NORMAL CR-SUBMANIFOLDS
OF A NEARLY KAELHERIAN MANIFOLD

AI HAIHUA, XIE LI AND WAN YONG

Abstract. In this paper, we give some sufficient and necessary conditions for normal CR-submanifold of a nearly Kaehlerian manifold, and generalize Bejancu’s research work.

1. Introduction

Let $\overline{M}$ be a real differentiable manifold. An almost complex structure on $\overline{M}$ is a tensor field $J$ of type (1, 1) on $\overline{M}$ such that at every point $x \in \overline{M}$, we have $J^2 = -I$, where $I$ denotes the identify transformation of $T_x \overline{M}$. A manifold $\overline{M}$ endowed with an almost complex structure is called an almost complex manifold.

A linear connection $\nabla$ on $\overline{M}$ is said to be a Riemannian connection if Riemannian metric $g$ satisfies
\begin{equation}
Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)
\end{equation}
for any $X, Y \in \Gamma(T \overline{M})$. More, we define the torsion tensor of $J$ or the Nijenhuis tensor of $J$ by
\begin{equation}
[J, J](X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]
\end{equation}
for any $X, Y \in \Gamma(T \overline{M})$, where $[X, Y]$ is the Lie bracket of vector fields $X$ and $Y$, that is, $[X, Y] = \nabla_X Y - \nabla_Y X$.

A Hermitian metric on an almost complex manifold $\overline{M}$ is a Riemannian metric $g$ satisfying
\begin{equation}
g(JX, JY) = g(X, Y)
\end{equation}
for any $X, Y \in \Gamma(T \overline{M})$.

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An almost complex manifold endowed with a Hermitian metric is said to be an almost Hermitian manifold.

**Definition 1.1** ([1]). An almost Hermitian manifold $\mathcal{M}$ with Levi-Civita connection $\nabla$ is called a nearly Kaehlerian manifold if we have
\begin{equation}
(\nabla_X J)Y + (\nabla_Y J)X = 0 \tag{1.5}
\end{equation}
for any $X, Y \in \Gamma(T\mathcal{M})$.

**Definition 1.2** ([1]). An almost Hermitian manifold $\mathcal{M}$ with Levi-Civita connection $\nabla$ is called a Kaehlerian manifold if we have
\begin{equation}
\nabla_X J = 0 \tag{1.6}
\end{equation}
for any $X \in \Gamma(T\mathcal{M})$.

Obviously, a Kaehlerian manifold is a nearly Kaehlerian manifold.

Let $M$ be an $m$-dimensional Riemannian submanifold of an $n$-dimensional Riemannian manifold $\mathcal{M}$. By $T^\perp M$, we denote the normal bundle to $M$ and by $g$ metric on $M$ and $\mathcal{M}$. Also, we denote the Levi-Civita connection on $M$ by $\nabla$, the induced connection on $M$ by $\nabla$, and the induced normal connection on $M$ by $\nabla^\perp$.

Then, for any $X, Y \in \Gamma(TM)$, we have
\begin{equation}
\nabla_X Y = \nabla_X Y + h(X, Y), \tag{1.7}
\end{equation}
where $h: \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.7) is called the Gauss formula and $h$ is called the second fundamental form of $M$.

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$ by $-A_V X$ and $\nabla^\perp_X V$, we denote the tangent part and normal part of $\nabla_X V$, respectively. Then we have
\begin{equation}
\nabla_X V = -A_V X + \nabla^\perp_X V. \tag{1.8}
\end{equation}
Thus, for any $V \in \Gamma(TM^\perp)$, we have a linear operator satisfying
\begin{equation}
g(A_V X, Y) = g(X, A_V Y) = g(h(X, Y), V). \tag{1.9}
\end{equation}
The equation (1.8) is called the Weingarten formula.

An $m$-dimensional distribution on a manifold $\mathcal{M}$ is a mapping $D$ defined on $\mathcal{M}$, which assigns to each point $x$ of $\mathcal{M}$ an $m$-dimensional linear subspace $D_x$ of $T_x \mathcal{M}$. A vector field $X$ on $\mathcal{M}$ belongs to $D$ if we have $X_x \in D_x$ for each $x \in \mathcal{M}$. When this happens, we write $X \in \Gamma(D)$. The distribution $D$ is said to be differentiable if for any $x \in \mathcal{M}$, there exist $m$ differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of $x$. From now on, all distributions are supposed to be differentiable of class $C^\infty$.

**Definition 1.3** ([1]). Let $\mathcal{M}$ be a real $n$-dimensional almost Hermitian manifold with almost complex structure $J$ and with Hermitian metric $g$. Let $M$ be a real $m$-dimensional Riemannian manifold isometrically immersed in $\mathcal{M}$. Then $M$ is called a CR-submanifold of $\mathcal{M}$ if there exists a differentiable distribution
\[D: x \to D_x \subset T_x M\]
on $M$ satisfying the following conditions:
(1) $D$ is holomorphic, that is, $J(D_x) = D_x$ for each $x \in M$,
(2) the complementary orthogonal distribution

$$D^\perp : x \rightarrow D^\perp_x \subset T_x M$$

is anti-invariant, that is, $J(D^\perp_x) \subset T_x M^\perp$, for each $x \in M$.

Now let $M$ be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold $\overline{M}$. For each vector field $X$ tangent to $M$, we put

$$(1.10) \quad JX = \phi X + \omega X,$$

where $\phi X$ and $\omega X$, respectively, are the tangent part and the normal part of $JX$. Also, for each vector field $V$ normal to $M$, we put

$$(1.11) \quad JV = BV + CV,$$

where $BV$ and $CV$ are respectively the tangent part and the normal part of $JV$.

We take account of the decomposition $T\overline{M} = D \oplus D^\perp \oplus JD^\perp \oplus \nu$. Obviously, we have $\phi X \in \Gamma(D)$, $\omega X \in \Gamma(JD^\perp)$, $BV \in \Gamma(D^\perp)$, $CV \in \Gamma(\nu)$ for any $X \in \Gamma(TM)$, $V \in \Gamma(JD^\perp \oplus \nu)$.

The covariant derivative of $\phi$ is defined by

$$(1.12) \quad (\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y$$

for any $X, Y \in \Gamma(TM)$. On the other hand, the covariant derivative of $\omega$ is defined by

$$(1.13) \quad (\nabla_X \omega)Y = \nabla_X \omega Y - \omega \nabla_X Y$$

for any $X, Y \in \Gamma(TM)$.

The exterior derivative of $\omega$ is defined by

$$(1.14) \quad d\omega(X,Y) = \frac{1}{2} \{ \nabla_X \omega Y - \nabla_Y \omega X - \omega([X,Y]) \}$$

for any $X, Y \in \Gamma(TM)$. The Nijenhuis tensor of $\phi$ is defined by

$$(1.15) \quad [\phi, \phi](X,Y) = [\phi X, \phi Y] + \phi^2[X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

for any $X, Y \in \Gamma(TM)$, where $[X,Y]$ is the Lie bracket of vector fields $X$ and $Y$.

We define two tensor fields $S$ and $S^*$, respectively, by

$$(1.16) \quad S(X,Y) = [\phi, \phi](X,Y) - 2Bd\omega(X,Y)$$

and

$$(1.17) \quad S^*(Y,X) = (L_Y \phi)X = [Y, \phi X] - \phi[Y, X]$$

for any $X, Y \in \Gamma(TM)$.

**Definition 1.4** ([1]). The CR-submanifold $M$ is said to be normal if the tensor fields $S$ vanishes identically on $M$. 

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$J(D_x) = D_x$ for each $x \in M$,

$J(D^\perp_x) \subset T_x M^\perp$, for each $x \in M$.

$\phi X$ and $\omega X$, respectively, are the tangent part and the normal part of $JX$.

$\phi X \in \Gamma(D)$, $\omega X \in \Gamma(JD^\perp)$, $BV \in \Gamma(D^\perp)$, $CV \in \Gamma(\nu)$ for any $X \in \Gamma(TM)$, $V \in \Gamma(JD^\perp \oplus \nu)$.

$JX = \phi X + \omega X,$

$BV + CV,$

$\nabla_X \phi Y = \nabla_X \phi Y - \phi \nabla_X Y$

$\nabla_X \omega Y - \omega \nabla_X Y$

$\frac{1}{2} \{ \nabla_X \omega Y - \nabla_Y \omega X - \omega([X,Y]) \}$

$[\phi, \phi](X,Y) = [\phi X, \phi Y] + \phi^2[X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$

$S(X,Y) = [\phi, \phi](X,Y) - 2Bd\omega(X,Y)$

$(L_Y \phi)X = [Y, \phi X] - \phi[Y, X]$

$M$ is said to be normal if the tensor fields $S$ vanishes identically on $M$. 

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2. Main Results

**Lemma 2.1** ([1]). Let $M$ be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then we have

$$J = \begin{bmatrix} 0 & J \end{bmatrix}$$

for any $X, Y \in \Gamma(TM)$.

**Lemma 2.2** ([1]). Let $\overline{M}$ be a nearly Kaehlerian manifold. Then we have

$$\nabla_X J Y = \frac{1}{4} J [J, J](X, Y)$$

for any $X, Y \in \Gamma(T\overline{M})$.

**Lemma 2.3** (Frobenius [1, 2]). The distribution $D^\perp$ is integrable if and only if

$$[X, Y] \in \Gamma(D^\perp)$$

for any $X, Y \in \Gamma(D^\perp)$.

**Lemma 2.4** (Urbano [1]). Let $M$ be a CR-submanifold of a nearly Kaehlerian manifold $\overline{M}$. Then the distribution $D^\perp$ is integrable if and only if

$$g(\nabla_X J Y, Z) = 0$$

for any $X, Y \in \Gamma(D^\perp), Z \in \Gamma(D)$.

**Lemma 2.5.** Let $M$ be a CR-submanifold of a nearly Kaehlerian manifold $\overline{M}$. Then we have

$$(\nabla_X \phi) Y = A_\omega Y X + Bh(X, Y) + \frac{1}{4} (J [J, J](X, Y))^\perp$$

and

$$(\nabla_X \omega) Y = -h(X, \phi Y) + Ch(X, Y) + \frac{1}{4} (J [J, J](X, Y))^\perp$$

for any $X, Y \in \Gamma(TM)$.

**Proof.** For any $X, Y \in \Gamma(TM)$, using (1.7), (1.8), (1.10) and (1.11) in (1.2), we have

$$(\nabla_X J Y) = \nabla_X \phi Y + h(X, \phi Y) - A_\omega Y X + \nabla_X \omega Y - \phi \nabla_X Y - \omega \nabla_X Y - Bh(X, Y) - Ch(X, Y).$$

Taking into account (1.12) and (1.13), (2.6) becomes

$$(\nabla_X J Y) = (\nabla_X \phi Y) + h(X, \phi Y) - A_\omega Y X + (\nabla_X \omega) Y - Bh(X, Y) - Ch(X, Y).$$

Taking account of (2.2) and (2.7), we obtain

$$\frac{1}{4} J [J, J](X, Y) = (\nabla_X \phi Y) + h(X, \phi Y) - A_\omega Y X + (\nabla_X \omega) Y - Bh(X, Y) - Ch(X, Y).$$
By comparing the tangent part and the normal part in (2.8), we get (2.4) and (2.5).

**Theorem 2.1.** Let $M$ be a CR-submanifold of a nearly Kaehlerian manifold $\mathcal{M}$. Then $M$ is normal if and only if we have

$$0 = A_{\omega Y} \phi X - \phi A_{\omega Y} X - A_{\omega X} \phi Y + \phi A_{\omega X} Y + \frac{1}{4} (J[J, J]([\phi X, Y]))^\top$$

$$- \frac{1}{4} (J[J, J]([\phi Y, X]))^\top - \frac{1}{2} \phi (J[J, J](X, Y))^\top - \frac{1}{2} B(J[J, J](X, Y))^\bot,$$

(2.9)

for any $X, Y \in \Gamma(TM)$.

**Proof.** For any $X, Y \in \Gamma(TM)$, using (2.4) and (2.5), (2.1) becomes

$$S(X, Y) = A_{\omega Y} \phi X - \phi A_{\omega Y} X - A_{\omega X} \phi Y + \phi A_{\omega X} Y$$

$$+ \frac{1}{4} (J[J, J]([\phi X, Y]))^\top - \frac{1}{4} (J[J, J]([\phi Y, X]))^\top$$

$$- \frac{1}{2} \phi (J[J, J](X, Y))^\top - \frac{1}{2} B(J[J, J](X, Y))^\bot.$$

(2.10)

Taking account of Definition 1.4, $M$ is normal if and only if (2.9) holds.

**Theorem 2.2.** Let $M$ be a CR-submanifold of a nearly Kaehlerian manifold $\mathcal{M}$ and

$$[J, J](X, Y) \in \Gamma(\nu),$$

(2.11)

for any $X, Y \in \Gamma(TM)$. Then $M$ is normal if and only if we have

$$A_{\omega Y} \phi X = \phi A_{\omega Y} X$$

for any $X \in \Gamma(D), Y \in \Gamma(D^\bot)$.

**Proof.** By using (2.11) in (2.10), we obtain

$$S(X, Y) = A_{\omega Y} \phi X - \phi A_{\omega Y} X - A_{\omega X} \phi Y + \phi A_{\omega X} Y$$

(2.13)

for any $X, Y \in \Gamma(TM)$.

Suppose $M$ is normal CR-submanifold of $\mathcal{M}$. Then (2.12) follows from (2.13) since $A_{\omega X} = 0$ for any $X \in \Gamma(D)$.

Now, if (2.12) is satisfied, we prove $S = 0$ by means of the decomposition $TM = D \oplus D^\bot$.

First, for any $X, Y \in \Gamma(D)$, from (2.13), we have

$$S(X, Y) = 0.$$

Next, for any $X \in \Gamma(D), Y \in \Gamma(D^\bot)$, using (2.12) in (2.13), we obtain

$$S(X, Y) = A_{\omega Y} \phi X - \phi A_{\omega Y} X = 0.$$

(2.14)

Finally, for any $X, Y \in \Gamma(D^\bot)$, from (2.11), Lemma 2.2 and Lemma 2.4, the distribution $D^\bot$ is integrable.

Next, from (1.8) and (1.2), we have

$$g(A_{\omega X} Y - A_{\omega Y} X, Z) = g(J\nabla_X Y, Z) + g((\nabla_X J) Y, Z)$$

$$- g(J\nabla_Y X, Z) - g((\nabla_Y J) X, Z).$$

(2.15)
for any $X, Y \in \Gamma(D^\perp)$, $Z \in \Gamma(TM)$. 

Using (2.2) in (2.15), we get

$$g(A_\omega XY - A_\omega YX, Z) = g(J\nabla_X Y, Z) - g(J\nabla_Y X, Z) + g(\frac{1}{2}J[J, J](X, Y), Z).$$

(2.16)

Using (1.4) and (2.11) in (2.16), we obtain

$$g(A_\omega XY - A_\omega YX, Z) = g(-\nabla_X Y + \nabla_Y X, JZ) = -g([X, Y], \phi Z).$$

(2.17)

From (2.17), we know $A_\omega XY - A_\omega YX = 0$ since the distribution $D^\perp$ is integrable. More, (2.13) becomes

$$S(X, Y) = \phi(A_\omega XY - A_\omega YX) = 0$$

(2.18)

From the above three conclusions, we know $S(X, Y) = 0$ for any $X, Y \in \Gamma(D^\perp)$, that is, the CR-submanifold $M$ is normal.

Corollary 2.1 (Bejancu [1]). Let $M$ be a CR-submanifold of a Kaehlerian manifold $\overline{M}$. Then $M$ is normal if and only if we have

$$A_\omega Y \phi X = \phi A_\omega Y X,$$

(2.19)

for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$.

Proof. Since a Kaehlerian manifold $\overline{M}$ is satisfied,

$$[J, J](X, Y) = 0 \in \Gamma(\nu)$$

for any $X, Y \in \Gamma(TM)$. Taking account of Theorem 2.2, Corollary 2.1 holds.

Theorem 2.3. Let $M$ be a CR-submanifold of a nearly Kaehlerian manifold $\overline{M}$ with following conditions satisfying

1. (2.11) holds,
2. for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$, we have

$$\nabla_X Y \in \Gamma(D).$$

(2.20)

Then $M$ is normal if and only if we have

$$S^*(Y, X) = 0,$$

(2.21)

for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$.

Proof. From the proof Theorem 2.2 follows that $M$ satisfying condition (2.11) is normal CR-submanifold if and only if $S(X, Y) = 0$ for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$. Using (1.12) and (2.5) in (2.1), we have

$$S(X, Y) = (\nabla_{\phi X} \phi Y - \phi \nabla_{\phi X} Y) - (\nabla_{\phi Y} \phi X - \phi \nabla_{\phi Y} X) + \phi(\nabla_X \phi Y - \phi \nabla_Y X) - \phi(\nabla_Y \phi X - \phi \nabla_Y X) - B(h(Y, \phi X) + \frac{1}{2}(J[J, J](X, Y))^\perp)$$

(2.22)

for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$. 

□
Using (2.11) in (2.22), we get
\begin{equation}
S(X,Y) = \phi(\phi(X,Y) - [\phi X, Y]) - Bh(Y, \phi X).
\end{equation}

Next, using (1.13) and (2.11) in (2.5), we obtain
\begin{equation}
h(Y, \phi X) = \omega \nabla Y X + Ch(X, Y) + \frac{1}{4} (J[J, J](Y, X))^\perp.
\end{equation}

That implies
\begin{equation}
J \omega \nabla Y X = Bh(Y, \phi X) \in \Gamma(D^\perp).
\end{equation}

Taking account of (1.17) and (2.25), (2.23) becomes
\begin{equation}
S(X, Y) = \phi S^*(Y, X) - J \omega \nabla Y X.
\end{equation}

Now suppose \( M \) is normal. Then by comparing \( D \) part and \( D^\perp \) part in (2.26), we have
\begin{equation}
\phi S^*(Y, X) = 0 \quad \text{and} \quad J \omega \nabla Y X = 0
\end{equation}

More, we get
\begin{equation}
S^*(Y, X) \in \Gamma(D^\perp)
\end{equation}
and
\begin{equation}
\nabla Y X \in \Gamma(D).
\end{equation}

On the other hand, using (2.20) and (2.28) in (1.17), we obtain
\begin{equation}
S^*(Y, X) = (\nabla Y \phi X - \nabla \phi X Y) - \phi[Y, X] \in \Gamma(D).
\end{equation}

for any \( X \in \Gamma(D), Y \in \Gamma(D^\perp) \). Combining (2.27) with (2.29), it follows that \( S^*(Y, X) = 0 \) for any \( X \in \Gamma(D), Y \in \Gamma(D^\perp) \).

Conversely, suppose (2.21) is satisfied. Then from (1.17), (2.20) and (2.21) we have
\begin{equation}
\nabla Y \phi X = \nabla \phi X Y + \phi[Y, X] \in \Gamma(D).
\end{equation}

Using (2.21) and (2.30) in (2.26), we get \( S(X, Y) = 0 \) for any \( X \in \Gamma(D), Y \in \Gamma(D^\perp) \). Moreover, taking into account Theorem 2.2, \( M \) is normal. \( \square \)

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