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STRONG CONGRUENCE RELATION IN ATOMISTIC LATTICES

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Abstract. In this paper, for an atomistic lattice L satisfying the ascending and descending chain conditions, it is proved that:

- the congruence lattice Con L is isomorphic to the sublattice of the set of all standard elements of L;
- every congruence relation of L is representable.

Further, the notion of a strong congruence relation in lattices is introduced, some examples of strong congruence relations are given and it is proved that in a sectionally complemented lattice satisfying the ascending and descending chain conditions, every congruence relation is strong.

1. Introduction

Atomistic lattices are proved to be very useful class of Lattices, mainly in software engineering ([**10**], atomistic lattices are called atomic partitions). L. Libkin[**6**], S. Radeleczki [**8**] studied atomistic lattices and obtained some structure theorems for atomistic algebraic lattices. B. Šešelja and A. Tepavčevič [**11**] studied atomistic weak congruence lattices and proved that in an atomistic algebraic lattice, every codistributive element has a complement which is standard.

We observe that an atomistic lattice satisfying the ascending and descending chain condition has some interesting properties. In the first section of this paper, we prove some results for a particular type of atomistic lattices and the results used to build a theory for strong congruence relation in the second section.

Let L be lattice and $CS(L)$ the set of all (nonempty) convex sublattices of L. S. Lavanya and S. Parameshwara Bhatta proved that there is a partial order on $CS(L)$ with respect to which $CS(L)$ is a lattice such that both L and $CS(L)$ are in the same equational class ([**4**]). They also gave some characterization of the class K of all lattices L for which every quotient lattice is a sublattice of $CS(L)$. One of the important result they proved is that a finite distributive lattice L belongs to K if and only if it is isomorphic to a direct product of chains. Recently, Dwight Duffus et al. [**1**] studied a fixed point property on $CS(L)$.

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Continuing the study, in Section 4, the notion of a strong congruence relation in lattices is introduced and it is proved that in a sectionally complemented lattice satisfying the ascending and descending chain conditions, every congruence relation is strong.

2. Preliminaries

Let L be a lattice and $CS(L)$ the set of all convex sublattices of L. Define a partial order $\leq$ on $CS(L)$ by for A, $B \in CS(L)$, $A \leq B$ if for each $a \in A$ there exists $b \in B$ such that $a \leq b$, and for each $b \in B$, there exists $a \in A$ such that $b \geq a$. Then $(CS(L); \leq)$ is a lattice called the *lattice of convex sublattices* of L and it is denoted by $CS(L)$, see [**3**]. A simple structure L and its $CS(L)$ given in the Figure 1.

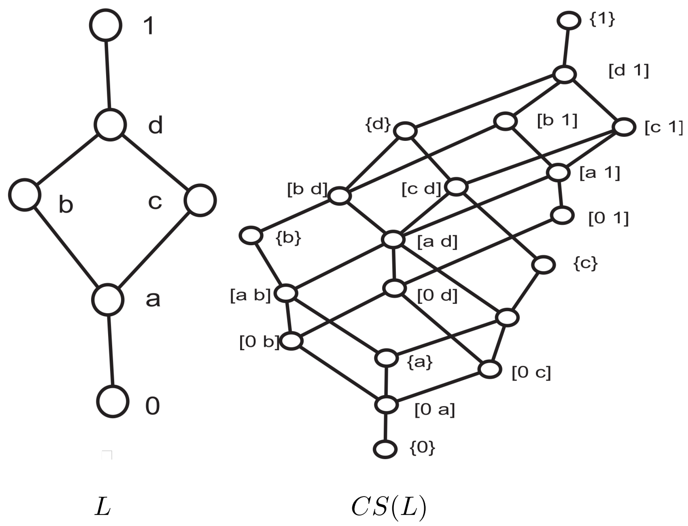

Figure 1.

Let L be a lattice and a be an element of L.

(1) The element a is called *distributive* if

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$$

for all $x, y \in L$.

($1^{\perp}$) The element a is called *codistributive*(or *dually distributive*) if

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$$

for all $x, y \in L$.

(2) The element a is called *standard* if

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$$

for all $x, y \in L$.

($2^{\perp}$) The element a is called *costandard* (or *dually standard*) if

$$x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y)$$

for all $x, y \in L$.

An ideal I of a lattice L is called standard if I is a standard element of $I(L)$, the lattice of all ideals of L.

$A(L)$ denotes set of atoms of a lattice L and $J(L)$ denotes set of join-irreducibles of L.

For $x \in L$, $A(x) := A(L) \cap (x]$.

For a, $b \in L$, $\Theta(a, b)$ denotes the principal congruence relation of L generated by a, b. For an element s of a lattice L, the congruence relation $\Theta_s = \Theta[(s]]$=the principal congruence relation generated by the ideal $(s]$.

Let L be a lattice and Θ be a congruence relation of L. If the quotient latticeL/Θ has the minimum element, say $[a]\Theta$, then $[a]\Theta$ as a subset of L is an ideal called the ideal kernel of the congruence relation Θ.

A congruence relation Θ of a lattice L is said to be *representable* if there is a sublattice L_1 of L such that the map $f\colon L_1 \to L/\Theta$, $a \mapsto [a]\Theta$, defines an isomorphism, see [**2**].

A lattice L with a minimum element 0 is called: (i) *atomistic* if every non-zero element a of L is the join of atoms contained in it, (ii) *sectionally complimented* if every interval [0 a] complimented. A sectionally complimented lattice is always atomistic.

Following results can be easily verified and will be often used in this paper.

Lemma 2.1. *If s is a distributive element of a lattice L, then*

(1) *for any $y \in L$, $y \vee s = \max[y]\Theta_s$,*
(2) $[s) = \{a \in L | \ a = \max[y]\Theta_s$ *for some* $y \in L\}$.

Lemma 2.2. *If s is a distributive element of a lattice L, then Θ_s is representable.*

Lemma 2.3.

(1) *In an atomistic lattice, $J(L) = A(L)$.*
(2) *Let L be a lattice. Then for any $a, b \in L$,*

$$J(a \wedge b) = J(a) \cap J(b) \text{ and } J(a \vee b) \supseteq J(a) \cup J(b).$$

(3) *An atomistic lattice L satisfying the ascending chain condition is algebraic.*

3. Atomistic lattices

Throughout this section, L is an atomistic lattice satisfying the ascending and descending chain conditions. Note that by Lemma 2.3, L is an algebraic lattice.

The following result of [**6**] will be frequently used in this paper.

Lemma 3.1. *For an element a of $\in L$, the following are equivalent:*

(1) *a is distributive;*
(2) *a is standard;*

(3) *for any* $y \in L$, $A(a \vee y) = A(a) \cup A(y)$.

The following result ([**7**]) on trellises will be used to prove our next result. Note that the notion of trellis introduced by H. Skala [**?**] is a natural generalization of a lattice. Therefore, the results on trellises also hold good for lattices.

Theorem 3.2. *The following statements are equivalent for a trellis* L*:*

1. *Every congruence relation has a kernel and every ideal is a congruence class under, at most, one congruence relation,*
2. L *is bounded below and if* $I = [0](\Theta(a, b))$ *for some* a, $b \in L$, *then* $\Theta[I] = \Theta(a, b)$.

Lemma 3.3. *Every ideal of* L *is the kernel of at most one congruence relation.*

Proof. Let $I = [0](\Theta(a, b))$ for some $a, b \in L$. To prove the theorem, by 2) of the above theorem, it is required to prove that $\Theta[I] = \Theta(a, b)$.

Evidently $\Theta[I] \subseteq \Theta(a, b)$. To prove the reverse inclusion, consider $x, y \in L$ with $x \equiv y(\Theta(a, b))$. Clearly, it can be assumed that $x \leq y$. Since L satisfies the ascending and descending chain condition, every chain connecting x and y is finite. Let $x = a_0 \prec a_1 \prec \cdots \prec a_n = y$ be any maximal chain connecting x and y; we prove that $x \equiv y(\Theta([I])$ by induction on n. For $n = 0$, nothing to prove.

Assume the result for $n = k$ for $k \geq 0$, i.e., if $x \equiv y(\Theta(a, b))$ and $x = a_0 \prec a_1 \prec \cdots \prec a_k = y$ any maximal chain connecting x and y, then $x \equiv y(\Theta([I])$.

We prove the result for $n = k + 1$.

Let $x \equiv y(\Theta(a, b))$ and $x = a_0 \prec a_1 \prec \cdots \prec a_{k+1} = y$ be any maximal chain connecting x and y. By induction hypothesis, $x = a_0 \equiv a_k(\Theta[I])$. Now, since $a_k < a_{k+1}$, there exists an atom $p \leq a_{k+1}$ such that $p \wedge a_k = 0$. Clearly, $0 = p \wedge a_k \equiv p \wedge a_{k+1} = p(\Theta(a, b))$. But then $p \in [0]\Theta(a, b) = I$ so that $0 \equiv p(\Theta[I])$. Now, $a_k \equiv p \vee a_k = a_{k+1} = y(\Theta[I])$. Thus $x = a_0 \equiv a_{k+1}(\Theta[I])$. Hence result is true for all n by mathematical induction. □

A congruence relation Θ of a lattice L is said to be standard if $\Theta = \Theta[I]$ where I is a standard ideal of L.

Lemma 3.4. *Every congruence relation of* L *is standard.*

Proof. Let I be the ideal kernel of a congruence relation. Since L satisfies the ascending chain condition, every ideal of L is principal. Therefore, $I = (a]$ for some element $a \in L$. We prove that the element a is standard. Suppose that a is not standard. By Lemma 3.1, there exists $b \in L$ such that $A(a \vee b) \neq A(a) \cup A(b)$. Let $p \in A(a \vee b) - A(a) \cup A(b)$.

We have

$$0 \equiv a(\Theta_a) \tag{3.1}$$

and

$$b = 0 \vee b \equiv a \vee b(\Theta_a). \tag{3.2}$$

Then $0 = b \wedge p \equiv (a \vee b) \wedge p = p(\Theta_a)$.

Now, from (3.1) and (3.2),

$$0 \equiv a \vee p(\Theta_a),$$

i.e,

$$a \vee p \in [0]\Theta_a = (a]. \tag{3.3}$$

Since $(a]$ is the ideal kernel, (3.3) implies that $a \vee p = a$ or equivalently $p \leq a$. Hence $p \in A(a)$, a contradiction. □

Theorem 3.5. *Every congruence relation of L is representable.*

Proof. Follows from Lemma 3.4, Lemma 3.1 and Lemma 2.2. □

We know that in any lattice, the set of all standard elements forms a sublattice of the lattice. One more result about standard elements is the following theorem.

Theorem 3.6 ([**2**])**.** *The map $a \to \Theta_a$ for standard elements is an embedding of the sublattice of standard elements into the congruence lattice.*

Now we prove that there is an isomorphism between sublattice of standard elements of L (an atomistic lattice satisfying the ascending and descending chain conditions) and the congruence lattice of L.

Theorem 3.7. *Let S denote the set of all standard elements of L. Then the map $f : S \to \operatorname{Con} L$ defined by $f(s) = \Theta_s$ is an isomorphism.*

Proof. Clearly, by Theorem 3.6, f is a one-to-one homomorphism. Also f is onto. In fact, if $(s] = [0]\Theta$, then by Lemma 3.4, $s \in S$ and by Lemma 3.3, $f(s) = \Theta_s$. □

4. Strong congruence relations

First we define new notion, strong congruence relation as follows:

Definition 4.1. Let L be a lattice. A congruence relation Θ is said to be *strong* if L/Θ is a sublattice of $CS(L)$. If L/Θ is a meet-subsemilattice or a join-subsemilattice of $CS(L)$, then we call Θ a *meet-strong* or *join-strong*, respectively.

It is known (see [**4**]) that in a relatively complemented lattice and in a finite distributive lattice which is direct product of chains, every congruence relation is strong. Further, it may be observed that even in a distributive lattice, a congruence relation need not be either meet-strong or join-strong (In the distributive lattice L of Figure 2, the congruence relation $\Psi = \Theta(0, a) \cup \Theta(d, 1)$ is neither meet-strong nor join-strong).

The following is a characterization theorem for strong congruence relation in complete lattices.

Theorem 4.2. *Let L be a complete lattice and let Θ be a congruence relation of L. Then following statements are equivalent:*

(1) *Θ is strong;*

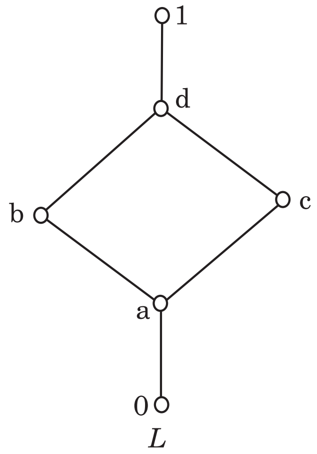

Figure 2.

(2) *Let* $P = \{a \in L|\ a = \max[x]\Theta$ *for some* $x \in L\}$
and $Q = \{a \in L|\ a = \min[x]\Theta$ *for some* $x \in L\}$.
Then P *and* Q *are sublattices of* L.

The proof of this theorem is immediate from the following Lemma.

Lemma 4.3. *Let* L *be a lattice and let* Θ *be a congruence relation of* L *such that each congruence class of* Θ *has a maximal element. Then following statements are equivalent:*

(1) Θ *is join-strong;*
(2) $P = \{a \in L|\ a = \max[x]\Theta$ *for some* $x \in L\}$ *is a sublattice of* L.

Proof. (1) $\Rightarrow$ (2): Let Θ be a join-strong congruence relation. Let

$$P = \{a \in L|\ a = \max[x]\Theta \text{ for some } x \in L\}$$

and let $a, b \in P$. To prove that $a \vee b = \max[a \vee b]\Theta$, consider $c \in [a \vee b]\Theta$. Since $[a \vee b]\Theta = [a]\Theta \underset{CS(L)}{\vee} [b]\Theta$, there exist $a_1 \in [a]\Theta, b_1 \in [b]\Theta$ such that $c \leq a_1 \vee b_1$. Now, since $a = \max[a]\Theta$ and $b = \max[b]\Theta$, $a_1 \leq a$ and $b_1 \leq b$ so that $c \leq a \vee b$.

To prove that $a \wedge b = \max[a \wedge b]\Theta$, consider $c \in [a \wedge b]\Theta$.

Since $c \equiv a \wedge b(\Theta)$, we get

$$a \vee c \equiv a(\Theta) \tag{4.1}$$

and

$$b \vee c \equiv b(\Theta). \tag{4.2}$$

Since $a = \max[a]\Theta, b = \max[b]\Theta$, (4.1) and (4.2) give $c \leq a \wedge b$.

(2) $\Rightarrow$ (1). We prove that for each $a, b \in L$,

$$[a \vee b]\Theta = [a]\Theta \underset{CS(L)}{\vee} [b]\Theta.$$

Clearly,

$$[a \vee b]\Theta \supseteq [a]\Theta \underset{CS(L)}{\vee} [b]\Theta.$$

On the other hand, let $c \in [a \vee b]\Theta$. Take $a_1 = c \wedge a$, $b_1 = c \wedge b$, $a_2 = \max[a]\Theta$ and $b_2 = \max[b]\Theta$. Then, by (2),

$$c \leq \max[a \vee b]\Theta = a_2 \vee b_2. \tag{4.3}$$

Clearly, $a_1 \vee b_1 \leq c$ and hence $a_1 \vee b_1 \leq c \leq a_2 \vee b_2$. We claim that $a_1 \in [a]\Theta$, $b_1 \in [b]\Theta$. Since

$$c \equiv a \vee b(\Theta), \tag{4.4}$$

clearly,

$$a_1 = c \wedge a \equiv a(\Theta) \tag{4.5}$$

and

$$b_1 = c \wedge b \equiv b(\Theta). \tag{4.6}$$

Therefore, $a_1 \in [a]\Theta$ and $b_1 \in [b]\Theta$. Thus claim holds.

Now, $a_1, a_2 \in [a]\Theta$, $b_1, b_2 \in [b]\Theta$. Therefore, $c \in [a]\Theta \underset{CS(L)}{\vee} [b]\Theta$. □

In a lattice satisfying the ascending and descending chain conditions, a congruence relation is strong which implies that it is representable. More generally, we prove the following theorem.

Theorem 4.4. *Let L be a lattice and let Θ be a join-strong congruence relation of L such that each congruence class of Θ has a maximal element. Then Θ is representable.*

Proof. By Lemma 4.3, the set P of all maximal elements of blocks of Θ is a sublattice of L. Clearly, $f\colon P \to L/\Theta$, $a \mapsto [a]\Theta$, defines an isomorphism. Hence Θ is representable. □

By dual arguments, one can prove that if L is a lattice and Θ is a meet-strong congruence relation of L such that each congruence class of Θ has a minimal element, then also Θ is representable.

We give some examples of join-strong congruence relations.

In the following theorem, we prove that if s is a distributive element of a lattice L, then Θ_s is representable.

Theorem 4.5. *If s is a distributive element of a lattice L, then Θ_s is join-strong.*

Proof. Since s is distributive, by Lemma 2.1, every congruence class of Θ_s has a maximal element and by the Lemma 2.2, the set of all maximal elements of blocks of Θ_s is a sublattice of L. Therefore, by Lemma 4.3, Θ is join-strong. □

Theorem 4.6. *In an atomistic lattice satisfying the ascending and descending chain conditions, every congruence relation is join-strong.*

Proof. Let Θ be a congruence relation of L. From Theorem 3.7, $\Theta = \Theta_s$ for some standard element s. Hence, by Theorem 4.5, Θ is join-strong. $\square$

Remark 4.7. Even in a finite atomistic lattice, every congruence relation may not be meet-strong (In the lattice of Figure 3, consider the congruence relation $\Theta = \Theta(0,a)$. $\Theta[j] = \{j\}$ and $\Theta[k] = \{k\}$. Further, $\Theta[j] \wedge \Theta[k] = \{0, a\}$. But $\{j\} \underset{CS(L)}{\wedge} \{k\} = \{a\}$. Hence Θ is not meet-strong). But if the lattice is sectionally complemented, then every congruence relation is strong as proved in the following theorem.

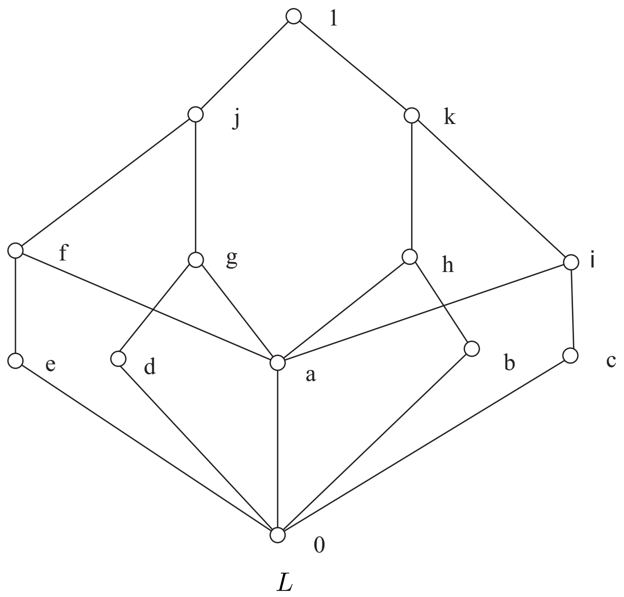

Figure 3. An atomistic lattice with a congruence relation $\Theta(0,a)$ which is not meet-strong

Theorem 4.8. *In a sectionally complemented lattice L, satisfying the ascending and descending chain conditions, every congruence relation is strong.*

Proof. Let Θ be a congruence relation of L. Since L is atomistic, from Theorem 4.6, Θ is join-strong. To prove that Θ is meet-strong, it is enough to prove, by the dual of Lemma 4.3, that $Q = \{a \in L |\ a = \min[x]\Theta \text{ for some } x \in L\}$ is a sublattice of L.

Let $a, b \in Q$ and $c = \min([a \wedge b]\Theta)$. We shall prove that $c = a \wedge b$.
Suppose that $c < a \wedge b$. Let $p \in A(a \wedge b) - A(c)$. We have $a \wedge b \equiv c(\Theta)$. Therefore,

$$p = (a \wedge b) \wedge p \equiv c \wedge p = 0(\Theta) \tag{4.7}$$

Let q be a complement of p in $[0, 1]$. Now, from (4.7),

$$1 = p \vee q \equiv 0 \vee q = q(\Theta).$$

Then

$$a = 1 \wedge a \equiv q \wedge a(\Theta). \tag{4.8}$$

Since $a = \min[a]\Theta$, from (4.8), we get $a \leq q$. But $p \leq a \wedge b \leq a \leq q$ imply that $p = 0$, a contradiction to the fact that p is an atom.

Let $a, b \in Q$. We prove that $a \vee b = \min[a \vee b]\Theta$.
Let $c \in [a \vee b]\Theta$.
Since $c \equiv a \vee b(\Theta)$, we get

$$a \wedge c \equiv a(\Theta) \tag{4.9}$$

and

$$b \wedge c \equiv b(\Theta). \tag{4.10}$$

Further, $a = \min[a]\Theta, b = \min[b]\Theta$, (4.9) and (4.10) give $c \geq a \vee b$. □

One more example of a strong congruence relation is given in the next theorem. First we prove a lemma.

In any lattice, a standard element can have at most one complement. Another interesting property of the complement of a standard element is the following lemma.

Lemma 4.9. *In a lattice L, if a standard element has a complement, then it is codistributive.*

Proof. Let L be a bounded lattice. Let s be a standard element and s' be the complement of s. Also let $x, y \in L$. We prove that $s' \wedge (x \vee y) = (s' \wedge x) \vee (s' \wedge y)$. Take $a = s' \wedge (x \vee y)$ and $b = (s' \wedge x) \vee (s' \wedge y)$. Evidently $b \leq a$. Consider

$$\begin{aligned}
a \wedge (s \vee b) &= a \wedge (s \vee ((s' \wedge x) \vee (s' \wedge y))) \\
&= a \wedge (((s \vee s') \wedge (s \vee x)) \vee ((s \vee s') \wedge (s \vee y))) && \text{since } s \text{ is distributive} \\
&= a \wedge ((s \vee x) \vee (s \vee y)) && \text{since } s \vee s' = 1 \\
&= a \wedge (s \vee (x \vee y)) \\
&= a && \text{since } a \leq x \vee y \leq s \vee (x \vee y),
\end{aligned}$$

and since $b \leq a$, we get

$$\begin{aligned}
(a \wedge s) \vee (a \wedge b) &= ((s' \wedge (x \vee y)) \wedge s) \vee b \\
&= b && \text{since } s' \wedge (x \vee y)) \wedge s \leq s \wedge s' = 0 \leq b.
\end{aligned}$$

Now, since s is standard, we get

$$a = a \wedge (s \vee b) = (a \wedge s) \vee (a \wedge b) = b.$$

□

Theorem 4.10. *Let L be a complemented lattice and let s be a standard element of L. Then Θ_s is a strong congruence relation.*

Proof. Let s' be the complement of s. By Lemma 4.9, s' is codistributive. By Theorem 4.5, Θ_s is join-strong and by the dual of Theorem 4.5, $\Theta[[s')]$ is meet-strong. To prove the theorem, it is enough to prove that $\Theta_s = \Theta[[s')]$.

Let $x \equiv y(\Theta_s)$. Then $x \vee s = y \vee s$. Now

$$\begin{aligned} x \wedge s' &= (x \wedge s') \vee (s \wedge s') = (x \vee s) \wedge s' \\ &= (y \vee s) \wedge s' = (y \wedge s') \vee (s \wedge s') \\ &= (y \wedge s'). \end{aligned}$$

Therefore, $x \equiv y(\Theta[[s')])$ so that $\Theta_s \subseteq \Theta[[s')]$.
On the other hand, let $x \equiv y(\Theta[[s')])$. Then $x \wedge s' = y \wedge s'$.

Now

$$\begin{aligned} x \vee s &= (x \vee s) \wedge (s \vee s') = (x \wedge s') \vee s \qquad \text{since } s \text{ distributive} \\ &= (y \wedge s') \vee s = (y \vee s) \wedge (s \vee s') \\ &= y \vee s. \end{aligned}$$

Therefore, $x \equiv y(\Theta_s)$ so that $\Theta[[s')] \subseteq \Theta_s$. □

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