Acta Math. Univ. Comenianae
Vol. LXXXV, 2 (2016), pp. 311–318

EXISTENCE RESULTS FOR THIRD-ORDER DIFFERENTIAL INCLUSIONS WITH THREE-POINT BOUNDARY VALUE PROBLEMS

A. REZAIGUIA and S. KELAIAIA

Abstract. In this paper, we investigate the solutions for a third-order differential inclusion with three-point boundary value problem. First, we apply the Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo. Secondly, our result is based on the fixed point theorem for multivalued maps due to Covitz and Nadler.

1. Introduction

Realistic problems arising from economics, optimal control, stochastic analysis can be modelled as differential inclusions. So much attention has been paid by many autors to study this kind of problems, see [**5, 6**] and the references therein.

By using Shaefer's fixed point theorem and some results on selections for lower semicontinuous multivalued maps, in this work, we prove some existence results for the three point boundary value problem of the third-order differential inclusion

$$-u'''(t) \in F(t,u(t)), \qquad t \in (0,1), \tag{1.1}$$

$$u'(0) = u'(1) = \alpha u(\eta), \qquad u(0) = \beta u(\eta), \tag{1.2}$$

where α, β and η are constants with $\alpha \in [0, \frac{1}{\eta})$, $0 < \eta < 1$, $\beta \neq 1 - \alpha\eta$, $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$. In [**12**], the authors discussed the existence of positive solutions to the problem (1.1) and (1.2) with $F(t,u(t))$ as a single-valued map ($F(t,u(t)) = a(t)f(t,u(t))$) and $\beta = 0$.

This paper is organized as follows. In Section 2, we present some theorems and lemmas that are used to prove our main results. In Section 3, we present existence results for the problem (1.1) and (1.2) when the right-hand side is nonconvex, where at first we apply the Schaefer's fixed point theorem (see [**1**, p. 29]) combined with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values [**2**]. The second

Received June 23, 2015; revised May 12, 2016.
2010 *Mathematics Subject Classification.* Primary .
Key words and phrases. Third-order differential inclusion; three point boundary value problem; fixed point theorem; selection theorem.

result is based on the fixed point theorem contraction multivalued maps due to Covitz and Nadler [**3**].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Here $C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\| = \sup\{|u(t)| : \text{ for all } t \in [0,1]\}$, $L^1([0,1],\mathbb{R})$, the Banach space of measurable functions $u\colon [0,1] \to \mathbb{R}$ which are Lebesgue integrable, normed by $\|u\|_{L^1} = \int_0^1 |u(t)|\mathrm{d}t$, and $AC^i([0,1],\mathbb{R})$ the space of i-times differentiable functions $u\colon [0,1] \to \mathbb{R}$ whose i-th derivative $u^{(i)}$ is absolutely continuous.

Let A be a subset of $[0,1] \times\mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{I} \times \mathcal{D}$, where $\mathcal{I}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is said to be decomposable if for all $u, v \in \mathcal{A}$ and $\mathcal{I} \subset [0,1] = I$ measurable, the function $u\chi_{\mathcal{I}} + v\chi_{I-\mathcal{I}} \in \mathcal{A}$ where χ denotes the characteristic function.

Let (X,d) be a metric space induced from the normed space $(X, \|\cdot\|)$. We denote

$$\mathcal{P}_0(X) = \{A \in \mathcal{P}(X) : A \neq \phi\}\,,$$
$$\mathcal{P}_{\mathrm{cl}}(X) = \{A \in \mathcal{P}_0(X) : A \text{ is closed}\}\,,$$
$$\mathcal{P}_{\mathrm{b}}(X) = \{A \in \mathcal{P}_0(X) : A \text{ is bounded}\}\,,$$
$$\mathcal{P}_{\mathrm{comp}}(X) = \{A \in \mathcal{P}_0(X) : A \text{ is compact}\}\,.$$

Consider $H_d\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$ and $d(b,A) = \inf_{a\in A} d(a,b)$. Then $(\mathcal{P}_{\mathrm{b,cl}}(X), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space see [**7**].

Let E be a separable Banach space, Y a nonempty closed subset of E and $G\colon Y \to \mathcal{P}_{\mathrm{cl}}(E)$ a multivalued operator. G is said to be lower semicontinuous (l.s.c.) if the set $\{x \in Y : G(x) \cap U \neq \emptyset\}$ is open for any open set U in E. G has a fixed point if there is $x \in Y$ such that $x \in G(x)$.

For more details on the multi-valued maps, see the books of Aubin and Cellina [**4**], Aubin and Frankowska [**8**], Deimling [**9**], Górniewicz [**10**] and Hu and Papageorgiou [**11**].

Definition 1. Let Y be a separable metric space and let $N\colon Y \to \mathcal{P}_0(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has the property (BC) if

1. N is lower semi-continuous (l.s.c),
2. N has nonempty closed and decomposable values.

Let $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}_{\mathrm{comp}}(\mathbb{R})$ be a multivalued map. Assign to F the multivalued operator

$$\mathcal{F}\colon C([0,1],\mathbb{R}) \to \mathcal{P}_0(L^1([0,1],\mathbb{R}))$$

by letting

$$\mathcal{F}(u) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,u(t)) \text{ for a.e. } t \in [0,1]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F. We say F is of the lower semi-continuous type (l.s.c type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Lemma 1 ([**2**]). *Let Y be a separable metric space and let $N\colon Y \to \mathcal{P}_0(L^1([0,1],\mathbb{R}))$ be a multivalued operator which has the property (BC). Then N has a continuous selection, i.e., there exists a continuous function (single-valued) $g\colon Y \to L^1([0,1],\mathbb{R})$ such that $g(u) \in N(u)$ for every $u \in Y$.*

Definition 2. A multivalued operator $N\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(Nx, Ny) \le \gamma d(x,y)$ for each $x, y \in X$,

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 2 ([**3**]). *Let (X,d) be a complete metric space. If $N\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

3. Existence Results

By the help of Schaefer's theorem combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, first we shall present an existence result for the problem (1.1) and (1.2). Before this, let us introduce the following hypotheses which are assumed hereafter:

(H_1) $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}_{\mathrm{comp}}(\mathbb{R})$ be a multivalued map verifying:

a) $(t,u) \to F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable.

b) $u \to F(t,u)$ is lower semicontinuous for a.e. $t \in [0,1]$.

(H_2) F is integrably bounded, that is, there exists a function $m \in L^1([0,1],\mathbb{R}_+)$ such that $\|F(t,u)\| = \sup\{\|v\| : v \in F(t,u)\} \leqslant m(t)$ for almost all $t \in [0,1]$.

Lemma 3 ([**13**]). *Let $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}_{\mathrm{comp}}(\mathbb{R})$ be a multivalued map. Assume* (H_1) *and* (H_2) *hold. Then F is of the* l.s.c. *type.*

Definition 3. A function $u \in AC^2([0,1],\mathbb{R})$ is called a solution to the BVP (1.1) and (1.2) if u satisfies the differential inclusion (1.1) a.e. on $[0,1]$ and the condition (1.2).

In the first result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problem is based on Schaefer's fixed point theorem with the selection theorem of Bressan and Colombo [**2**] for lower semicontinuous maps with decomposable values.

Theorem 1. *Suppose that hypothesis* (H_1) *and* (H_2) *hold. Then the problem* (1.1) *and* (1.2) *has at least one solution.*

Proof. (H_1) and (H_2) imply by Lemma 3 that F is of the lower semi-continuous type. Then from Lemma 1, there exists a continuous function $g\colon C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $g(u) \in \mathcal{F}(u)$ for all $u \in C([0,1],\mathbb{R})$.

We consider the problem

$$-u''' = g(u), \qquad \text{a.e. } t \in [0,1], \tag{3.1}$$

$$u'(0) = u'(1) = \alpha u(\eta), \qquad u(0) = \beta u(\eta). \tag{3.2}$$

Remark 1. If $u \in C([0,1],\mathbb{R})$ is a solution to the problem (3.1) and (3.2), then u is a solution to the problem (1.1) and (1.2).

Transform problem (3.1) and (3.2) into a fixed point problem. Consider the operator $T\colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$, defined by

$$\begin{aligned} T(u)(t) = & -\frac{1}{2}\int_0^t (t-s)^2 g(u)\mathrm{d}s + \frac{1}{2}[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}] \int_0^1 (1-s) g(u)\mathrm{d}s \\ & -\frac{1}{2}\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 g(u)\mathrm{d}s. \end{aligned}$$

We show that T is a compact operator.

Step 1. T is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned} |T(u_n)(t) - T(u)(t)| \le & \frac{1}{2}\int_0^t (t-s)^2 \left|g(u_n) - g(u)\right| \mathrm{d}s \\ & + \frac{1}{2}[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}] \int_0^1 (1-s)|g(u_n) - g(u)|\mathrm{d}s \\ & + \frac{1}{2}\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 |g(u_n) - g(u)|\mathrm{d}s, \end{aligned}$$

Since g is continuous, then

$$\|T(u_n) - T(u)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. T is bounded on bounded sets of $C([0,1],\mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in T(u), u \in B_r = \{u \in C([0,1],\mathbb{R}) : \|u\| \le r\}$, one has $\|h\| \le c$. By (H_2), we have for each $t \in [0,1]$,

$$|h(t)| \le \left[1 + \frac{\eta^2}{2}\Big|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\Big|\right] \int_0^1 m(s)\mathrm{d}s + \frac{\eta^2}{2}\Big|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\Big| \int_0^\eta m(s)\mathrm{d}s = c.$$

Then $|h| \le c$.

Step 3. T sends bounded sets of $C([0,1],\mathbb{R})$ into equicontinuous sets.

Let $t_1, t_2 \in [0,1], t_1 < t_2$, and B_r be a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$\begin{aligned}
&|h(t_2)-h(t_1)|\\
&\le \frac{1}{2}\int_{t_1}^{t_2}(t_2-s)^2|g(u)|\mathrm{d}s+\frac{1}{2}\left[t_2^2-t_1^2+\eta^2\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\right]\int_0^1(1-s)|g(u)|\mathrm{d}s\\
&\quad+\frac{1}{2}\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\int_0^\eta(\eta-s)^2|g(u)|\mathrm{d}s+\frac{1}{2}\int_0^{t_1}((t_1-s)^2-(t_2-s)^2)|g(u)|\mathrm{d}s,\\
&\le \frac{1}{2}\int_{t_1}^{t_2}(t_2-s)^2m(s)\mathrm{d}s+\left[\frac{t_2^2-t_1^2}{2}+\frac{\alpha\eta^2(t_2-t_1)}{2|1-\alpha\eta-\beta|}\right]\int_0^1(1-s)m(s)\mathrm{d}s\\
&\quad+\frac{1}{2}\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\int_0^\eta(\eta-s)^2m(s)\mathrm{d}s+\frac{1}{2}\int_0^{t_1}((t_1-s)^2-(t_2-s)^2)m(s)\mathrm{d}s.
\end{aligned}$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of *Steps 1* to *3*, together with the Arzela-Ascoli theorem, we can conclude that T is completely continuous.

In order to apply Schaefer's theorem, it remains to show next step.

Step 4 The set

$$\Omega=\{u\in C([0,1],\mathbb{R}):\lambda u=T(u)\text{ for some }\lambda>1\}$$

is bounded.

Let $u \in \Omega$. Then $\lambda u = T(u)$ for some $\lambda > 1$ and

$$\begin{aligned}
u(t)=&-\frac{\lambda^{-1}}{2}\int_0^t(t-s)^2g(u)\mathrm{d}s+\frac{\lambda^{-1}}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1(1-s)g(u)\mathrm{d}s\\
&-\frac{\lambda^{-1}}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta(\eta-s)^2g(u)\mathrm{d}s,
\end{aligned}$$

this implies by (H_2) that for each $t\in[0,1]$, we have

$$\begin{aligned}
|u(t)|\le&\ \frac{1}{2}\int_0^t(t-s)^2m(s)\mathrm{d}s+\frac{1}{2}\left[t^2+\eta^2\left|\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right|\right]\int_0^1(1-s)m(s)\mathrm{d}s\\
&+\frac{1}{2}\left|\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right|\int_0^\eta(\eta-s)^2m(s)\mathrm{d}s,
\end{aligned}$$

thus

$$\begin{aligned}
|u|\le&\ \frac{1}{2}\int_0^1(t-s)^2m(s)\mathrm{d}s+\frac{1}{2}\left[1+\eta^2\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right]\int_0^1(1-s)m(s)\mathrm{d}s\\
&+\frac{1}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\int_0^\eta(\eta-s)^2m(s)\mathrm{d}s=K.
\end{aligned}$$

This shows that Ω is bounded.

As a consequence of Schaefer's theorem (see [**1**, p. 29]), we deduce that T has a fixed point which is a solution to (3.1) and (3.2), and hence from Remark 1, a solution to the problem (1.1) and (1.2). □

Now we prove the existence of solutions to the problem (1.1) and (1.2) with a non convex valued right-hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [**3**].

Theorem 2. *Assume that*

(H_3) $F\colon [0,1]\times\mathbb{R}\to\mathcal{P}_{\text{comp}}(\mathbb{R})$ *is such that* $F(\cdot,u)[0,1]\times\mathbb{R}\to\mathcal{P}_{\text{comp}}(\mathbb{R})$ *is measurable for each* $t\in[0,1]$.

(H_4) $H_d(F(t,u),F(t,\overline{u}))\le p(t)|u-\overline{u}|$ *for almost all* $t\in[0,1]$ *and* $u,\overline{u}\in\mathbb{R}$ *with* $p\in L^1([0,1],\mathbb{R}^+)$ *and* $d(0,F(t,0))\le p(t)$ *for almost all* $t\in[0,1]$.

Then the problem (1.1) *and* (1.2) *has at least one solution on* $[0,1]$ *if*

$$\left[1+\frac{1+\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right]\|p\|_{L^1}<1.$$

Proof. For each $u\in C([0,1]\times\mathbb{R})$, define the set of selections of F by

$$S_{F,u}:=\{w\in L^1([0,1],\mathbb{R}):w(t)\in F(t,u(t))\text{ for a.e. }t\in[0,1]\},$$

and the multi-valued operator $\Omega: C([0,1]\times\mathbb{R})\to\mathcal{P}_{\text{cl}}(C([0,1]\times\mathbb{R}))$ by

$$\begin{aligned}\Omega(u)=\bigg\{h\in C([0,1]\times\mathbb{R}):h(t)&=-\frac{1}{2}\int_0^t(t-s)^2f(u)\mathrm{d}s\\&+\frac{1}{2}\Big[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\Big]\int_0^1(1-s)f(u)\mathrm{d}s.\\&-\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta(\eta-s)^2f(u)\mathrm{d}s,\ t\in[0,1]\bigg\}\end{aligned}$$

for $f\in S_{F,u}$. Observe that the set $S_{F,u}$ is nonempty for each $u\in C([0,1]\times\mathbb{R})$, by the assumption ($H_3$), so F has a measurable selection (see [**14**, Theorem III.6]). Now we show that the operator Ω satisfies the assumptions of Lemma 2. To show that $\Omega(u)\in\mathcal{P}_{\text{cl}}C([0,1]\times\mathbb{R})$, for each $u\in C([0,1]\times\mathbb{R})$, let $\{v_n\}_{n\ge0}\in\Omega(u)$ be such that $v_n\to v$ ($n\to\infty$) in $C([0,1]\times\mathbb{R})$. Then $v\in C([0,1]\times\mathbb{R})$ and there exists $w_n\in S_{F,u}$ such that for each $t\in[0,1]$,

$$\begin{aligned}v_n(t)=&-\frac{1}{2}\int_0^t(t-s)^2w_n(s)\mathrm{d}s+\frac{1}{2}\Big[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\Big]\int_0^1(1-s)w_n(s)\mathrm{d}s\\&-\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta(\eta-s)^2w_n(s)\mathrm{d}s.\end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that w_n converges to w in $L^1([0,1]\times\mathbb{R})$. Thus, $w\in S_{F,u}$ and for each $t\in[0,1]$

$$\begin{aligned}v_n(t)\to v(t)=&-\frac{1}{2}\int_0^t(t-s)^2w(s)\mathrm{d}s+\frac{1}{2}\Big[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\Big]\int_0^1(1-s)w(s)\mathrm{d}s\\&-\frac{1}{2}\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^\eta(\eta-s)^2w(s)\mathrm{d}s,\end{aligned}$$

Hence, $v\in\Omega(u)$.

Next we show that there exists $\gamma<1$ such that

$$H_d(\Omega u,\Omega\overline{u})\le\gamma\|u-\overline{u}\|\qquad\text{for each }u,\overline{u}\in C([0,1]\times\mathbb{R}).$$

Let $u, \overline{u} \in C([0,1] \times \mathbb{R})$ and $h_1 \in \Omega(u)$. Then there exists $v_1(t) \in S_{F,u}$ such, that for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) = & -\frac{1}{2}\int_0^t (t-s)^2 v_1(s)\mathrm{d}s + \frac{1}{2}\Big[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\Big]\int_0^1 (1-s)v_1(s)\mathrm{d}s \\ & -\frac{1}{2}\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 v_1(s)\mathrm{d}s. \end{aligned}$$

By H_4, we have

$$H_d(F(t,u), F(t,\overline{u})) \le p(t)|u(t) - \overline{u}(t)|.$$

So, there exists $w \in S_{F,\overline{u}}$ such that

$$|v_1 - w| \le p(t)|u - \overline{u}|, \qquad t \in [0,1].$$

Define $\mathcal{U}\colon [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{U}(t) = \{w \in \mathbb{R} : |v_1 - w| \le p(t)|u(t) - \overline{u}(t)|\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}(t) \cap F(t, \overline{u}(t))$ is measurable ([**14**, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for $\mathcal{V}$. So $v_2(t) \in S_{F,\overline{u}}$, and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \le p(t)|u(t) - \overline{u}(t)|$. For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) = & -\frac{1}{2}\int_0^t (t-s)^2 v_2(s)\mathrm{d}s + \frac{1}{2}\Big[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\Big]\int_0^1 (1-s)v_2(s)\mathrm{d}s \\ & -\frac{1}{2}\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 v_2(s)\mathrm{d}s, \\ h_1(t) = & -\frac{1}{2}\int_0^t (t-s)^2 v_1(s)\mathrm{d}s + \frac{1}{2}\Big[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\Big]\int_0^1 (1-s)v_1(s)\mathrm{d}s \\ & -\frac{1}{2}\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\int_0^\eta (\eta-s)^2 v_1(s)\mathrm{d}s. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| \le \frac{1}{2}\int_0^t & (t-s)^2 |v_1(s) - v_2(s)|\mathrm{d}s + \frac{1}{2}\Big| t^2 \\ & + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\Big| \int_0^1 (1-s)\,|v_1(s) - v_2(s)|\,\mathrm{d}s \\ & + \frac{1}{2}\Big|\frac{\alpha t + \beta}{1-\alpha\eta-\beta}\Big| \int_0^\eta (\eta-s)^2 |v_1(s) - v_2(s)|\mathrm{d}s, \\ \le & \ \Big[1 + \frac{1+\eta^2}{2}\Big|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\Big|\Big] \int_0^1 p(s)\,|u(s) - \overline{u}(s)|\,\mathrm{d}s. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le \Big[1 + \frac{1+\eta^2}{2}\Big|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\Big|\Big] \|p\|_{L^1}\|u - \overline{u}\|.$$

Analogously, interchanging the roles of u and $\overline{u}$, we obtain

$$H_d(\Omega(u),\Omega(\overline{u})) \leq \gamma\|u-\overline{u}\| \leq \left[1+\frac{1+\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right]\|p\|_{L^1}\|u-\overline{u}\|.$$

Since Ω is a contraction, from Lemma 2, it follows that Ω has a fixed point u which is a solution to the problem (1.1) and (1.2). This completes the proof. □

References

1. Smart D. R., *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1974.

2. Bressan A. and Colombo G., *Extensions and selections of maps with decomposable values*, Studia Math., **90(1)** (1988), 69–86.

3. Covitz, H. and Nadler S. B. Jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., **8** (1970), 5–11.

4. Aubin J. P. and Cellina A., *Differential Inclusions*, Springer-Verlag, 2012.

5. Smirnov G. V., *Introduction to the Theory of Differential Inclusions*, vol. 41 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, USA, 2002.

6. Simsen J. and Gentile C. B., *Systems of p-Laplacian differential inclusions with large diffusion*, J. Math. Anal. Appl., **368(2)** (2010), 525–537.

7. Kisielewicz M., *Differential Inclusions and Optimal Control*, Mathematics and Its Applications (East European Series), vol. 44, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

8. Aubin J. P. and Frankowska H., *Set-Valued Analysis*, Birkhauser, Boston 1990.

9. Deimling K., *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York 1992.

10. Górniewicz L., *Topological Fixed Point Theory of Multivalued Map pings*, Mathematics and Its Applications vol. 495, Kluwer Academic Publishers, Dordrecht 1999.

11. Hu S. and Papageorgiou N., *Handbook of Multivalued Analysis*, *Volume I: Theory*, Kluwer Academic Publishers, Dordrecht-Boston-London 1997.

12. Torres F. J., Positive Solution For A Third-Order three point boundary value Problem, Electronic J. of Differential Equations, **2013** (2013), No. 147, 1–11.

13. Frigon, M. and Granas, A.: *Théorèmes d'existence pour des inclusions différentielles sans convexité*, C. R. Acad. Sci. Paris, Ser. I., **310**, (1990), 819–822.

14. Castaing C. and Valadier M., *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, Springer, Berlin, 1977.

A. Rezaiguia, Univ. Souk Ahras Fact sci Dep MI 41000 Souk Ahras Algeria, Department of Mathematics, Faculty of Sciences, University of Annaba, P.O. Box12 Annaba, Algeria,
e-mail: `ali_rezaig@yahoo.fr`

S. Kelaiaia, Department of Mathematics, Faculty of Sciences, University of Annaba, P.O. Box12 Annaba, Algeria,
e-mail: `kelaiaiasmail@yahoo.fr`