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NORMAL CR-SUBMANIFOLDS OF A NEARLY KAEHLERIAN MANIFOLD

AI HAIHUA, XIE LI AND WAN YONG

Abstract. In this paper, we give some sufficient and necessary conditions for normal CR-submanifold of a nearly Kaehlerian manifold, and generalize Bejancu's research work.

1. Introduction

Let $\overline{M}$ be a real differentiable manifold. An almost complex structure on $\overline{M}$ is a tensor field J of type (1, 1) on $\overline{M}$ such that at every point $x \in \overline{M}$, we have $J^2 = -I$, where I denotes the identify transformation of $T_x\overline{M}$. A manifold $\overline{M}$ endowed with an almost complex structure is called an almost complex manifold.

A linear connection $\overline{\nabla}$ on $\overline{M}$ is said to be a Riemannian connection if Riemannian metric g satisfies

$$Xg(Y,Z) = g(\overline{\nabla}_X Y, Z) + g(Y, \overline{\nabla}_X Z) \tag{1.1}$$

for any $X, Y \in \Gamma(T\overline{M})$.

The covariant derivative of J is defined by

$$(\overline{\nabla}_X J)Y = \overline{\nabla}_X JY - J\overline{\nabla}_X Y \tag{1.2}$$

for any $X, Y \in \Gamma(T\overline{M})$. More, we define the torsion tensor of J or the Nijenhuis tensor of J by

$$[J,J](X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] \tag{1.3}$$

for any $X, Y \in \Gamma(T\overline{M})$, where $[X,Y]$ is the Lie bracket of vector fields X and Y, that is, $[X,Y] = \overline{\nabla}_X Y - \overline{\nabla}_Y X$.

A Hermitian metric on an almost complex manifold $\overline{M}$ is a Riemannian metric g satisfying

$$g(JX,JY) = g(X,Y) \tag{1.4}$$

for any $X, Y \in \Gamma(T\overline{M})$.

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An almost complex manifold endowed with a Hermitian metric is said to be an almost Hermitian manifold.

Definition 1.1 ([**1**])**.** An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a nearly Kaehlerian manifold if we have

$$(\overline{\nabla}_X J)Y + (\overline{\nabla}_Y J)X = 0 \tag{1.5}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.2 ([**1**])**.** An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a Kaehlerian manifold if we have

$$\overline{\nabla}_X J = 0 \tag{1.6}$$

for any $X \in \Gamma(T\overline{M})$.

Obviously, a Kaehlerian manifold is a nearly Kaehlerian manifold.

Let M be an m-dimensional Riemannian submanifold of an n-dimensional Riemannian manifold $\overline{M}$. By $TM^{\perp}$, we denote the normal bundle to M and by g metric on M and $\overline{M}$. Also, we denote the Levi-Civita connection on $\overline{M}$ by $\overline{\nabla}$, the induced connection on M by ∇, and the induced normal connection on M by $\nabla^{\perp}$.

Then, for any $X, Y \in \Gamma(TM)$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.7}$$

where $h\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^{\perp})$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.7) is called the Gauss formula and h is called the second fundamental form of M.

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$ by $-A_V X$ and $\nabla^{\perp}_X V$, we denote the tangent part and normal part of $\overline{\nabla}_X V$, respectively. Then we have

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V. \tag{1.8}$$

Thus, for any $V \in \Gamma(TM^{\perp})$, we have a linear operator satisfying

$$g(A_V X, Y) = g(X, A_V Y) = g(h(X, Y), V). \tag{1.9}$$

The equation (1.8) is called the Weingarten formula.

An m-dimensional distribution on a manifold $\overline{M}$ is a mapping D defined on $\overline{M}$, which assignes to each point x of $\overline{M}$ an m-dimensional linear subspace D_x of $T_x\overline{M}$. A vector field X on $\overline{M}$ belongs to D if we have $X_x \in D_x$ for each $x \in \overline{M}$. When this happens, we write $X \in \Gamma(D)$. The distribution D is said to be differentiable if for any $x \in \overline{M}$, there exist m differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of x. From now on, all distributions are supposed to be differentiable of class C^{∞}.

Definition 1.3 ([**1**])**.** Let $\overline{M}$ be a real n-dimensional almost Hermitian manifold with almost complex structure J and with Hermitian metric g. Let M be a real m-dimensional Riemannian manifold isometrically immersed in $\overline{M}$. Then M is called a CR-submanifold of $\overline{M}$ if there exists a differentiable distribution

$$D\colon x \to D_x \subset T_x M$$

on M satisfying the following conditions:

(1) D is holomorphic, that is, $J(D_x) = D_x$ for each $x \in M$,

(2) the complementary orthogonal distribution

$$D^{\perp}\colon x \to D_x^{\perp} \subset T_xM$$

is anti-invariant, that is, $J(D_x^{\perp}) \subset T_xM^{\perp}$, for each $x \in M$.

Now let M be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold $\overline{M}$. For each vector field X tangent to M, we put

$$JX = \phi X + \omega X, \tag{1.10}$$

where ϕX and ωX, respectively, are the tangent part and the normal part of JX. Also, for each vector field V normal to M, we put

$$JV = BV + CV, \tag{1.11}$$

where BV and CV are respectively the tangent part and the normal part of JV.

We take account of the decomposition $T\overline{M} = D \oplus D^{\perp} \oplus JD^{\perp} \oplus \nu$. Obviously, we have $\phi X \in \Gamma(D)$, $\omega X \in \Gamma(JD^{\perp})$, $BV \in \Gamma(D^{\perp})$, $CV \in \Gamma(\nu)$ for any $X \in \Gamma(TM)$, $V \in \Gamma(JD^{\perp} \oplus \nu)$.

The covariant derivative of ϕ is defined by

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y \tag{1.12}$$

for any $X, Y \in \Gamma(TM)$. On the other hand, the covariant derivative of ω is defined by

$$(\nabla_X \omega)Y = \nabla_X^{\perp} \omega Y - \omega \nabla_X Y \tag{1.13}$$

for any $X, Y \in \Gamma(TM)$.

The exterior derivative of ω is defined by

$$d\omega(X, Y) = \frac{1}{2}\{\nabla_X^{\perp} \omega Y - \nabla_Y^{\perp} \omega X - \omega([X, Y])\} \tag{1.14}$$

for any $X, Y \in \Gamma(TM)$.

The Nijenhuis tensor of ϕ is defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \tag{1.15}$$

for any $X, Y \in \Gamma(TM)$, where $[X, Y]$ is the Lie bracket of vector fields X and Y.

We define two tensor fields S and S^*, respectively, by

$$S(X, Y) = [\phi, \phi](X, Y) - 2Bd\omega(X, Y) \tag{1.16}$$

and

$$S^*(Y, X) = (Ł_Y \phi)X = [Y, \phi X] - \phi[Y, X] \tag{1.17}$$

for any $X, Y \in \Gamma(TM)$.

Definition 1.4 ([**1**])**.** The CR-submanifold M is said to be normal if the tensor fields S vanishes identically on M.

2. Main Results

Lemma 2.1 ([**1**]). *Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then we have*

$$S(X,Y) = (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X + \phi\{(\nabla_Y\phi)X - (\nabla_X\phi)Y\} - B\{(\nabla_X\omega)Y - (\nabla_Y\omega)X\} \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$.

Lemma 2.2 ([**1**]). *Let $\overline{M}$ be a nearly Kaehlerian manifold. Then we have*

$$(\overline{\nabla}_X J)Y = \frac{1}{4}J[J,J](X,Y) \tag{2.2}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Lemma 2.3 (Frobenius [**1, 2**]). *The distribution $D^\perp$ is integrable if and only if*

$$[X,Y] \in \Gamma(D^\perp)$$

for any $X, Y \in \Gamma(D^\perp)$.

Lemma 2.4 (Urbano [**1**]). *Let M be a CR-submanifold of a nearly Kaehlerian manifold $\overline{M}$. Then the distribution $D^\perp$ is integrable if and only if*

$$g(\overline{\nabla}_X \mathbf{J})Y, Z) = 0 \tag{2.3}$$

for any $X, Y \in \Gamma(D^\perp), Z \in \Gamma(D)$.

Lemma 2.5. *Let M be a CR-submanifold of a nearly Kaehlerian manifold $\overline{M}$. Then we have*

$$(\nabla_X\phi)Y = A_{\omega Y}X + Bh(X,Y) + \frac{1}{4}(J[J,J](X,Y))^\top \tag{2.4}$$

and

$$(\nabla_X\omega)Y = -h(X,\phi Y) + Ch(X,Y) + \frac{1}{4}(J[J,J](X,Y))^\perp \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, using (1.7), (1.8), (1.10) and (1.11) in (1.2), we have

$$(\overline{\nabla}_X J)Y = \nabla_X\phi Y + h(X,\phi Y) - A_{\omega Y}X + \nabla^\perp_X\omega Y - \phi\nabla_X Y - \omega\nabla_X Y - Bh(X,Y) - Ch(X,Y). \tag{2.6}$$

Taking into account (1.12) and (1.13), (2.6) becomes

$$(\overline{\nabla}_X J)Y = (\nabla_X\phi)Y + h(X,\phi Y) - A_{\omega Y}X + (\nabla_X\omega)Y - Bh(X,Y) - Ch(X,Y). \tag{2.7}$$

Taking account of (2.2) and (2.7), we obtain

$$\frac{1}{4}J[J,J](X,Y) = (\nabla_X\phi)Y + h(X,\phi Y) - A_{\omega Y}X + (\nabla_X\omega)Y - Bh(X,Y) - Ch(X,Y). \tag{2.8}$$

By comparing the tangent part and the normal part in (2.8), we get (2.4) and (2.5). □

Theorem 2.1. *Let M be a CR-submanifold of a nearly Kaehlerian manifold $\overline{M}$. Then M is normal if and only if we have*

$$\begin{aligned} 0 = {} & A_{\omega Y}\phi X - \phi A_{\omega Y}X - A_{\omega X}\phi Y + \phi A_{\omega X}Y + \frac{1}{4}(J[J,J](\phi X, Y))^{\top} \\ & - \frac{1}{4}(J[J,J](\phi Y, X))^{\top} - \frac{1}{2}\phi(J[J,J](X,Y))^{\top} - \frac{1}{2}B(J[J,J](X,Y))^{\perp}, \end{aligned} \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, using (2.4) and (2.5), (2.1) becomes

$$\begin{aligned} S(X,Y) = {} & A_{\omega Y}\phi X - \phi A_{\omega Y}X - A_{\omega X}\phi Y + \phi A_{\omega X}Y \\ & + \frac{1}{4}(J[J,J](\phi X, Y))^{\top} - \frac{1}{4}(J[J,J](\phi Y, X))^{\top} \\ & - \frac{1}{2}\phi(J[J,J](X,Y))^{\top} - \frac{1}{2}B(J[J,J](X,Y))^{\perp}. \end{aligned} \tag{2.10}$$

Taking account of Definition 1.4, M is normal if and only if (2.9) holds. □

Theorem 2.2. *Let M be a CR-submanifold of a nearly Kaehlerian manifold $\overline{M}$ and*

$$[J,J](X,Y) \in \Gamma(\nu), \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$. Then M is normal if and only if we have

$$A_{\omega Y}\phi X = \phi A_{\omega Y}X \tag{2.12}$$

for any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$.

Proof. By using (2.11) in (2.10), we obatin

$$S(X,Y) = A_{\omega Y}\phi X - \phi A_{\omega Y}X - A_{\omega X}\phi Y + \phi A_{\omega X}Y \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$.

Suppose M is normal CR-submanifold of $\overline{M}$. Then (2.12) follows from (2.13) since $A_{\omega X} = 0$ for any $X \in \Gamma(D)$.

Now, if (2.12) is sastified, we prove $S = 0$ by means of the decomposition $TM = D \oplus D^{\perp}$.

First, for any $X, Y \in \Gamma(D)$, from (2.13), we have

$$S(X,Y) = 0.$$

Next, for any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$, using (2.12) in (2.13), we obtain

$$S(X,Y) = A_{\omega Y}\phi X - \phi A_{\omega Y}X = 0. \tag{2.14}$$

Finally, for any $X, Y \in \Gamma(D^{\perp})$, from (2.11), Lemma 2.2 and Lemma 2.4, the distribution $D^{\perp}$ is integrable.

Next, from (1.8) and (1.2), we have

$$\begin{aligned} g(A_{\omega X}Y - A_{\omega Y}X, Z) = {} & g(J\overline{\nabla}_X Y, Z) + g((\overline{\nabla}_X J)Y, Z) \\ & - g(J\overline{\nabla}_Y X, Z) - g((\overline{\nabla}_Y J)X, Z) \end{aligned} \tag{2.15}$$

for any $X, Y \in \Gamma(D^\perp)$, $Z \in \Gamma(TM)$.

Using (2.2) in (2.15), we get

$$g(A_{\omega X}Y - A_{\omega Y}X, Z) = g(J\overline{\nabla}_X Y, Z) - g(J\overline{\nabla}_Y X, Z) + g(\frac{1}{2}J[J,J](X,Y), Z). \tag{2.16}$$

Using (1.4) and (2.11) in (2.16), we obtain

$$g(A_{\omega X}Y - A_{\omega Y}X, Z) = g(-\overline{\nabla}_X Y + \overline{\nabla}_Y X, JZ) = -g([X,Y], \phi Z). \tag{2.17}$$

From (2.17), we know $A_{\omega X}Y - A_{\omega Y}X = 0$ since the distribution $D^\perp$ is integrable. More, (2.13) becomes

$$S(X,Y) = \phi(A_{\omega X}Y - A_{\omega Y}X) = 0 \tag{2.18}$$

for any $X, Y \in \Gamma(D^\perp)$.

From the above three conclusions, we know $S(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$, that is, the CR-submanifold M is normal. □

Corollary 2.1 (Bejancu [**1**]). *Let M be a CR-submanifold of a Kaehlerian manifold $\overline{M}$. Then M is normal if and only if we have*

$$A_{\omega Y}\phi X = \phi A_{\omega Y}X, \tag{2.19}$$

for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$.

Proof. Since a Kaehlerian manifold $\overline{M}$ is satisfied,

$$[J,J](X,Y) = 0 \in \Gamma(\nu)$$

for any $X, Y \in \Gamma(T\overline{M})$. Taking account of Theorem 2.2, Corollary 2.1 holds. □

Theorem 2.3. *Let M be a CR-submanifold of a nearly Kaehlerian manifold $\overline{M}$ with following conditions satisfing*

(1) (2.11) *holds,*

(2) *for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$, we have*

$$\nabla_X Y \in \Gamma(D). \tag{2.20}$$

Then M is normal if and only if we have

$$S^*(Y,X) = 0, \tag{2.21}$$

for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$.

Proof. From the proof Theorem 2.2 follows that M sastifying condition (2.11) is normal CR-submainifold if and only if $S(X,Y) = 0$ for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$. Using (1.12) and (2.5) in (2.1), we have

$$\begin{aligned} S(X,Y) &= (\nabla_{\phi X}\phi Y - \phi\nabla_{\phi X}Y) - (\nabla_{\phi Y}\phi X - \phi\nabla_{\phi Y}X) \\ &\quad + \phi(\nabla_Y\phi X - \phi\nabla_Y X) - \phi(\nabla_X\phi Y - \phi\nabla_X Y) \\ &\quad - B(h(Y,\phi X) + \frac{1}{2}(J[J,J](X,Y))^\perp) \end{aligned} \tag{2.22}$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$.

Using (2.11) in (2.22), we get

$$S(X,Y)=\phi(\phi[X,Y]-[\phi X,Y])-Bh(Y,\phi X). \tag{2.23}$$

Next, using (1.13) and (2.11) in (2.5), we obtain

$$h(Y,\phi X)=\omega\nabla_Y X+Ch(X,Y)+\frac{1}{4}(J[J,J](Y,X))^{\perp}. \tag{2.24}$$

That implies

$$J\omega\nabla_Y X=Bh(Y,\phi X)\in\Gamma(D^{\perp}). \tag{2.25}$$

Taking account of (1.17) and (2.25), (2.23) becomes

$$S(X,Y)=\phi S^*(Y,X)-J\omega\nabla_Y X. \tag{2.26}$$

Now suppose M is normal. Then by comparing D part and $D^{\perp}$ part in (2.26), we have

$$\phi S^*(Y,X)=0 \qquad \text{and} \qquad J\omega\nabla_Y X=0$$

More, we get

$$S^*(Y,X)\in\Gamma(D^{\perp}) \tag{2.27}$$

and

$$\nabla_Y X\in\Gamma(D). \tag{2.28}$$

On the other hand, using (2.20) and (2.28) in (1.17), we obtain

$$S^*(Y,X)=(\nabla_Y\phi X-\nabla_{\phi X}Y)-\phi[Y,X]\in\Gamma(D). \tag{2.29}$$

for any $X\in\Gamma(D)$, $Y\in\Gamma(D^{\perp})$. Combining (2.27) with (2.29), it follows that $S^*(Y,X)=0$ for any $X\in\Gamma(D)$, $Y\in\Gamma(D^{\perp})$,

Conversely, suppose (2.21) is saitisfied. Then from (1.17), (2.20) and (2.21) we have

$$\nabla_Y\phi X=\nabla_{\phi X}Y+\phi[Y,X]\in\Gamma(D). \tag{2.30}$$

Using (2.21) and (2.30) in (2.26), we get $S(X,Y)=0$ for any $X\in\Gamma(D)$, $Y\in\Gamma(D^{\perp})$. Moreover, taking into account Theorem 2.2, M is normal. □

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Ai Haihua, Dept. of Math.and Computing Science, Changsha University of Science and Technology, Changsha, Hunan, P. R. China, *e-mail*: aihhua888@163.com

Xie Li, Dept. of Math.and Computing Science, Changsha University of Science and Technology, Changsha, Hunan, P. R. China, *e-mail*: pkums900@163.com

Wan Yong, Dept. of Math.and Computing Science, Changsha University of Science and Technology, Changsha, Hunan, P. R. China, *e-mail*: wanyong870901@foxmail.com