixed point operator

A mapping is completely continuous if it is continuous and maps bounded sets into relatively compact sets. A set of functions $Y \in \Omega \subset \mathbb{X}$ is almost equi-continuous if it is equi-continuous on each interval $[0,T],\ 0 \leq T < +\infty$. The following result is an extension of Arzéla-Ascoli compactness criterion to unbounded intervals (see [**4**]).

Lemma 3.1. *Let $M \subseteq C_l(\mathbb{R}^+,\mathbb{R}^n)$. Then the set M is relatively compact in $C_l(\mathbb{R}^+,\mathbb{R}^n)$ if the following conditions hold:*
(a) *M is uniformly bounded in $C_l(\mathbb{R}^+,\mathbb{R}^n)$.*
(b) *The functions belonging to M are almost equi-continuous on $\mathbb{R}^+$.*
(c) *The functions from M are equi-convergent at $+\infty$, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $\sum_{i=1}^n |x_i(t) - x_i(\infty)| < \varepsilon$ for all $t \geq T(\varepsilon)$ and $x \in M$.*

As a consequence, we easily derive the following compactness criterion

Lemma 3.2. *A subset $\mathcal{M} \subseteq \mathbb{X}$ is relatively compact in $\mathbb{X}$ if the following conditions hold:*
(a) *$\mathcal{M}$ is uniformly bounded in $\mathbb{X}$.*
(b) *The functions belonging to the sets*

$$\mathcal{M} \text{ and } \mathcal{B} = \{z \mid z(t) = p(t)x'(t),\ x \in \mathcal{M}\}$$

are almost equi-continuous on $\mathbb{R}^+$.
(c) *The functions from $\mathcal{M}$ and $\mathcal{B}$ are equi-convergent at $+\infty$.*

Now, we prove the following lemma.

Lemma 3.3. *Assume that Assumption $(\mathcal{C}_0)$ is satisfied and*

$$f(t,y,z) \geq h(t) \qquad \text{for } t \in \mathbb{R}_+,\ y \geq \omega(t),\ \text{and}\ \ z \in \mathbb{R}^n. \tag{3.5}$$

Then, for any bounded subset $\Omega \subset \mathbb{X}$, the mapping F given by (3.4) sends $\overline{\Omega} \cap \mathcal{P}$ into $\mathcal{P}$ and is completely continuous.

Proof.
Claim 1. $F\colon \overline{\Omega} \cap \mathcal{P} \to \mathbb{X}$ is well defined. First there exists a positive constant r_2 such that $\max\{\|y\|, \|\omega\|\} \leq r_2$ for all $y \in \overline{\Omega} \cap \mathcal{P}$. Since f_i, $(i \in \{1,\dots,n\})$, are L^1-Carathéodory functions, then by Lemma 2.5, we get the estimates for $t \geq 0$,

$$|(F_i y)(t)| \leq \int_0^\infty \mathcal{H}_i(s)|\varphi_{r_2}(s) - h_i(s)|\mathrm{d}s \leq \int_0^{+\infty} \mathcal{A}_i|\varphi_{r_2}(s) - h_i(s)|\mathrm{d}s < \infty$$

and

$$|p(t)(F_i y)'(t)| \leq \int_0^\infty \mathcal{K}_i(s)|\varphi_{r_2}(s) - h_i(s)|\mathrm{d}s \leq \int_0^{+\infty} \mathcal{B}_i|\varphi_{r_2}(s) - h_i(s)|\mathrm{d}s < \infty.$$

Therefore, $Fy \in C^1([0,+\infty),\mathbb{R}^n)$ and $F(\overline{\Omega} \cap \mathcal{P}) \subset \mathbb{X}$.

Claim 2. $F\colon \overline{\Omega}\cap\mathcal{P}\to\mathcal{P}$. Let $y\in\overline{\Omega}\cap\mathcal{P}$. By Lemma 2.5, for $t\geq 0$,

$$\begin{aligned}(F_iy)(t) &= \int_0^{+\infty} H_i(t,s)\Big[g_i\Big(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s)\Big)-h_i(s)\Big]\mathrm{d}s\\ &\geq \int_0^{+\infty}\sigma\mathcal{G}(t)\mathcal{H}_i(s)\Big[g_i\Big(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s)\Big)-h_i(s)\Big]\mathrm{d}s\\ &\geq \int_0^{+\infty}\sigma\mathcal{G}(t)H_i(s,\tau)\Big[g_i\Big(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s)\Big)-h_i(s)\Big]\mathrm{d}s\\ &= \sigma\mathcal{G}(t)(F_iy)(\tau)\end{aligned}$$

for all $\tau\geq 0$. Taking the least upper bound over $\tau\in\mathbb{R}^+$ yields

$$(F_iy)(t)\geq\sigma\mathcal{G}(t)\sup_{\tau\in\mathbb{R}^+}|(F_iy)(\tau)|,\qquad\text{for all } t\geq 0.$$

Also

$$\begin{aligned}(F_iy)(0) &= \int_0^{+\infty} H_i(0,s)\Big[g_i\Big(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s)\Big)-h_i(s)\Big]\mathrm{d}s\\ &\geq \int_0^{+\infty}\Lambda|p(t)\frac{\partial}{\partial t}H_i(t,s)|\Big[g_i\Big(s,y_1(s),\dots,y_n(s)y_1'(s),\dots,y_n'(s)\Big)-h_i(s)\Big]\mathrm{d}s\\ &= \Lambda|p(t)(F_iy)'(t)\end{aligned}$$

for all $t\geq 0$. Since $t\geq 0$ is arbitrary, passing again to the supremum over t, we get

$$(F_iy)(0)\geq\Lambda\sup_{t\in\mathbb{R}^+}|p(t)(F_iy)'(t)|\cdot$$

Claim 3. A is continuous on $\overline{\Omega}\cap\mathcal{P}$. Let a sequence $y_m=\big(y_{1,m},\dots y_{n,m}\big)$ converge to a limit $y=\big(y_1,\dots,y_n\big)$ in $\overline{\Omega}\cap\mathcal{P}$ as $m\to+\infty$. Then there exists $r>0$ independent of n such that $\max\{\|\omega\|,\|y\|,\sup_{m\geq 1}\|y_m\|\}\leq r$. Since, for $i\in\{1,\dots,n\}$, f_i are L^1-Carathéodory functions, then by Lemma 2.5, for $t\geq 0$,

$$|(F_iy_m)(t)|\leq\int_0^{\infty}\mathcal{H}_i(s)|\varphi_r(s)-h_i(s)|\mathrm{d}s\leq\int_0^{+\infty}\mathcal{A}_i|\varphi_r(s)-h_i(s)|\mathrm{d}s<\infty$$

and

$$|p(t)(F_iy_m)'(t)|\leq\int_0^{\infty}\mathcal{K}_i(s)|\varphi_r(s)-h_i(s)|\mathrm{d}s\leq\int_0^{+\infty}\mathcal{B}_i|\varphi_r(s)-h_i(s)|\mathrm{d}s<\infty.$$

In addition, the functions f_i ($i\in\{1,\dots,n\}$) are continuous in the second and third arguments, thus

$$\lim_{m\to+\infty}|g(s,y_m(s),y_m'(s))-g(s,y(s),y'(s))|=0.$$

Hence for each $i \in \{1, \dots, n\}$, the Lebesgue dominated convergence theorem guarantees that

$$\begin{aligned}\|F_i y_m - F_i y\|_1 &= \sup_{t\in\mathbb{R}_+} \Big|(F_i y_m)(t) - (F_i y)(t)\Big| \\ &\le \sup_{t\in\mathbb{R}_+} \int_0^{+\infty} \mathcal{A}_i |g_i(s, y_{1,m}(s), \dots, y_{n,m}(s), y'_{1,m}(s), \dots, y'_{n,m}(s)) \\ &\quad - g_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s))| \mathrm{d}s \to 0, \quad \text{as } m \to +\infty\end{aligned}$$

and

$$\begin{aligned}\|F_i y_m - F_i y\|_2 &= \sup_{t\in\mathbb{R}_+} \Big|p(t)((F_i y_m)'(t) - (F_i y)'(t))\Big| \\ &\le \sup_{t\in\mathbb{R}_+} \int_0^{+\infty} \mathcal{B}_i |g_i(s, y_{1,m}(s), \dots, y_{n,m}(s), y'_{1,m}(s), \dots, y'_{n,m}(s)) \\ &\quad - g_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s))| \mathrm{d}s \to 0, \quad \text{as } m \to +\infty.\end{aligned}$$

As a result

$$\|Fy_m - Fy\| = \sum_{i=1}^n \max\{\|F_i y_m - F_i y\|_1, \|F_i y_m - F_i y\|_2\} \to 0, \qquad \text{as } \ m \to +\infty.$$

Claim 4. F maps bounded sets into relatively compact sets. Let Ω be a bounded subset of $\mathbb{X}$; then there exists $r > 0$ such that $\max\{\|y\|, \|\omega\|\} \le r$, for all $y \in \overline{\Omega} \cap \mathcal{P}$. It is easy to prove that $F(\overline{\Omega} \cap \mathcal{P})$ is uniformly bounded.

(a) The family of functions $\{Fy \mid \ y \in \overline{\Omega} \cap \mathcal{P}\}$ are almost equi-continuous on $\mathbb{R}_+$. Indeed, for any $y \in \overline{\Omega} \cap \mathcal{P}$, $T > 0$, $t_1, t_2 \in [0, T]$ $(t_1 < t_2)$, and for $i \in \{1, \dots, n\}$, we have the estimates

$$\begin{aligned}\Big|(F_i y)(t_1) - (F_i y)(t_2)\Big| &\le \int_0^{+\infty} \Big|H_i(t_1, s) - H_i(t_2, s)\Big| \\ &\quad \times \Big|g_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s)) - h_i(s)\Big| \mathrm{d}s \\ &= \int_0^{t_1} \Big|H_i(t_1, s) - H_i(t_2, s)\Big| \\ &\quad \times \Big|g_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s)) - h_i(s)\Big| \mathrm{d}s \\ &\quad + \int_{t_1}^{t_2} \Big|H_i(t_1, s) - H_i(t_2, s)\Big| \\ &\quad \times \Big|g_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s)) - h_i(s)\Big| \mathrm{d}s \\ &\quad + \int_{t_2}^{+\infty} \Big|H_i(t_1, s) - H_i(t_2, s)\Big| \\ &\quad \times \Big|g_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s)) - h_i(s)\Big| \mathrm{d}s.\end{aligned}$$

Each of the terms in the right-hand side is estimated as follows:

$$\begin{aligned}
&\int_0^{t_1} \Big|H_i(t_1,s)-H_i(t_2,s)\Big| \Big|g_i(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s))-h_i(s)\Big|\mathrm{d}s\\
&\le \frac{1}{\rho}|\phi_2(t_1)-\phi_2(t_2)|\int_0^{t_1}\phi_1(s)|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\qquad+\frac{1}{k^{(i)}\rho^2}(\beta+\alpha A)(\delta+\gamma A)\int_0^{+\infty}\mu_i(\tau)\mathrm{d}\xi_i(\tau)\\
&\qquad\times\Big[k_3^{(i)}|\phi_2(t_1)-\phi_2(t_2)|+k_4^{(i)}|\phi_1(t_1)-\phi_1(t_2)|\Big]\int_0^{t_1}|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\qquad+\frac{1}{k^{(i)}\rho^2}(\beta+\alpha A)(\delta+\gamma A)\int_0^{+\infty}\nu_i(\tau)\mathrm{d}\eta_i(\tau)\\
&\qquad\times\Big[k_1^{(i)}|\phi_1(t_1)-\phi_1(t_2)|+k_2^{(i)}|\phi_2(t_1)-\phi_2(t_2)|\Big]\int_0^{t_1}|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\to 0,\quad \text{as } |t_1-t_2|\to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big|H_i(t_1,s)-H_i(t_2,s)\Big| \times g_i(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s))-h_i(s)\mathrm{d}s\\
&\le \frac{1}{\rho}\phi_1(t_1)\int_{t_1}^{t_2}\phi_2(s)|\varphi_r(s)-h_i(s)|\mathrm{d}s+\frac{1}{\rho}\phi_2(t_2)\int_{t_1}^{t_2}\phi_1(s)|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\qquad+\frac{1}{k^{(i)}\rho^2}(\beta+\alpha A)(\delta+\gamma A)\int_0^{+\infty}\mu_i(\tau)\mathrm{d}\xi_i(\tau)\\
&\qquad\times\Big[k_3^{(i)}|\phi_2(t_1)-\phi_2(t_2)|+k_4^{(i)}|\phi_1(t_1)-\phi_1(t_2)|\Big]\int_{t_1}^{t_2}|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\qquad+\frac{1}{k^{(i)}\rho^2}(\beta+\alpha A)(\delta+\gamma A)\int_0^{+\infty}\nu_i(\tau)\mathrm{d}\eta_i(\tau)\\
&\qquad\times\Big[k_1^{(i)}|\phi_1(t_1)-\phi_1(t_2)|+k_2^{(i)}|\phi_2(t_1)-\phi_2(t_2)|\Big]\int_{t_1}^{t_2}|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\to 0,\quad \text{as } |t_1-t_2|\to 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_{t_2}^{+\infty} \Big|H_i(t_1,s)-H_i(t_2,s)\Big|g_i(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s))\mathrm{d}s\\
&\le \frac{1}{\rho}|\phi_1(t_1)-\phi_1(t_2)|\int_{t_2}^{+\infty}\phi_2(s)|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\qquad+\frac{1}{k^{(i)}\rho^2}(\beta+\alpha A)(\delta+\gamma A)\int_0^{+\infty}\mu_i(\tau)\mathrm{d}\xi_i(\tau)\\
&\qquad\times\Big[k_3^{(i)}|\phi_2(t_1)-\phi_2(t_2)|+k_4^{(i)}|\phi_1(t_1)-\phi_1(t_2)|\Big]\int_{t_2}^{+\infty}|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\qquad+\frac{1}{k^{(i)}\rho^2}(\beta+\alpha A)(\delta+\gamma A)\int_0^{+\infty}\nu_i(\tau)\mathrm{d}\eta_i(\tau)\\
&\qquad\times\Big[k_1^{(i)}|\phi_1(t_1)-\phi_1(t_2)|+k_2^{(i)}|\phi_2(t_1)-\phi_2(t_2)|\Big]\int_{t_2}^{+\infty}|\varphi_r(s)-h_i(s)|\mathrm{d}s\\
&\to 0,\quad \text{as } |t_1-t_2|\to 0.
\end{aligned}$$

Then

$$\|(Fy)(t_1)-(Fy)(t_2)\| = \sum_{i=1}^{n} \Big|(F_iy)(t_1)-(F_iy)(t_2)\Big| \to 0, \quad \text{as } |t_1-t_2| \to 0,$$

proving equi-continuity of the family $\{z|\, z(t) = (Fy)(t),\ y \in \overline{\Omega} \cap \mathcal{P}\}$. Similarly, we obtain that for any $i \in \{1,\dots,n\}$ and for all $y \in \overline{\Omega} \cap \mathcal{P}$, the difference $\Big|p(t_1)(F_iy)'(t_1) - p(t_2)(F_iy)'(t_2)\Big|$ tends to 0 as $|t_1-t_2| \to 0$. Then

$$\|p(t_1)(Fy)'(t_1)-p(t_2)(Fy)'(t_2)\| = \sum_{i=1}^{n} \Big|p(t_1)(F_iy)'(t_1)-p(t_2)(F_iy)'(t_2)\Big|$$

tends to 0 as $|t_1-t_2| \to 0$. This shows that the functions $\{z|\, z(t) = p(t)(Fy)'(t),\ y \in \overline{\Omega} \cap \mathcal{P}\}$ are almost equi-continuous on $\mathbb{R}_+$. Consequently, the family $\{Fy|\ y \in \overline{\Omega} \cap \mathcal{P}\}$ is equi-continuous.

(b) The functions $\{Fy \mid\ y \in \overline{\Omega} \cap \mathcal{P}\}$ are equi-convergent at $+\infty$. Indeed, since f_i are L^1-Carathéodory functions, then for $y \in \overline{\Omega} \cap \mathcal{P}$, $t \geq 0$, and $i \in \{1,\dots,n\}$, we have

$$|(F_iy)(t)| \leq \int_0^{+\infty} \mathcal{H}_i(s)|\varphi_r(s)-h_i(s)|\mathrm{d}s \leq \int_0^{+\infty} \mathcal{A}_i|\varphi_r(s)-h_i(s)|\mathrm{d}s < \infty$$

and

$$|p(t)(F_iy)'(t)| \leq \int_0^{+\infty} \mathcal{K}_i(s)|\varphi_r(s)-h_i(s)|\mathrm{d}s \leq \int_0^{+\infty} \mathcal{B}_i|\varphi_r(s)-h_i(s)|\mathrm{d}s < \infty.$$

Hence for all $i \in \{1,\dots,n\}$,

$$\begin{aligned}
\lim_{t\to+\infty} F_iy(t) &= \lim_{t\to+\infty} \int_0^{+\infty} H_i(t,s)\Big[g_i\Big(s, y_1(s),\dots,y_n(s), y_1'(s),\dots,y_n'(s)\Big) - h_i(s)\Big]\mathrm{d}s \\
&= \int_0^{+\infty} \Big[\frac{\delta}{\rho}\phi_1(s) + \frac{k_4^{(i)}(\beta+\alpha A)+k_3^{(i)}\delta}{k^{(i)}\rho}\int_0^{+\infty} \mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau) \\
&\quad + \frac{k_1^{(i)}(\beta+\alpha A)+k_2^{(i)}\delta}{k^{(i)}\rho}\int_0^{+\infty} \nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau)\Big] \\
&\quad \times \Big[g_i\Big(s, y_1(s),\dots,y_n(s), y_1'(s),\dots,y_n'(s)\Big) - h_i(s)\Big]\mathrm{d}s < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to+\infty} p(t)(F_iy)'(t) \\
&= \lim_{t\to+\infty} \int_0^{+\infty} p(t)\frac{\partial}{\partial t}H_i(t,s)\Big[g_i\Big(s, y_1(s),\dots,y_n(s), y_1'(s),\dots,y_n'(s)\Big) - h_i(s)\Big]\mathrm{d}s \\
&= \int_0^{+\infty} \Big[-\frac{\gamma}{\rho}\phi_1(s) + \frac{k_4^{(i)}\alpha - k_3^{(i)}\gamma}{k^{(i)}\rho}\int_0^{+\infty} \mu_i(\tau)G(\tau,s)\mathrm{d}\xi_i(\tau) \\
&\quad + \frac{k_1^{(i)}\alpha - k_2^{(i)}\gamma}{k^{(i)}\rho}\int_0^{+\infty} \nu_i(\tau)G(\tau,s)\mathrm{d}\eta_i(\tau)\Big] \\
&\quad \times \Big[g_i\Big(s, y_1(s),\dots,y_n(s), y_1'(s),\dots,y_n'(s)\Big) - h_i(s)\Big]\mathrm{d}s < \infty.
\end{aligned}$$

Therefore, the integrals involved in the definitions of $F_iy(t)$, $p(t)(F_iy)'(t)$, on the one hand, and in the limits $\lim\limits_{t\to+\infty} F_iy(t)$ and $\lim\limits_{t\to+\infty} p(t)(F_iy)'(t)$, on the other one, are convergent uniformly for $y \in \overline{\Omega} \cap \mathcal{P}$. As a consequence, we have

$$\begin{aligned}
&\lim_{t\to+\infty} \Big(F_iy(t) - \lim_{x\to+\infty} F_iy(x)\Big)\\
&= \lim_{t\to+\infty} \int_0^{+\infty} \Big(H_i(t,s) - \lim_{t\to+\infty} H_i(t,s)\Big)\\
&\quad \times \Big[g_i\Big(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)\Big) - h_i(s)\Big]\mathrm{d}s\\
&= \int_0^{+\infty} \lim_{t\to+\infty} \Big(H_i(t,s) - \lim_{t\to+\infty} H_i(t,s)\Big)\\
&\quad \times \Big[g_i\Big(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)\Big) - h_i(s)\Big]\mathrm{d}s = 0.
\end{aligned}$$

Similarly, we can check that

$$\lim_{t\to+\infty} \Big(p(t)(F_iy)'(t) - \lim_{x\to+\infty} p(x)(F_iy)'(x)\Big) = 0.$$

Hence

$$\lim_{t\to+\infty} \|Fy(t) - \lim_{x\to+\infty} Fy(x)\| = \sum_{i=1}^{n} \lim_{t\to+\infty} |F_iy(t) - \lim_{x\to+\infty} F_iy(x)| = 0$$

and

$$\lim_{t\to+\infty} \|p(t)(Fy)'(t) - \lim_{x\to+\infty} p(x)(Fy)'(x)\| = 0,$$

uniformly in $y \in \overline{\Omega} \cap \mathcal{P}$. This means that the family $\{y \mid y \in F(\overline{\Omega} \cap \mathcal{P})\}$ is equi-convergent at $+\infty$. Using Lemma 3.2, we conclude that the range $F(\overline{\Omega} \cap \mathcal{P})$ is relatively compact, ending the proof of the lemma. □

4. Main Results

4.1. General Assumptions

This section deals with problem (3.2). The notations $y_1, \dots, y_n, z_1, \dots, z_n, \omega_1, \dots, \dots, \omega_n$ stand for $y_1(t), \dots, y_n(t), z_1(t), \dots, z_n(t), \omega_1(t), \dots, \omega_n(t)$, respectively. For each $i \in \{1, \dots, n\}$, we set the main assumptions:

($\mathcal{C}_1$) $f_i\colon \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}^n \to \mathbb{R}_+$ is an L^1-Carathéodory function, nondecreasing with respect to the second argument, and there exist continuous functions $a_i, b_i \in C(\mathbb{R}_+^n, \mathbb{R}_+)$, c_i, $d_i \in C(\mathbb{R}^n, \mathbb{R}_+)$ and $q_i \in C(I, \mathbb{R}_+)$ such that for all $t \in \mathbb{R}_+$, $y_i \geq \omega_i$, and $z_i \in \mathbb{R}$,

$$\begin{aligned}
f_i(t, y_1, \dots, y_n, z_1, \dots, z_n) \leq{}& q_i(t)\Big[a_i\Big(y_1, \dots, y_n\Big) + b_i\Big(y_1, \dots, y_n\Big)\Big]\\
&\times\Big[c_i\Big(p(t)z_1, \dots, p(t)z_n\Big) + d_i\Big(p(t)z_1, \dots, p(t)z_n\Big)\Big],
\end{aligned}$$

where c_i are nonincreasing, $\frac{b_i}{a_i}$, $\frac{d_i}{c_i}$ are nondecreasing functions, and the functions $\mathcal{H}_i q_i$ are Lebesgue integrable functions in $\mathbb{R}_+$.

($\mathcal{C}_2$) There exists $R > 0$ such that for $y_1, \dots, y_n$, satisfying $\sum\limits_{i=1}^{n} \omega_i \leq \sum\limits_{i=1}^{n} y_i \leq R$, we have

$$\frac{a_i(y_1, \dots, y_n)}{\sum\limits_{i=1}^{n} y_i} \leq M_i \tag{4.1}$$

and

$$R\mathbb{G}_i + \mathbb{H}_i \leq R\Lambda/n, \tag{4.2}$$

where

$$\mathbb{G}_i = \int_0^{+\infty} \mathcal{H}_i(s) q_i(s) \mathrm{d}s, \qquad \mathbb{H}_i = \int_0^{+\infty} \mathcal{H}_i(s) h_i(s) \mathrm{d}s,$$

and

$$M_i^{-1} = \Big(1 + \frac{b_i(R, \dots, R)}{a_i(R, \dots, R)}\Big)\Big(1 + \frac{d_i(R, \dots, R)}{c_i(R, \dots, R)}\Big) c_i\Big(-R, \dots, -R\Big).$$

($\mathcal{C}_2'$) There exists $R > 0$ such that

$$\Big(1 + \frac{b_i(R, \dots, R)}{a_i(R, \dots, R)}\Big)\Big(1 + \frac{d_i(R, \dots, R)}{c_i(R, \dots, R)}\Big) c_i\Big(-R, \dots, -R\Big)\mathbb{K}_i + \mathbb{H}_i \leq R\frac{\Lambda}{n}, \tag{4.3}$$

where

$$\mathbb{K}_i = \int_0^{+\infty} \mathcal{H}_i(s) q_i(s) a_i\Big(\omega_1(s), \dots, \omega_n(s)\Big) \mathrm{d}s.$$

($\mathcal{C}_3$) There exist positive constants r, μ, η with $\frac{\rho}{\sigma\Lambda\beta\delta} < r < \min\Big\{\frac{R}{n}, \ \frac{\sigma\Lambda\beta\delta}{\rho}\frac{R}{n}\Big\}$ and $0 < \mu < \eta$ such that for all $t \in [\mu, \eta]$, $\frac{\sigma\Lambda\beta\delta}{\rho} r \leq \sum\limits_{i=1}^{n} y_i \leq \frac{R}{n}$ and $z_i \in \mathbb{R}$, we have

$$f_i(t, y_1, \dots, y_n, z_1, \dots, z_n) \geq h(t) + \mathcal{L}_i, \tag{4.4}$$

where the constant $\mathcal{L}_i$ satisfies $\sum\limits_{i=1}^{n} \mathcal{L}_i \int_\mu^\eta \mathcal{H}_i(s) \mathrm{d}s \geq \frac{\rho}{\sigma\beta\delta} r$.

Remark 4.1.

(a) The constant r in ($\mathcal{C}_3$) exists if, e.g.,

$$R > n \max\Big\{\frac{\rho}{\sigma\beta\delta\Lambda}, \ \Big(\frac{\rho}{\sigma\beta\delta\Lambda}\Big)^2\Big\}.$$

(b) Monotonic functions defined on $\mathbb{R}^n$ are understood in the following sense

Definition 4.1. Let $\mathcal{P}$ be a cone in a real Banach space X and $\preceq$ be the partial ordering defined by $\mathcal{P}$. Let D be a subset of X and $F\colon D \to X$ be a mapping. Then the operator F is said to be nondecreasing on D (resp., nonincreasing on D) provided $x_1, x_2 \in D$ with $x_1 \preceq x_2$ that implies $Fx_1 \preceq Fx_2$ (resp., $Fx_1 \succeq Fx_2$).

We are now ready to state and prove our first existence result.

Theorem 4.1. *Assume that Assumptions* $(\mathcal{C}_0)$–$(\mathcal{C})_2$ *and* $(\mathcal{C}_3)$ *hold together with* (3.5) *and*

$$f\Big(t,\omega(t),z\Big) \geq 2h(t) \qquad \text{for all } \ t\in\mathbb{R}_+, \quad z\in[-R,R]^n, \tag{4.5}$$

where R *is given as in* $(\mathcal{C}_2)$.

Then problem (1.1) *has at least one positive solution* $y\in\mathcal{P}$ *satisfying*

$$0<\|y\|\leq R, \quad \sum_{i=1}^n y_i(t)\geq \frac{\sigma\Lambda\beta\delta}{\rho}r, \quad \text{and} \quad y(t)\geq\omega(t)>0_{\mathbb{R}^n} \quad \text{for all } t\in\mathbb{R}_+.$$

Let ψ be a nonnegative continuous concave functional on $\mathcal{P}$. For positive numbers τ_1,τ_2, define the following subsets

$$\mathcal{P}_\tau=\mathcal{P}\cap\overline{B}(0,\tau)=\{x\in\mathcal{P}\mid \ \|x\|\leq\tau\},$$
$$\mathcal{P}(\psi,\tau_1,\tau_2)=\{x\in\mathcal{P}\mid \ \psi(x)\geq\tau_1 \text{ and } \|x\|\leq\tau_2\}.$$

The following fixed point theorem is needed in the proof of our main results.

Lemma 4.1. [**15**, Theorem 3.2] *Let* $F\colon\mathcal{P}_\lambda\to\mathcal{P}$ *be a completely continuous operator. Assume that there exists a concave functional* ψ *satisfying* $\psi(x)\leq\|x\|$ *for all* $x\in\mathcal{P}$ *as well as real numbers* $\lambda\geq\nu_1>\nu_2>0$ *such that*
(1) $\{x\in\mathcal{P}(\psi,\nu_2,\nu_1)\mid \ \psi(x)>\nu_2\}\neq\emptyset$ *and* $\psi(Fx)>\nu_2$ *if* $x\in\mathcal{P}(\psi,\nu_2,\nu_1)$.
(2) $Fx\in\mathcal{P}_\lambda$ *if* $x\in\mathcal{P}(\psi,\nu_2,\lambda)$.
(3) $\psi(Fx)>\nu_2$ *for any* $x\in\mathcal{P}(\psi,\nu_2,\lambda)$ *with* $\|Fx\|>\nu_1$.
Then F *has at least one positive fixed point in* $\mathcal{P}(\psi,\nu_2,\lambda)$.

Proof of Theorem 4.1. Define the nonnegative continuous concave function ψ by

$$\psi(y)=\inf_{t\in[0,+\infty)}\sum_{i=1}^n y_i(t).$$

It is clear that $\psi(y)\leq\|y\|$ for all $y\in\mathcal{P}$, and the mapping $F\colon\mathcal{P}_R\to\mathcal{P}$ is completely continuous by Lemma 3.3. Let r be such that $\frac{\rho}{\sigma\Lambda\beta\delta}<r<\min\left\{\frac{R}{n},\frac{\sigma\Lambda\beta\delta}{\rho}\frac{R}{n}\right\}$. In Claims 1–4, we check the validity of the assumptions of Lemma 4.1, and then we prove the existence of solutions to problem (3.2).
Claim 1: $F(\mathcal{P}_R)\subset\mathcal{P}_R$. Let $y\in\mathcal{P}_R$ then for all, $t\geq 0$

$$\sum_{i=1}^n y_i(t)\leq\sum_{i=1}^n\|y_i\|_1\leq\|y\|\leq R \text{ and } \ p(t)y_i'(t)\geq-\|y\|\geq -R.$$

Assumptions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, and the definition of the function g in (3.3) guarantee that for all positive t,

$$\begin{aligned}|(F_iy)(t)| = & \int_0^{+\infty} H_i(t,s)\Big|g_i\Big(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s)\Big)-h_i(s)\Big|\mathrm{d}s\\ \le & \int_0^{+\infty}\mathcal{H}_i(s)(q_i(s)\Big[a_i\Big(y_1(s),\dots,y_n(s)\Big)+b_i\Big(y_1(s),\dots,y_n(s)\Big)\Big]\\ & \quad\times\Big[c_i\Big(p(s)y_1'(s),\dots,p(s)y_n'(s)\Big)+d_i\Big(p(s)y_1'(s),\dots,p(s)y_n'(s)\Big)\Big]\mathrm{d}s\\ & +\int_0^{+\infty}\mathcal{H}_i(s)h_i(s)\mathrm{d}s\\ \le & \Big(1+\frac{b_i(\|y\|,\dots,\|y\|)}{a_i(\|y\|,\dots\|y\|)}\Big)\Big(1+\frac{d_i(\|y\|,\dots,\|y\|)}{c_i(\|y\|,\dots\|y\|)}\Big)\times c_i(-\|y\|,\dots,-\|y\|)\\ & \quad\times\int_0^{+\infty}\mathcal{H}_i(s)(q_i(s)a_i\Big(y_1(s),\dots,y_n(s)\Big)\mathrm{d}s+\int_0^{+\infty}\mathcal{H}_i(s)h_i(s)\mathrm{d}s\\ \le & \frac{1}{M_i}\int_0^{+\infty}\mathcal{H}_i(s)q_i(s)M_i\sum_{i=1}^n y_i(s)\mathrm{d}s+\int_0^{+\infty}\mathcal{H}_i(s)h_i(s)\mathrm{d}s\\ \le & R\mathbb{G}_i+\mathbb{H}_i\le\Lambda\frac{R}{n}\cdot\end{aligned}$$

Passing to the supremum over t, we get

$$\|F_iy\|_1\le\Lambda\frac{R}{n}.$$

Moreover,

$$\|F_iy\|\le\Lambda^{-1}\|F_iy\|_1\le\frac{R}{n}.$$

As a consequence

$$\|Fy\|=\sum_{i=1}^n\|F_iy\|\le R\qquad\text{for all } y\in\mathcal{P}_R.$$

Claim 2: The set $\{y\in\mathcal{P}(\psi,r,\frac{R}{n})\mid\psi(y)>r\}$ is nonempty because it contains the constant function $y_0\equiv(\frac{R}{n^2},\dots,\frac{R}{n^2})$. Indeed, $\|y_0\|=\sum_{i=1}^n\|y_{i,0}\|_1=\frac{R}{n}$ and $\psi(y_0)=\inf_{t\in[0,+\infty)}\sum_{i=1}^n y_{i,0}(t)=\frac{R}{n}>r$.

Claim 3: $\psi(Fy)>r$ if $y\in\mathcal{P}(\psi,r,\frac{R}{n})$. Let $y\in\mathcal{P}(\psi,r,\frac{R}{n})$ then $\sum_{i=1}^n y_i(t)\le\|y\|\le\frac{R}{n}\cdot$ Moreover, Lemma 2.1 yields

$$\sum_{i=1}^n y_i(t)\ge\sigma\Lambda\mathcal{G}(t)\|y\|\ge\sigma\Lambda\frac{\beta\delta}{\rho}\psi(y)\ge\frac{\sigma\Lambda\beta\delta}{\rho}r.$$

From inequality (4.4), we derive the estimates

$$
\begin{aligned}
\psi(Fy) &= \inf_{t\in\mathbb{R}_+} \sum_{i=1}^{n} (F_i y)(t) \\
&\geq \inf_{t\in\mathbb{R}_+} \sum_{i=1}^{n} \int_{\mu}^{\eta} H_i(t,s) \times \Big(g_i\Big(s, y_1(s), \dots, y_n, y_1'(s), \dots, y_n'(s)\Big) - h_i(s)\Big) \mathrm{d}s \\
&\geq \inf_{t\in\mathbb{R}_+} \sum_{i=1}^{n} \int_{\mu}^{\eta} \sigma(t)\mathcal{H}_i(s)\mathcal{L}_i \mathrm{d}s = \frac{\sigma\beta\delta}{\rho} \sum_{i=1}^{n} \mathcal{L}_i \int_{\mu}^{\eta} \mathcal{H}_i(s)\mathrm{d}s > r.
\end{aligned}
$$

Claim 4: $\psi(Fy) > r$ for all $y \in \mathcal{P}(\psi, r, R)$ with $\|Fy\| > \frac{R}{n}$. Let y be an element of $\mathcal{P}(\psi, r, R)$. By Lemma 3.3, $Fy \in \mathcal{P}$. Therefore, Lemma 2.1 implies that $\sum\limits_{i=1}^{n} (F_i y)(t) \geq \sigma\Lambda\, \mathcal{G}(t)\|Fy\| \geq \sigma\Lambda \frac{\beta\delta}{\rho}\|Fy\|$ for all $t \in \mathbb{R}_+$. Consequently, the following lower bounds hold

$$
\psi(Fy) = \inf_{t\in[0,+\infty)} \sum_{i=1}^{n} (F_i y)(t) \geq \sigma\Lambda\, \mathcal{G}(t)\|Fy\| > \sigma\Lambda\, \frac{\beta\delta}{\rho}\frac{R}{n} \geq r.
$$

Therefore, all of the conditions of Lemma 4.1 are satisfied with $\lambda = R$, $\nu_1 = \frac{R}{n}$, and $\nu_2 = r$. Hence, F has at least one positive fixed point $y = (y_1, \dots, y_n) \in \mathcal{P}(\psi, r, R)$. More precisely, problem (3.2) has at least one positive solution y satisfying

$$
\|y\| \leq R \qquad \text{and} \qquad \sum_{i=1}^{n} y_i(t) \geq \frac{\sigma\Lambda\beta\delta}{\rho} r \quad \text{for all } t \geq 0.
$$

Claim 5: For the fixed point y, we have that $Fy(t) \geq \omega(t)$ for all $t \geq 0$. Otherwise, there would exist some $i_0 \in \{1, \dots, n\}$ and $t_0 \in \mathbb{R}^+$ such that

$$
\omega_{i_0}(t_0) - (F_{i_0} y)(t_0) > 0.
$$

Since $Fy = y$ and $\|y\| \leq R$, taking into account (3.3), (4.5), and the fact that f is nondecreasing with respect to the second argument, we have $y_{i_0}(t_0) < \omega_{i_0}(t_0)$ and

$$
\begin{aligned}
0 &< \omega_{i_0}(t_0) - (F_{i_0} y)(t_0) \\
&= \int_{0}^{+\infty} H_{i_0}(t_0,s) \Big[h_{i_0}(s) - g_{i_0}(s, y(s), y'(s)) + h_{i_0}(s) \Big] \mathrm{d}s \\
&\leq - \int_{0}^{+\infty} H_{i_0}(t_0,s) \Big[f_{i_0}(s, \omega(s), y'(s)) - 2h_{i_0}(s) \Big] \mathrm{d}s \leq 0,
\end{aligned}
$$

which is contradictory. Indeed, at every point $s \in [0, \infty)$,

- either there exists some $i_1 \in \{1, \dots, n\}$ such that $y_{i_1}(s) < \omega_{i_1}(s)$, and so, by (3.3),

$$
g_{i_0}(s, y_1(s), y_1'(s)) = f_{i_0}(s, \omega_{i_1}(s), y_1'(s)),
$$

- or $y(s) \geq \omega(s)$ where
$$g_{i_0}(s, y(s), y'(s)) = f_{i_0}(s, y(s), y'(s)) \geq f_{i_0}(s, \omega(s), y'(s))$$
for f nondecreasing with respect to the second argument.

We have proved that $Fy \geq \omega > 0_{\mathbb{R}^n}$. Therefore, $y = F(y)$ and y is a positive solution to problem (1.1) and satisfies

$$0 < \|y\| \leq R, \quad \sum_{i=1}^{n} y_i(t) \geq \frac{\sigma\Lambda\beta\delta}{\rho} r \quad \text{and} \quad y(t) \geq \omega(t) > 0_{\mathbb{R}^n} \quad \text{for all } t \in \mathbb{R}^+.$$

□

Theorem 4.2. *Assume that* $(\mathcal{C}_1)$ *holds with nondecreasing functions* a_i, $i \in \{1, \dots, n\}$, *and that* $(\mathcal{C}_0)$, $(\mathcal{C}_2')$, *and* $(\mathcal{C}_3)$ *hold together with* (3.5) *and* (4.5).

Then problem (1.1) *has at least one positive solution* $y \in \mathcal{P}$ *satisfying the same properties as in Theorem 4.1.*

Proof. The proof is similar to that of Theorem 4.1 with slight modifications in *Claim 1.* The other steps are identical. Given $y \in \mathcal{P}_R$, for all positive t, we have $\sum_{i=1}^{n} y_i(t) \leq \|y\| \leq R$ and $p(t)y_i'(t) \geq -\|y\| \geq -R$. Thus, for all positive t, from Assumptions $(\mathcal{C}_1)$, $(\mathcal{C}_2')$, and the definition of g, we get the estimates

$$\begin{aligned}
|(F_i y)(t)| &= \int_0^{+\infty} H_i(t,s)\Big|g_i\Big(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)\Big) - h_i(s)\Big|\mathrm{d}s \\
&\leq \int_0^{+\infty} \mathcal{H}_i(s)(q_i(s)\Big[a_i\Big(y_1(s), \dots, y_n(s)\Big) + b_i\Big(y_1(s), \dots, y_n(s)\Big)\Big] \\
&\qquad \times \Big[c_i\Big(p(s)y_1'(s), \dots, p(s)y_n'(s)\Big) + d_i\Big(p(s)y_1'(s), \dots, p(s)y_n'(s)\Big)\Big]\mathrm{d}s \\
&\quad + \int_0^{+\infty} \mathcal{H}_i(s)\, h_i(s)\mathrm{d}s \\
&\leq (1 + \frac{b_i(R, \dots, R)}{a_i(R, \dots R)})(1 + \frac{d_i(R, \dots, R)}{c_i(R, \dots R)}) \times c_i(-R, \dots, -R)\mathbb{K}_i + \mathbb{H}_i \\
&\leq \Lambda \frac{R}{n}.
\end{aligned}$$

□

Next, we apply the Leggett-Williams fixed point theorem to prove the existence of three nonnegative solutions to problem (1.1). This a variant of Lemma 4.1.

Lemma 4.2. [**15**, Theorem 3.3] *Let* $F\colon \mathcal{P}_\lambda \to \mathcal{P}_\lambda$ *be a completely continuous operator. Assume that there exists a concave positive functional* ψ *satisfying* $\psi(x) \leq \|x\|$ *for all* $x \in \mathcal{P}$, *and the constants* ν_1, ν_2 *and* ν_3 *with* $0 < \nu_3 < \nu_2 < \nu_1 \leq \lambda$ *satisfy the following conditions:*

(1) $\{x \in \mathcal{P}(\psi, \nu_2, \nu_1) \mid \psi(x) > \nu_2\} \neq \emptyset$ *and* $\psi(Fx) > \nu_2$ *if* $x \in \mathcal{P}(\psi, \nu_2, \nu_1)$.

(2) $\|Fx\| < \nu_3$ *if* $x \in \mathcal{P}_{\nu_3}$.

(3) $\psi(Fx) > \nu_2$ *for all* $x \in \mathcal{P}(\psi, \nu_2, \lambda)$ *with* $\|Fx\| > \nu_1$.

Then F has at least three fixed points x_1, x_2, x_3 in $\mathcal{P}_\lambda$ such that

$$\|x_1\| < \nu_3, \quad \psi(x_2) > \nu_2, \qquad \text{and} \qquad \|x_3\| > \nu_3 \quad \text{with } \psi(x_3) < \nu_2.$$

We have the following theorem.

Theorem 4.3. *Suppose that $(\mathcal{C}_0)$-$(\mathcal{C}_2)$ and $(\mathcal{C}_3)$ hold and there exists a constant $0 < \widetilde{R} < r$, where r is defined as in $(\mathcal{C}_3)$ such that*

$$\widetilde{R}\mathbb{G}_i + \mathbb{H}_i \leq \widetilde{R}\Lambda/n, \tag{4.6}$$

and

$$\frac{a_i(y_1, \dots, y_n)}{\sum\limits_{i=1}^{n} y_i} \leq N_i \qquad \text{for all } \sum_{i=1}^{n} \omega_i \leq \sum_{i=1}^{n} y_i \leq \widetilde{R}, \tag{4.7}$$

where

$$N_i^{-1} = \Big(1 + \frac{b_i(\widetilde{R}, \dots, \widetilde{R})}{a_i(\widetilde{R}, \dots, \widetilde{R})}\Big)\Big(1 + \frac{d_i(\widetilde{R}, \dots, \widetilde{R})}{c_i(\widetilde{R}, \dots, \widetilde{R})}\Big)c_i\Big(- \widetilde{R}, \dots, -\widetilde{R}\Big).$$

Then problem (1.1) *has at least three positive solutions $y_1, y_2, y_3 \in \mathcal{P}$ such that $0 < \|y_1\| \leq \widetilde{R}$, and for all positive t, $y_i(t) \geq \omega(t) > 0_{\mathbb{R}^n}$, $(i = 1, 2, 3)$,*

$$\sum_{i=1}^{n} y_{2,i}(t) \geq r \qquad \text{and} \qquad \|y_3\| \geq \widetilde{R} \quad \text{with } \sum_{i=1}^{n} y_{3,i}(t) \leq r.$$

Proof. The proof of these results follows from Lemma 4.2 and Theorem 4.1, with $\lambda = R$, $\nu_1 = \frac{R}{n}$, $\nu_2 = r$ and $\nu_3 = \widetilde{R}$ by following the same arguments as in Claims 1–5. □

Remark 4.2. Notice that all of solutions in Theorems 4.1, 4.2, and 4.3 are positive, hence nontrivial; indeed,

$$y(t) \geq \omega(t), \qquad \text{for all } t \in \mathbb{R}^+$$

for every solution y.

We close this section with a nonexistence result.

Theorem 4.4. *For each $i \in \{1, \dots, n\}$, assume that*

$$f_i(t, y_1, \dots, y_n, z_1, \dots, z_n) \geq h(t) \qquad \text{for all } t \in \mathbb{R}^+,\ y_i \in \mathbb{R}^+ \text{ and } z_i \in \mathbb{R}.$$

$$f_i(t, y_1, \dots, y_n, z_1, \dots, z_n) - h(t) \geq \mathcal{L}_i \sum_{i=1}^{n} y_i \text{ for all } t \in [\mu, \eta], y_i \in \mathbb{R}^+ \text{ and } z_i \in \mathbb{R},$$

where $\mathcal{L}_i = \dfrac{\rho}{\sigma\beta\delta}\big(\int_\mu^\eta \mathcal{H}_i(s)\mathrm{d}s\big)^{-1}$.

Then problem (1.1) *has no positive solution.*

Proof. On the contrary, assume that system (1.1) has a positive solution $y \in \mathbb{X}$. Then for all $t \in [\mu, \eta]$, $\sum_{i=1}^{n} y_i \geq 0$, $z_i \in \mathbb{R}$ ($i \in \{1, \ldots, n\}$), we have

$$\begin{aligned}
\sum_{i=1}^{n} y_i(t) \geq y_i(t) \\
&= \int_0^{+\infty} H_i(t,s)\Big(f_i\Big(s, y_1(s), \ldots, y_n(s), y_1'(s), \ldots, y_n'(s)\Big) - h_i(s)\Big)\mathrm{d}s \\
&> \frac{\sigma\beta\delta}{\rho}\int_\mu^\eta \mathcal{H}_i(s)\mathcal{L}_i \sum_{i=1}^{n} y_i(s)\mathrm{d}s \\
&\geq \min_{t\in[\mu,\eta]} \sum_{i=1}^{n} y_i(t),
\end{aligned}$$

contradicting the continuity of the functions y_i on the compact interval $[\mu, \eta]$. □

Remark 4.3. The results of this paper still hold true if in problem (1.1), we replace the four parameters α, β, γ, and δ by $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, $\mathrm{diag}(\beta_1, \ldots \beta_n)$, $\mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, and $\mathrm{diag}(\delta_1, \ldots, \delta_n)$, respectively.

5. Example

Consider the six-point boundary value problem

$$(5.1)\qquad \begin{cases}
-(p(t)y_1'(t))' = f_1(t, y_1(t), y_2(t), y_1'(t), y_2'(t)) - h_1(t), \quad t > 0, \\
-(p(t)y_2'(t))' = f_2(t, y_1(t), y_2(t), y_1'(t), y_2'(t)) - h_2(t), \quad t > 0, \\
\frac{3}{2}y_1(0) - p(0)y_1'(0) = \frac{1}{4}y_1(\frac{1}{2}) + \frac{1}{8}y_1(1), \\
\frac{1}{2}\lim_{t\to+\infty} y_1(t) + \frac{3}{2}\lim_{t\to+\infty} p(t)y_1'(t) = \frac{1}{9}y_1(\frac{1}{2}) + \frac{1}{3}y_1(1), \\
\frac{3}{2}y_2(0) - p(0)y_2'(0) = \frac{1}{4}y_2(\frac{1}{3}) + \frac{1}{9}y_2(\frac{2}{3}), \\
\frac{1}{2}\lim_{t\to+\infty} y_2(t) + \frac{3}{2}\lim_{t\to+\infty} p(t)y_2'(t) = \frac{3}{8}y_2(\frac{1}{3}) + y_2(\frac{2}{3}),
\end{cases}$$

where

$$\begin{aligned}
f_1(t, y_1, y_2, z_1, z_2) &= q_1(t)\Big[a_1\Big(y_1, y_2\Big) + b_1\Big(y_1, y_2\Big)\Big] \\
&\quad \times [c_1\Big(p(t)z_1, p(t)z_2\Big) + d_1\Big(p(t)z_1, p(t)z_2\Big)], \\
f_2(t, y_1, y_2, z_1, z_2) &= q_2(t)\Big[a_2\Big(y_1, y_2\Big) + b_2\Big(y_1, y_2\Big)\Big] \\
&\quad \times [c_2\Big(p(t)z_1, p(t)z_2\Big) + d_2\Big(p(t)z_1, p(t)z_2\Big)],
\end{aligned}$$

$$p(t) = 100 + t^2, \quad q_1(t) = \mathrm{e}^{-8t}, \quad q_2(t) = \mathrm{e}^{-8t}, \quad h_1(t) = \mathrm{e}^{-12t}, \quad h_2(t) = \mathrm{e}^{-14t},$$

$$a_1\Big(y_1,y_2\Big)=\mathrm{e}^{-\frac{11}{2}}(y_1+y_2)^{\frac{4}{3}}, \qquad a_2\Big(y_1,y_2\Big)=\mathrm{e}^{-10}(y_1+y_2)^{\frac{5}{4}},$$
$$b_1\Big(y_1,y_2\Big)=\mathrm{e}^{-6}(y_1+y_2+e^9)^{\frac{5}{3}}, \qquad b_2(y_2,y_2)=\mathrm{e}^{-10}(y_1+y_2+e^8)^2,$$
$$c_1\Big(z_1,z_2\Big)=\frac{\mathrm{e}^{-10}}{(z_1+z_2+e^{10})^2}, \qquad d_1\Big(z_1,z_2\Big)=\frac{\mathrm{e}^{-5}}{(z_1+z_2+e^{10})^4}+\frac{1}{40},$$
$$c_2(z_1,z_2)=\frac{\mathrm{e}^{-2}}{(z_1+z_2)^2}, \qquad d_1\Big(z_1,z_2\Big)=\frac{\mathrm{e}^{-10}}{(z_1+z_2+100)^4}+\frac{1}{60}.$$

Then problem (5.1) has at least one positive solution.

Proof. BVP (5.1) can be regarded as a boundary value problem of the form (1.1). Obviously $(\mathcal{C}_1)$ holds and $\alpha=\frac{3}{2}$, $\beta=1$, $\gamma=\frac{1}{2}$, $\delta=\frac{3}{2}$, and $\mu_i\equiv\nu_i\equiv 1$, $i=1,2$. Moreover,

$$\xi_1(t)=\begin{cases}0, & t\in\Big[0,\frac{1}{2}\Big),\\ \frac{1}{4}, & t\in\Big[\frac{1}{2},1\Big),\\ \frac{3}{8}, & t\in[1,+\infty),\end{cases} \qquad \eta_1(t)=\begin{cases}0, & t\in\Big[0,\frac{1}{2}\Big),\\ \frac{1}{9}, & t\in[\frac{1}{2},1),\\ \frac{4}{9}, & t\in[1,+\infty),\end{cases}$$

$$\xi_2(t)=\begin{cases}0, & t\in\Big[0,\frac{1}{3}\Big),\\ \frac{1}{4}, & t\in\Big[\frac{1}{3},\frac{2}{3}\Big),\\ \frac{13}{36}, & t\in\Big[\frac{2}{3},+\infty\Big),\end{cases} \qquad \eta_2(t)=\begin{cases}0, & t\in\Big[0,\frac{1}{3}\Big),\\ \frac{3}{8}, & t\in\Big[\frac{1}{3},\frac{2}{3}\Big),\\ \frac{11}{8}, & t\in\Big[\frac{2}{3},+\infty\Big).\end{cases}$$

Then

$$\begin{aligned}
\int_0^{+\infty}\mu_1(s)y_1(s)\mathrm{d}\xi_1(s)&=y_1\Big(\frac{1}{2}\Big)\Big(\xi_1\Big(\frac{1^+}{2}\Big)-\xi_1\Big(\frac{1^-}{2}\Big)\Big)+y_1(1)(\xi_1(1^+)-\xi_1(1^-))\\
&=y_1\Big(\frac{1}{2}\Big)\Big(\frac{1}{4}-0\Big)+y_1(1)\Big(\frac{3}{8}-\frac{1}{4}\Big)=\frac{1}{4}y_1\Big(\frac{1}{2}\Big)+\frac{1}{8}y_1(1),\\
\int_0^{+\infty}\mu_2(s)y_2(s)\mathrm{d}\xi_2(s)&=y_2\Big(\frac{1}{3}\Big)\Big(\xi_2(\frac{1^+}{3}\Big)-\xi_2\Big(\frac{1^-}{3}\Big)\Big)+y_2\Big(\frac{2}{3}\Big)(\xi_2\Big(\frac{2^+}{3}\Big)-\xi_2\Big(\frac{2^-}{3}\Big)\Big)\\
&=y_2\Big(\frac{1}{2}\Big)\Big(\frac{1}{4}-0\Big)+y_2(1)\Big(\frac{13}{36}-\frac{1}{4}\Big)\\
&=\frac{1}{4}y_2\Big(\frac{1}{3}\Big)+\frac{1}{9}y_2\Big(\frac{2}{3}\Big),\\
\int_0^{+\infty}\nu_1(s)y_1(s)\mathrm{d}\eta_1(s)&=y_1\Big(\frac{1}{2}\Big)\Big(\eta_1\Big(\frac{1^+}{2}\Big)-\eta_1\Big(\frac{1^-}{2}\Big)\Big)+y_1(1)(\eta_1(1^+)-\eta_1(1^-))\\
&=y_1\Big(\frac{1}{2}\Big)\Big(\frac{1}{9}-0\Big)+y_1(1)\Big(\frac{4}{9}-\frac{1}{9}\Big)\\
&=\frac{1}{9}y_1\Big(\frac{1}{2}\Big)+\frac{1}{3}y_1(1),
\end{aligned}$$

$$\begin{aligned}\int_0^{+\infty} \nu_2(s)y_2(s)\mathrm{d}\eta_2(s) &= y_2\Big(\frac{1}{3}\Big)\Big(\eta_2(\frac{1^+}{3}\Big)-\eta_2\Big(\frac{1^-}{3}\Big)\Big)+y_2\Big(\frac{2}{3}\Big)\Big(\eta_2\Big(\frac{2^+}{3}\Big)-\eta_2\Big(\frac{2^-}{3}\Big)\Big)\\ &= y_2\Big(\frac{1}{2}\Big)\Big(\frac{3}{8}-0\Big)+y_2(1)\Big(\frac{11}{8}-\frac{3}{8}\Big)\\ &= \frac{3}{8}y_2\Big(\frac{1}{3}\Big)+y_2\Big(\frac{2}{3}\Big),\end{aligned}$$

and so

$$\int_0^{+\infty}\mu_1(s)\mathrm{d}\xi_1(s)=\frac{1}{4}+\frac{1}{8}=\frac{3}{8}<\infty,\qquad \int_0^{+\infty}\mu_2(s)\mathrm{d}\xi_2(s)=\frac{1}{4}+\frac{1}{9}=\frac{13}{36}<\infty,$$

$$\int_0^{+\infty}\nu_1(s)\mathrm{d}\eta_1(s)=\frac{1}{9}+\frac{1}{3}=\frac{4}{9}<\infty,\qquad \int_0^{+\infty}\nu_2(s)\mathrm{d}\eta_2(s)=\frac{3}{8}+1=\frac{11}{8}<\infty.$$

Therefore, $(\mathcal{C}_0)$ holds true. We can also compute

$$k_1^{(1)}\simeq 0.3847,\ k_3^{(1)}\simeq 0.2933,\ k^{(1)}\simeq 0,0892,$$

and

$$k_1^{(2)}\simeq 0.3707,\ k_3^{(2)}\simeq 0.9033,\ k^{(2)}\simeq 0.2641,$$

so that (2.7) is satisfied.

In order to check (4.2) and (4.1) in Assumption $(\mathcal{C}_2)$, we calculate $A\simeq 0.1571$, $\rho\simeq 2.8678$, $\sigma\simeq 1.4703$, $\Lambda=2/3$, and

$$2\max\Big\{\frac{\rho}{\sigma\beta\delta\Lambda},\ \Big(\frac{\rho}{\sigma\beta\delta\Lambda}\Big)^2\Big\}\simeq 76.0870.$$

Then we can choose $R=100$. Moreover, we have

$$\Big(\mathbb{G}_1,\,\mathbb{G}_2\Big)\le\Big(\mathcal{A}_1\int_0^{+\infty}q_1(s)\mathrm{d}s,\ \mathcal{A}_2\int_0^{+\infty}q_2(s)\mathrm{d}s\Big)\simeq(0.1381,\,0.2964),$$

$$\Big(\mathbb{H}_1,\,\mathbb{H}_2\Big)\le\Big(\mathcal{A}_1\int_0^{+\infty}q_1(s)\mathrm{d}s,\ \mathcal{H}_2\int_0^{+\infty}q_2(s)\mathrm{d}s\Big)\simeq(0.0921,\,0.1694).$$

Hence

$$\begin{aligned}\Big(\mathbb{G}_1\,R+\mathbb{H}_1,\mathbb{G}_2\,R+\mathbb{H}_2\Big) &\le (13.9014,\,29.8064)\\ &\le (100/3,\,100/3)=(R\tfrac{\Lambda}{2},R\tfrac{\Lambda}{2})\cdot\end{aligned}$$

In addition, for $y_1+y_2\le R$,

$$\frac{a_1(y_1,y_2)}{\sum_{i=1}^{2}y_i}=\frac{\mathrm{e}^{-\frac{11}{2}}(y_1+y_2)^{\frac{4}{3}}}{(y_1+y_2)}\le 0.0190\le M_1=0.0218$$

and

$$\frac{a_2(y_1,y_2)}{\sum_{i=1}^{2}y_i}=\frac{\mathrm{e}^{-10}(y_1+y_2)^{\frac{5}{4}}}{(y_1+y_2)}\le 1.1404\times10^{-4}\le M_2=4.9546\times10^{-4}.$$

Consequently, (4.2) and (4.1) in Assumption ($\mathcal{C}_2$) are both satisfied. Since $\frac{\rho}{\sigma\Lambda\beta\delta} \simeq 1.9505$ and $\min\left\{\frac{R}{2},\ \frac{\sigma\Lambda\beta\delta}{\rho}\frac{R}{2}\right\} \simeq 25.6348$, we can choose $r = 2$. Then for all $t \in [\mu, \eta] = [\frac{1}{10}, \frac{1}{3}]$, $\frac{\sigma\Lambda\beta\delta}{\rho} r \leq \sum_{i=1}^{2} y_i \leq \frac{R}{2}$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\Big(f_1(t, y_1, y_2, z_1, z_2) - h_1(t),\ f_2(t, y_1, y_2, z_1, z_2) - h_2(t)\Big) \geq \Big(\mathcal{L}_1,\ \mathcal{L}_2\Big), \tag{5.2}$$

where $(\mathcal{L}_1,\ \mathcal{L}_2) \simeq (13.7746, 0.2206)$. Hence (4.4) in Assumption ($\mathcal{C}_3$) is satisfied. Also

$$\sum_{i=1}^{2} \xi_i \int_{\mu}^{\eta} \mathcal{H}_i(s) \mathrm{d}s \geq 2.6883 \geq 2.6006 \simeq r \frac{\rho}{\sigma\beta\delta}.$$

Finally, for all $t \in I$, we have $(y_1, y_2) \in \mathbb{R}^2_+$, and $(z_1, z_2) \in \mathbb{R}^2$,

$$\begin{aligned}\Big(f_1(t, y_1, y_2, z_1, z_2),\ f_2(t, y_1, y_2, z_1, z_2)\Big) &\geq \Big(\frac{1}{40}\,\mathrm{e}^{-8t+9},\ \frac{1}{60}\,\mathrm{e}^{-8t+6}\Big)\\ &\geq \Big(2h_1(t),\ 2h_2(t)\Big).\end{aligned}$$

Therefore, (3.5), (4.5) together with Assumptions ($\mathcal{C}_0$), ($\mathcal{C}_1$), ($\mathcal{C}_2$), and ($\mathcal{C}_3$) in Theorem 4.1 are fulfilled. All the computations have been performed using Matlab 7.9. As a consequence, the boundary value problem (5.1) has at least one nontrivial positive solution $y = (y_1, y_2)$ such that $0 < \|y\| \leq 100$ and $y_1(t) + y_2(t) \geq 1.0254$ for all positive t. □

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