ON FRACTIONAL DELAY INTEGRODIFFERENTIAL EQUATIONS WITH FOUR-POINT MULTITERM FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

K. SHRI AKILADEVI and K. BALACHANDRAN

Abstract. In this paper, we study the existence and uniqueness of solutions for the fractional delay integrodifferential equations with four-point multiterm fractional integral boundary conditions by using fixed point theorems. The fractional derivative considered here is in the Caputo sense. Examples are provided to illustrate the results.

1. Introduction

Fractional differential equations have been receiving greater attention during the past few decades due to their varied applications to various fields of science and engineering. For a detailed study, one can refer to the books [18, 25, 27, 33]. In recent years, fractional differential equations involving a variety of boundary conditions have been investigated by several researchers. In particular, fractional boundary value problems with integral boundary conditions form a very important class of problems which includes two, three, multi-point and nonlocal boundary conditions as special cases. Multi-point boundary conditions arise in problems related to heat conduction, nonlinear elasticity, electric power networks, electric railway systems, telecommunication lines, and so on. For some recent contributions to fractional boundary value problems, see [1, 3, 5, 12, 24, 35, 40, 41].

On the other hand, delay differential equations are often used as tools in several areas of applied mathematics including the study of epidemics, population dynamics, automation, control theory, industrial robotics, traffic flow and so on. The literature related to the existence of solutions of integer order delay differential equations is very extensive; see, for instance, [2, 6, 7, 8, 9, 21, 29, 34, 36] and the references therein. For fractional order initial value problems with delay, one can refer [11, 13, 15, 22, 26, 28, 39]. But for fractional boundary value problems with delay, the theory is relatively less developed and many aspects of these problems are yet to be explored. Some recent works on fractional boundary
value problems with delay can be found in [14, 16, 17, 23, 31, 32, 37, 38] and the references therein.

Balachandran [10] studied the existence of mild solutions for a class of abstract fractional integrodifferential equation with nonlocal condition of the form

\[ CD^q \left( u(t) + e(t, u(t)) \right) = Au(t) + f \left( t, u(t), u(\alpha(t)) \right) + \int_0^t k(t, s, u(s), u(\beta(s))) ds, \]

\[ u(0) + g(u) = u_0, \]

where \( CD^q \) is the Caputo fractional derivative of order \( 0 < q < 1 \), \( t \in J = [0, a] \), \( A \) is a closed linear unbounded operator in a Banach space \( X \) with dense domain \( D(A) \), \( u_0 \in X \) and \( f : J \times X^3 \to X \), \( e : J \times X \to X \), \( k : \Delta \times X^2 \to X \), \( \gamma : C(J; X) \to X \), \( \alpha, \beta : J \to J \) are continuous with \( \Delta = \{(t, s) : 0 \leq s \leq t \leq a\} \). The result is obtained using Krasnoselskii’s fixed point theorem.

Ntouyas [30] studied the existence results for the following fractional differential equation with fractional integral boundary condition

\[ CD^q x(t) = f \left( t, x(t) \right), \quad 0 < t < 1, \quad 0 < q \leq 2, \]

\[ x(0) = 0, \quad x(1) = \alpha I^\beta x(\eta), \quad 0 < \eta < 1, \]

where \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a given continuous function, \( \alpha \in \mathbb{R} \) is such that \( \alpha \neq \Gamma(p + 2)/\eta^{p+1} \) and \( I^p \) is the Riemann-Liouville fractional integral of order \( 0 < p < 1 \). The existence results are obtained by using Krasnoselskii’s fixed point theorem and Leray-Schauder degree theory.

Guezane-Lakoud and Khaldi [20] discussed the existence and uniqueness of solutions to the fractional differential equation with integral boundary condition

\[ CD^0_{\alpha+} u(t) = f \left( t, u(t), CD^\sigma_{\alpha+} u(t) \right), \quad 0 < t < 1, \]

\[ u(0) = 0, \quad u'(1) = I^\sigma_{\alpha+} u(1), \]

where \( 1 < q < 2, 0 < \sigma < 1 \) and \( f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a continuous function. The results are proved using Banach contraction principle and Leray-Schauder nonlinear alternative.

The existence and uniqueness of solutions to the fractional differential equation with four-point nonlocal Riemann-Liouville fractional integral boundary conditions of the form

\[ CD^q x(t) = f \left( t, x(t) \right), \quad 1 < q \leq 2, \quad t \in [0, 1], \]

\[ x(0) = a I^\beta x(\eta), \quad 0 < \beta \leq 1, \]

\[ x(1) = b I^\alpha x(\sigma), \quad 0 < \alpha \leq 1, \quad 0 < \eta, \sigma < 1. \]

was investigated by Ahmad in [4]. Here \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a given continuous function and \( a, b \) are real constants. The results are established using fixed point theorems.

Motivated by the above works, in this paper, we study the existence and uniqueness of solutions for the following nonlinear fractional integrodifferential boundary
value problem with delay of the form

\[ C D_0^q x(t) = f\left( t, x(t), x(\lambda(t)), \int_0^t k(t, s, x(s), x(\sigma(s)))ds \right), \]
\[ 1 < q \leq 2, \quad t \in J = [0, 1], \]

(1)

\[ x(0) = \sum_{i=1}^{n} a_i (I_{0+}^{\gamma_i} x)(\zeta), \]
\[ x(1) = \sum_{i=1}^{n} b_i (I_{0+}^{\delta_i} x)(\eta), \quad 0 < \zeta < \eta < 1, \]

where the functions \( f : J \times X^3 \rightarrow X \), \( k : \Omega \times X^2 \rightarrow X \), \( \lambda, \sigma : J \rightarrow J \) are continuous with \( 0 \leq \lambda(t), \sigma(t) \leq t, \ t \in J \). \( I_{0+}^{\mu} \) is the Riemann-Liouville fractional integral of order \( \mu > 0 \) for \( \mu = \gamma_i \) or \( \delta_i \), and \( a_i, b_i \) are suitably chosen real constants for \( i = 1, 2, \ldots, n \). Here \( \Omega = \{(t, s) : 0 \leq s \leq t \leq 1\} \). \( (X, \| \cdot \|) \) is a Banach space and \( Z = C(J, X) \) denotes the Banach space of all continuous functions from \( J \rightarrow X \) endowed with the topology of uniform convergence with the norm denoted by \( \| \cdot \|_C \).

The paper is organized as follows: In Section 2, we introduce definitions, notations and some preliminary notions. In Section 3, we present our main results on existence and uniqueness of solutions using Krasnoselskii’s fixed point theorem, Leray-Schauder nonlinear alternative and Banach contraction principle, respectively. Examples are presented in Section 4 illustrating the applicability of the imposed conditions. To the best of the authors’ knowledge, no paper has considered the existence of solutions to the fractional delay integrodifferential equations with four-point multiterm fractional integral boundary conditions in Banach spaces.

2. Preliminaries

In this section, we give some of the basic definitions, notations and lemmas [25] which will be used throughout the work.

**Definition 2.1.** The Riemann-Liouville fractional integral of a function \( f \in L^1(\mathbb{R}^+) \) of order \( q > 0 \) is defined by

\[ I_{0+}^{\mu} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \]

provided the integral exists.

**Definition 2.2.** The Caputo fractional derivative of order \( n-1 < q \leq n \) is defined by

\[ C D_0^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds, \]

where the function \( f(t) \) has absolutely continuous derivatives up to order \( (n-1) \). In particular, if \( 0 < q \leq 1 \),

\[ C D_0^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q}ds, \]
where \( f'(s) = Df(s) = \frac{df(s)}{ds} \).

For brevity of notation, \( Kx(t) = \int_0^t k(t, s, x(s), x(\sigma(s)))ds \), \( I_0^q \) is taken as \( I^q \) and \( C_{\alpha}D_0^q \) is taken as \( CD^q \).

**Lemma 2.1** ([1]). Let \( p, q \geq 0 \), \( f \in L^1[a, b] \). Then \( IP^q f(t) = IP^{q+q} f(t) = IP^q f(t) \) and \( C_{\alpha}D_0^q I^q f(t) = f(t) \) for all \( t \in [a, b] \).

**Definition 2.3.** A function \( x(t) \in C(J, X) \) is said to be a solution of (1) if it satisfies the equation
\[
CD^q x(t) = f(t, x(t), x(\lambda(t)), Kx(t)), \quad t \in J,
\]
and the boundary conditions
\[
x(0) = \sum_{i=1}^{n} a_i(I^{\gamma_i}x)(\zeta),
\]
\[
x(1) = \sum_{i=1}^{n} b_i(I^{\delta_i}x)(\eta), \quad 0 < \zeta < \eta < 1.
\]

To study the nonlinear problem (1), we first consider the linear problem and obtain its solution.

**Lemma 2.2.** For \( f(t) \in C(J, X) \), the unique solution of the fractional boundary value problem
\[
CD^q x(t) = f(t), \quad 1 < q \leq 2, \quad t \in J,
\]
\[
x(0) = \sum_{i=1}^{n} a_i(I^{\gamma_i}x)(\zeta), \quad x(1) = \sum_{i=1}^{n} b_i(I^{\delta_i}x)(\eta), \quad 0 < \zeta < \eta < 1,
\]
is given by
\[
x(t) = I^q f(t) + (A_4 - A_3) \sum_{i=1}^{n} a_i I^{\gamma_i + q} f(\zeta)
\]
\[
+ (A_2 + A_1) \{ \sum_{i=1}^{n} b_i I^{\delta_i + q} f(\eta) - I^q f(1) \},
\]
where
\[
A_1 = \frac{1}{A} \left( 1 - \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i}}{\Gamma(\gamma_i + 1)} \right), \quad A_2 = \frac{1}{A} \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i + 1}}{\Gamma(\gamma_i + 2)},
\]
\[
A_3 = \frac{1}{A} \left( 1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i}}{\Gamma(\delta_i + 1)} \right), \quad A_4 = \frac{1}{A} \left( 1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i + 1}}{\Gamma(\delta_i + 2)} \right),
\]
\[
A = \left( 1 - \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i}}{\Gamma(\gamma_i + 1)} \right) \left( 1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i + 1}}{\Gamma(\delta_i + 2)} \right)
\]
\[
+ \left( \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i + 1}}{\Gamma(\gamma_i + 2)} \right) \left( 1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i}}{\Gamma(\delta_i + 1)} \right).
\]
Proof. For some vector constants $c_0, c_1 \in X$, the general solution of (2) can be written as [25]

$$x(t) = I^q f(t) + c_0 + c_1 t. \tag{4}$$

Using the boundary condition $x(0) = \sum_{i=1}^{n} a_i(I^\gamma x)(\zeta)$ and Lemma 2.1 in (4), we have

$$\left(1 - \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i}}{\Gamma(\gamma_i + 1)}\right)c_0 - \left(\sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i + 1}}{\Gamma(\gamma_i + 2)}\right)c_1 = \sum_{i=1}^{n} a_i I^{\gamma_i + q} f(\zeta). \tag{5}$$

Next, using the boundary condition $x(1) = \sum_{i=1}^{n} b_i(I^\delta x)(\eta)$ and Lemma 2.1 in (4), we have

$$\left(1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i}}{\Gamma(\delta_i + 1)}\right)c_0 + \left(1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i + 1}}{\Gamma(\delta_i + 2)}\right)c_1 = \sum_{i=1}^{n} b_i I^{\delta_i + q} f(\eta) - I^q f(1). \tag{6}$$

Solving (5) and (6) for $c_0$ and $c_1$, we have

$$c_0 = \frac{1}{A} \left[\left(1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i + 1}}{\Gamma(\delta_i + 2)}\right) \sum_{i=1}^{n} a_i I^{\gamma_i + q} f(\zeta) \right.$$

$$\left. + \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i + 1}}{\Gamma(\gamma_i + 2)} \left\{ \sum_{i=1}^{n} b_i I^{\delta_i + q} f(\eta) - I^q f(1) \right\} \right],$$

$$c_1 = \frac{1}{A} \left[\left( \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i}}{\Gamma(\delta_i + 1)} - 1 \right) \sum_{i=1}^{n} a_i I^{\gamma_i + q} f(\zeta) \right.$$

$$\left. + \left(1 - \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i}}{\Gamma(\gamma_i + 1)}\right) \left\{ \sum_{i=1}^{n} b_i I^{\delta_i + q} f(\eta) - I^q f(1) \right\} \right].$$

Substituting the above values of $c_0$ and $c_1$ in (4), we get

$$x(t) = I^q f(t) + (A_4 - A_3 t) \sum_{i=1}^{n} a_i I^{\gamma_i + q} f(\zeta) + (A_2 + A_1 t) \left\{ \sum_{i=1}^{n} b_i I^{\delta_i + q} f(\eta) - I^q f(1) \right\}. \tag{7}$$

□

3. Main Results

In view of Lemma 2.2, we transform (1) as

$$x = F(x), \tag{7}$$
where \( F: Z \to Z \) is given by

\[
(Fx)(t) = \int_0^t (t-s)^{q-1} \frac{1}{\Gamma(q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, ds \\
+ \left( A_4 - A_4^t \right) \sum_{i=1}^n a_i \int_0^t \frac{(\zeta - s)^{\gamma_i + q - 1}}{\Gamma(\gamma_i + q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, ds \\
+ \left( A_2 + A_1^t \right) \sum_{i=1}^n b_i \int_0^t \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, ds \\
- \int_0^1 (1-s)^{q-1} \frac{1}{\Gamma(q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, ds
\]

for \( t \in J \). Observe that the problem (1) has solutions if the operator equation (7) has fixed points.

Assume that the following conditions hold:

(A1) There exist positive constants \( L_f \) and \( L_k \) such that
- (i) \( \|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \leq L_f (\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|), \) 
  \( t \in J, x_1, x_2, y_1, y_2, z_1, z_2 \in X \),
- (ii) \( \|k(t, s, x_1, y_1) - k(t, s, x_2, y_2)\| \leq L_k (\|x_1 - x_2\| + \|y_1 - y_2\|), t, s \in J, x_1, x_2, y_1, y_2 \in X \).

(A2) \( \|f(t, x, y, z)\| \leq l(t) \phi(\|x\|), (t, x, y, z) \in J \times X^3 \), where \( l \in L^1(J, \mathbb{R}^+) \) and \( \phi : [0, \infty) \to (0, \infty) \) is a continuous nondecreasing function.

(A3) Let \( \Delta = 2L_f (\theta_1 + L_k \theta_2) < 1 \), where
- \( \theta_1 = \frac{1 + |A_2| + |A_1|}{\Gamma(q + 1)} + (|A_4| + |A_3|) \rho_1 + (|A_2| + |A_1|) \rho_3 \) and
- \( \theta_2 = \frac{1 + |A_2| + |A_1|}{\Gamma(q + 2)} + (|A_4| + |A_3|) \rho_2 + (|A_2| + |A_1|) \rho_4 \) with
- \( \rho_1 = \sum_{i=1}^n |a_i| \frac{\zeta^{\gamma_i + q}}{\Gamma(\gamma_i + q + 1)} \),
- \( \rho_2 = \sum_{i=1}^n |a_i| \frac{\zeta^{\gamma_i + q + 1}}{\Gamma(\gamma_i + q + 2)} \),
- \( \rho_3 = \sum_{i=1}^n |b_i| \frac{\eta^{\delta_i + q}}{\Gamma(\delta_i + q + 1)} \),
- \( \rho_4 = \sum_{i=1}^n |b_i| \frac{\eta^{\delta_i + q + 1}}{\Gamma(\delta_i + q + 2)} \).

We prove the existence of solutions to (1) by applying Krasnoselskii’s fixed point theorem.

**Lemma 3.1** ([19, Krasnoselskii Theorem]). Let \( S \) be a closed, convex, nonempty subset of a Banach space \( X \). Let \( \mathcal{P}, \mathcal{Q} \) be two operators such that
- (i) \( \mathcal{P}x + \mathcal{Q}y \in S \), whenever \( x, y \in S \),
- (ii) \( \mathcal{P} \) is compact and continuous,
- (iii) \( \mathcal{Q} \) is a contraction mapping.

Then there exists \( z \in S \) such that \( z = \mathcal{P}z + \mathcal{Q}z \).
Theorem 3.1. Suppose that the assumptions (A1) and (A2) hold with

\[
L = 2L_f \left\{ \frac{|A_2| + |A_1|}{\Gamma(q+1)} + \frac{|A_3|}{\Gamma(q+2)} \right\} + L_k \left\{ \frac{|A_2| + |A_1|}{\Gamma(q+1)} + \frac{|A_3|}{\Gamma(q+2)} \right\} < 1.
\]

Then the boundary value problem (1) has at least one solution on \( J \).

Proof. Consider \( B_r = \{ x \in Z : \| x \| \leq r \} \). Now, for \( t \in J \), we decompose \( F \) as \( F_1 + F_2 \) on \( B_r \), where

\[
(F_1 x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, ds,
\]

\[
(F_2 x)(t) = (A_4 - A_3 t) \sum_{i=1}^n a_i \int_0^\zeta \frac{(\zeta-s)\gamma+q+1}{\Gamma(\gamma_1 + q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, ds
\]

\[
+ (A_2 + A_1 t) \left\{ \sum_{i=1}^n b_i \int_0^\eta \frac{(\eta-s)^\gamma+q-1}{\Gamma(\gamma_1 + q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, ds
\]

\[
- \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, ds \right\}.
\]

Choose \( r > \| l \| L^1(\rho(r)\theta_1) \). For \( x, y \in B_r \), we find that

\[
\|F_1 x + F_2 y\|
\]

\[
\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| \, ds
\]

\[
+ \sum_{i=1}^n \left\{ \int_0^\zeta \frac{(\zeta-s)^\gamma+q+1}{\Gamma(\gamma_1 + q)} \| f(s, y(s), y(\lambda(s)), Ky(s)) \| \, ds
\]

\[
+ \sum_{i=1}^n \left\{ \int_0^\eta \frac{(\eta-s)^\gamma+q-1}{\Gamma(\gamma_1 + q)} \| f(s, y(s), y(\lambda(s)), Ky(s)) \| \, ds
\]

\[
+ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \| f(s, y(s), y(\lambda(s)), Ky(s)) \| \, ds \right\}
\]

\[
\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} l(s) \phi(\| x \|) \, ds
\]

\[
+ \sum_{i=1}^n \left\{ \int_0^\zeta \frac{(\zeta-s)^\gamma+q+1}{\Gamma(\gamma_1 + q)} l(s) \phi(\| y \|) \, ds
\]

\[
+ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} l(s) \phi(\| y \|) \, ds \right\}
\]
Thus $F_1x + F_2y \in B_r$. Next we prove that $F_2$ is a contraction.

$$\|(F_2x)(t) - (F_2y)(t)\| \leq (|A_4| + |A_3|) \sum_{i=1}^{n} |a_i| \int_0^t (\zeta - s)^{q_i + q_i - 1} \frac{\|f(s, x(s), x(\lambda(s)), Kx(s)) - f(s, y(s), y(\lambda(s)), Ky(s))\|}{1 (\zeta - s)^{q_i + q_i - 1}} ds + (|A_2| + |A_1|) \sum_{i=1}^{n} |b_i| \int_0^t (\eta - s)^{q_i + q_i - 1} \frac{\|s, x(s), x(\lambda(s)), Kx(s)) - f (s, y(s), y(\lambda(s)), Ky(s))\| ds}{1 (\zeta - s)^{q_i + q_i - 1}}$$

Next we prove that $F_2$ is a contraction.
\[ \leq 2L_f \left\{ \frac{|A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_2 \right\} \\
+ L_k \left\{ \frac{|A_2| + |A_1|}{\Gamma(q + 2)} + (|A_4| + |A_3|)\rho_2 + (|A_2| + |A_1|)\rho_3 \right\} \|x - y\| \leq L\|x - y\|. \]

Hence \( F_2 \) is a contraction. Continuity of \( f \) and \( k \) implies that the operator \( F_1 \) is continuous. Also \( F_1 \) is uniformly bounded on \( B_r \) as

\[
\| (F_1x)(t) \| \leq \int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| ds \\
\leq \int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} l(s)\phi(\|x\|) ds \leq \| l \| \phi(r) \frac{\Gamma(q)}{\Gamma(q + 1)}. 
\]

To prove that the operator \( F_1 \) is compact, it remains to show that \( F_1 \) is equicontinuous. Now, for any \( t_1, t_2 \in J \) with \( t_1 < t_2 \) and \( x \in B_r \), we have

\[
\| (F_1x)(t_2) - (F_1x)(t_1) \| \\
\leq \int_0^{t_1} \frac{[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]}{\Gamma(q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| ds \\
+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| ds \\
\leq \int_0^{t_1} \frac{[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]}{\Gamma(q)} l(s)\phi(\|x\|) ds \\
+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} l(s)\phi(\|x\|) ds \\
\leq \phi(r) \left[ \int_0^{t_1} \frac{[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]}{\Gamma(q)} l(s) ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} l(s) ds \right]. 
\]

As \( t_2 \to t_1 \), the right hand side of the above inequality tends to zero independent of \( x \in B_r \). Thus \( F_1 \) is equicontinuous. By Arzela-Ascoli’s Theorem, \( F_1 \) is compact on \( B_r \). Hence, by the Krasnoselskii fixed point theorem, there exists a fixed point \( x \in Z \) such that \( Fx = x \) which is a solution to the fractional boundary value problem (1).

The next existence result is based on Leray-Schauder nonlinear alternative.

**Theorem 3.2** ([19. Leray-Schauder nonlinear alternative]). Let \( E \) be a Banach space, \( C \) a closed, convex subset of \( E \), \( U \) an open subset of \( C \) and \( 0 \in U \). Suppose that \( F: \overline{U} \to C \) is a continuous, compact (that is, \( F(\overline{U}) \) is a relatively compact subset of \( C \)) map. Then either

(i) \( F \) has a fixed point in \( \overline{U} \) or

(ii) there is a \( u \in \partial U \) (the boundary of \( U \) in \( C \)) and \( \lambda_0 \in (0, 1) \) with \( u = \lambda_0 F(u) \).
Theorem 3.3. Assume that the following hypotheses hold:

(A4) There exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and functions $n_1, n_2 \in L^1(J, \mathbb{R}^+)$ such that for each $(t, x, y, z) \in J \times X^3$,

$$\|f(t, x, y, z)\| \leq n_1(t)\psi(\|x\|) + n_2(t).$$

(A5) There exists a constant $M > 0$ such that $M A^{-1} > 1$, where

$$A = (\psi(M)\|n_1\|_{L^1} + \|n_2\|_{L^1})\theta_1.$$

Then the boundary value problem (1) has at least one solution on $J$.

Proof. Observe that the operator $F : Z \to Z$ defined by (8) is continuous. Next we show that $F$ maps bounded sets into bounded sets in $Z$.

For a positive number $r$, let $B_r = \{x \in Z : \|x\| \leq r\}$ be a bounded ball in $Z$. Then, for $x \in B_r$, we have

$$\|(Fx)(t)\|
\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s, x(s), x(\lambda(s)), Kx(s))\|ds
\quad + (|A_4| + |A_3|) \sum_{i=1}^n |a_i| \int_0^c \frac{(s-\zeta)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)} \|f(s, x(s), x(\lambda(s)), Kx(s))\|ds
\quad + (|A_2| + |A_1|) \left\{ \sum_{i=1}^n |b_i| \int_0^q \frac{(s-\zeta)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} \|f(s, x(s), x(\lambda(s)), Kx(s))\|ds
\quad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \|f(s, x(s), x(\lambda(s)), Kx(s))\|ds \right\}
\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left( n_1(s)\psi(\|x\|) + n_2(s) \right)ds + (|A_4| + |A_3|)
\times \sum_{i=1}^n |a_i| \int_0^c \frac{(s-\zeta)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)} (n_1(s)\psi(\|x\|) + n_2(s)ds + (|A_2| + |A_1|)
\times \left\{ \sum_{i=1}^n |b_i| \int_0^q \frac{(s-\zeta)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} (n_1(s)\psi(\|x\|) + n_2(s))ds
\quad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} (n_1(s)\psi(\|x\|) + n_2(s))ds \right\}
\leq \psi(\|x\|)\|n_1\|_{L^1} \left[ \frac{1 + |A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_3 \right]
\quad + \|n_2\|_{L^1} \left[ \frac{1 + |A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_3 \right]
\leq (\psi(r)\|n_1\|_{L^1} + \|n_2\|_{L^1})\theta_1.$$

Thus

$$\|(Fx)(t)\| \leq (\psi(r)\|n_1\|_{L^1} + \|n_2\|_{L^1})\theta_1.$$
Now we show that $F$ maps bounded sets into equicontinuous sets in $B_r$. For that, let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for $x \in B_r$,

$$
\| (Fx)(t_2) - (Fx)(t_1) \|
\leq \int_0^{t_2} \left[ \frac{(t_2 - s)^{\eta - 1} - (t_1 - s)^{\eta - 1}}{\Gamma(q)} \right] \| f(s, x(s), x(\lambda(s)), Kx(s)) \| ds
\leq \int_0^{t_2} \frac{(t_2 - s)^{\eta - 1} - (t_1 - s)^{\eta - 1}}{\Gamma(q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| ds + | A_1 | (t_2 - t_1)
\times \sum_{i=1}^{n} | a_i | \int_0^{c} \frac{\zeta^{\gamma_i + q - 1}}{\Gamma(\gamma_i + q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| ds + | A_1 |
\times (t_2 - t_1) \left\{ \sum_{i=1}^{n} | b_i | \int_0^{n} \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| ds
\right.
\leq \psi(r) \left[ \int_0^{t_2} \left[ \frac{(t_2 - s)^{\eta - 1} - (t_1 - s)^{\eta - 1}}{\Gamma(q)} \right] n_1(s) ds + \int_0^{t_1} \frac{(t_2 - s)^{\eta - 1} - (t_1 - s)^{\eta - 1}}{\Gamma(q)} n_1(s) ds
\right.
\left. + | A_1 | (t_2 - t_1) \sum_{i=1}^{n} | a_i | \int_0^{c} \frac{\zeta^{\gamma_i + q - 1}}{\Gamma(\gamma_i + q)} n_1(s) ds + | A_1 | (t_2 - t_1)
\times \left( \sum_{i=1}^{n} | b_i | \int_0^{n} \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} n_2(s) ds + \int_0^{t_2} \frac{(t_2 - s)^{\eta - 1}}{\Gamma(q)} n_2(s) ds + \int_0^{t_1} \frac{(t_2 - s)^{\eta - 1}}{\Gamma(q)} n_2(s) ds
\right.
\left. + | A_1 | (t_2 - t_1) \sum_{i=1}^{n} | a_i | \int_0^{c} \frac{\zeta^{\gamma_i + q - 1}}{\Gamma(\gamma_i + q)} n_2(s) ds + | A_1 | (t_2 - t_1)
\times \left( \sum_{i=1}^{n} | b_i | \int_0^{n} \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} n_2(s) ds + \int_0^{t_2} \frac{(1 - s)^{\eta - 1}}{\Gamma(q)} n_2(s) ds
\right.
\right].
$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero independent of $x \in B_r$. Thus $F$ maps bounded sets into equicontinuous sets in $B_r$. By Arzela-Ascoli’s Theorem, $F$ is completely continuous.

Now let $x = \lambda_0 Fx$, where $\lambda_0 \in (0, 1)$. Then, for $t \in J$, we have

$$
x(t) = \lambda_0 \int_0^{t} \frac{(t - s)^{\eta - 1}}{\Gamma(q)} f(s, x(s), x(\lambda(s)), Kx(s)) ds
\leq \lambda_0 (A_4 - A_2) \sum_{i=1}^{n} a_i \int_0^{n} \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} f(s, x(s), x(\lambda(s)), Kx(s)) ds
\leq \lambda_0 (A_4 - A_2) \left\{ \sum_{i=1}^{n} b_i \int_0^{n} \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} f(s, x(s), x(\lambda(s)), Kx(s)) ds
\right.
\leq \lambda_0 (A_4 - A_2) \left\{ \sum_{i=1}^{n} b_i \int_0^{n} \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} f(s, x(s), x(\lambda(s)), Kx(s)) ds
\right\}.
$$
Then, using the computations of the first step, we have
\[ \|x(t)\| \leq (\psi(\|x\|))(n_1\|L^1 + n_2\|L^1)\theta_1 \]
which can be written as
\[ \|x\| \left( (\psi(\|x\|))(n_1\|L^1 + n_2\|L^1)\theta_1 \right)^{-1} \leq 1. \]
In view of (A5), there exists M such that \( \|x\| \neq M \). We set
\[ U = \{ x \in Z : \|x\| < M \}. \]
Note that the operator \( F : \bar{U} \rightarrow Z \) is continuous and completely continuous. From the choice of \( U \), there is no \( x \in \partial U \) such that \( x = \lambda_0 Fx \) for some \( \lambda_0 \in (0, 1) \).
Consequently, by the Leray-Schauder nonlinear alternative, we deduce that \( F \) has a fixed point \( x \in \bar{U} \) which is a solution to the problem (1).

The next result is based on Banach contraction principle.

**Theorem 3.4.** Assume that the hypotheses (A1) and (A3) hold. Then the boundary value problem (1) has a unique solution on \( J \).

**Proof.** Let \( M_1 = \sup_{t \in J} \|f(t, 0, 0, 0)\| \) and \( M_2 = \sup_{t \in J} \|k(t, s, 0, 0)\| \). Consider \( B_r = \{ x \in Z : \|x\| \leq r \} \), where \( r \geq \frac{\Delta_2}{\Delta_1} \), with \( \Delta_2 = L_2 M_2 \theta_2 + M_1 \theta_1 \) and \( \Delta_1 \) is given by the assumption (A3). Now we show that \( FB_r \subset B_r \), where \( F : Z \rightarrow Z \) is defined by (8). For \( x \in B_r \), we have
\[ \|F(x)(t)\| \leq \int_0^1 \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \|f(s, x(s), x(\lambda(s)), Kx(s))\| ds \]
\[ + (|A_4| + |A_3|) \sum_{i=1}^n |a_i| \int_0^\zeta \frac{(s-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i + q)} \|f(s, x(s), x(\lambda(s)), Kx(s))\| ds \]
\[ + (|A_2| + |A_1|) \left\{ \sum_{i=1}^n |b_i| \int_0^\eta \frac{(s-s)^{\delta_i+q-1}}{\Gamma(\delta_i + q)} \|f(s, x(s), x(\lambda(s)), Kx(s))\| ds \right\} \]
\[ \leq \int_0^1 \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \left[ \|f(s, x(s), x(\lambda(s)), Kx(s))\| + \|f(s, 0, 0, 0)\| \right] ds \]
\[ + (|A_4| + |A_3|) \sum_{i=1}^n |a_i| \int_0^\zeta \frac{(s-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i + q)} \left[ \|f(s, x(s), x(\lambda(s)), Kx(s))\| \right. \]
\[ - \|f(s, 0, 0, 0)\| + \|f(s, 0, 0, 0)\| \right] ds + (|A_2| + |A_1|) \left\{ \sum_{i=1}^n |b_i| \int_0^\eta \frac{(s-s)^{\delta_i+q-1}}{\Gamma(\delta_i + q)} \right. \]
\[ \left[ \|f(s, x(s), x(\lambda(s)), Kx(s))\| - f(s, 0, 0, 0)\| + \|f(s, 0, 0, 0)\| \right] ds \]
\[ + \int_0^1 \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \left[ \|f(s, x(s), x(\lambda(s)), Kx(s))\| - f(s, 0, 0, 0)\| + \|f(s, 0, 0, 0)\| \right] ds \]
\[
\leq \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(q)} \left[ L_f(\|x(s)\| + \|x(\lambda(s))\| + \|K_x(s)\|) + M_1 \right] ds + (|A_4| + |A_3|) \\
\times \sum_{i=1}^n |a_i| \int_0^c \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i + q)} \left[ L_f(\|x(s)\| + \|x(\lambda(s))\| + \|K_x(s)\|) + M_1 \right] ds \\
+ (|A_2| + |A_1|) \left\{ \sum_{i=1}^n |b_i| \int_0^\eta \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i + q)} \left[ L_f(\|x(s)\| + \|x(\lambda(s))\| + \|K_x(s)\|) \\
+ M_1 \right] ds \right\} \\
\leq \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(q)} \left[ L_f(\|x(s)\| + \|x(\lambda(s))\| + \|K_x(s)\|) + \int_0^\eta \left[ \|k(s, \tau, x(\tau), x(\sigma(\tau))) - k(s, \tau, 0, 0)\| \\
+ \|k(s, \tau, 0, 0)\| \right] d\tau \right] ds + (|A_4| + |A_3|) \left\{ \sum_{i=1}^n |a_i| \int_0^c \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i + q)} \\
\times \left[ L_f(\|x(s)\| + \|x(\lambda(s))\|) + \int_0^\eta \left[ \|k(s, \tau, x(\tau), x(\sigma(\tau))) - k(s, \tau, 0, 0)\| \\
+ \|k(s, \tau, 0, 0)\| \right] d\tau \right] ds \right\} \\
\times \left[ \frac{1}{\Gamma(q)} \int_0^t \left[ L_f(\|x(s)\| + \|x(\lambda(s))\|) + \int_0^\eta \left[ \|k(s, \tau, x(\tau), x(\sigma(\tau))) - k(s, \tau, 0, 0)\| \\
+ \|k(s, \tau, 0, 0)\| \right] d\tau \right] ds \right] \\
\leq 2L_f(\theta_1 + L_2 \theta_2) + (L_f M_2 \theta_2 + M_1 \theta_1) \\
\leq \Delta_1 r + \Delta_2 \leq r.
\]

This shows that \( FB_r \subset B_r \). Next, for \( x, y \in Z \) and \( t \in J \), we obtain

\[
\|(Fx)(t) - (Fy)(t)\| \\
\leq \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(q)} \left| f(s, x(s), x(\lambda(s)), K_x(s)) - f(s, y(s), y(\lambda(s)), K_y(s)) \right| ds \\
+ (|A_4| + |A_3|) \left\{ \sum_{i=1}^n |a_i| \int_0^\eta \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i + q)} \left| f(s, x(s), x(\lambda(s)), K_x(s)) - f(s, y(s), y(\lambda(s)), K_y(s)) \right| ds \right\} \\
\times \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(q)} \left[ L_f(\|x(s)\| + \|x(\lambda(s))\| + \|K_x(s)\|) + M_1 \right] ds \\
\leq 2L_f(\theta_1 + L_2 \theta_2) \|x - y\| \\
\|Fx - Fy\| \leq \Delta_3 \|x - y\|.
\]
Here $\Delta_1$ depends only on the parameters involved in the problem. By assumption (A3), $\Delta_1<1$ and therefore, $F$ is a contraction. Hence, by the Banach contraction principle, the problem (1) has a unique solution on $J$. □

**Remark 3.1.** By fixing the parameters in the given problem, several special cases can be obtained.

(a) For $f = f(t,x(t))$ and taking $a_i$, $b_i = 0$, $i = 2,\ldots,n$ in the problem (1),
   (i) the results of [3] are got by taking $a_1 = 0$, $\delta_1 = 1$.
   (ii) the results of [4] appear as a special case with $X = \mathbb{R}$.
   (iii) the results of [30] are obtained as a special case by taking $a_1 = 0$ and $X = \mathbb{R}$.

(b) For $a_i$, $b_i = 0$, $i = 1,2,\ldots,n$, the problem (1) reduces to a Dirichlet problem.

(c) Taking $a_i$, $b_i = 0$, $i = 2,\ldots,n$, and $\gamma_1,\delta_1 = 1$, the problem (1) reduces to a fractional delay integrodifferential equation with integral boundary conditions.

(d) For $q = 2$, taking $a_i$, $b_i = 0$, $i = 2,\ldots,n$, and $\gamma_1,\delta_1 = 1$, we obtain new results for a second order delay differential equation with integral boundary conditions.

(e) The problem (1) can be generalized to fractional integrodifferential boundary value problem with multiple delay of the form

$$ C^D^q x(t) = f(t,x(t),x(\lambda_1(t)),x(\lambda_2(t)),\ldots,x(\lambda_{m_1}(t))), $$

$$ \int_0^t k(t,s,x(s),x(\sigma_1(s)),x(\sigma_2(s)),\ldots,x(\sigma_{m_2}(s)))ds, \quad t \in J = [0,1], $$

$$ x(0) = \sum_{i=1}^n a_i(I^{\gamma_i}x)(\zeta), $$

$$ x(1) = \sum_{i=1}^n b_i(I^{\delta_i}x)(\eta), \quad 0 < \zeta < \eta < 1, $$

where $1 < q \leq 2$, the functions $\lambda_i, \sigma_j : J \to J$, $i = 1,2,\ldots,m_1$, $j = 1,2,\ldots,m_2$, are continuous such that $0 \leq \lambda_i(t) \leq t$, $0 \leq \sigma_j(t) \leq t$, $t \in J$, and under suitable assumptions, we can establish the existence results.

4. Example

Consider the following fractional boundary value problem

$$ C^D^{3/2} x(t) = \frac{1}{(t+63)} \frac{|x(t)|}{1+|x(t)|} + \frac{e^{-t}}{62 + e^t} \frac{|x(t/2)|}{1+|x(t/2)|} $$

$$ + \frac{1}{63} \int_0^t e^{-s} \frac{|x(s^2)|}{1+|x(s^2)|} ds, \quad t \in [0,1], $$

$$ x(0) = \sum_{i=1}^5 a_i(I^{\gamma_i}x)(\zeta), $$

$$ x(1) = \sum_{i=1}^5 b_i(I^{\delta_i}x)(\eta), \quad 0 < \zeta < \eta < 1. $$


Here \( X = \mathbb{R} \), \( q = \frac{3}{2} \), \( n = 5 \), \( \zeta = \frac{1}{2} \), \( \eta = \frac{2}{3} \).

\[
a_1 = 3, \quad a_2 = 4, \quad a_3 = 7, \quad a_4 = 11, \quad a_5 = 17,
\]

\[
\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = \frac{4}{3}, \quad \gamma_4 = \frac{5}{3}, \quad \gamma_5 = \frac{8}{3},
\]

\[
b_1 = 2, \quad b_2 = 5, \quad b_3 = 16, \quad b_4 = 11, \quad b_5 = 10,
\]

\[
\delta_1 = \frac{1}{2}, \quad \delta_2 = \frac{3}{4}, \quad \delta_3 = \frac{5}{4}, \quad \delta_4 = \frac{3}{2}, \quad \delta_5 = \frac{7}{4}.
\]

From the above given data, we see that 
\( A = -3.126434, A_1 = 3.122104, A_2 = -0.914796, A_3 = 6.693181, A_4 = 1.858697, \)
\( \rho_1 = 1.264617, \rho_2 = 0.195105, \rho_3 = 3.284678, \rho_4 = 0.612445, \)
\( \theta_1 = 27.86379, \theta_2 = 5.656502. \)

(i) From (10), we have

\[
f(t, x(t), x(\lambda(t)), Kx(t)) = \frac{1}{(t + 63)} \frac{|x(t)|}{1 + |x(t)|} + \frac{e^{-t}}{62 + e^t} \frac{|x(t/2)|}{1 + |x(t/2)|} + \frac{1}{63} \int_0^t \frac{e^{-s}}{7} \frac{|x(s^2)|}{1 + |x(s^2)|} ds
\]

where \( Kx(t) = \int_0^t \frac{e^{-s}}{7} \frac{|x(s^2)|}{1 + |x(s^2)|} ds \), \( \lambda(t) = t/2, \sigma(t) = t^2 \). The condition (A1) is satisfied with \( L_1 = 1/63 \) and \( L_4 = 1/7 \). The condition (A2) is satisfied with \( L(t) = 1 \) and \( \phi(||x||) = 5/147 \). Computing the value of \( L \), we have \( L = 0.884972 < 1 \), thereby satisfying the condition (9). Thus all the assumptions of the Theorem 3.1 are satisfied. Hence the problem (10) with the given function \( f \) has at least one solution on \( J \).

(ii) Now we take

\[
f(t, x(t), x(\lambda(t)), Kx(t)) = \frac{\sin 2\pi x}{72\pi} \frac{1 + e^{-t}}{71 + e^t} \frac{x(t^3)}{1 + x(t^3)} + \frac{1}{36} \int_0^t \frac{e^{-s}}{1} \frac{x(s)}{1 + x(s)} ds
\]

in (10), where \( Kx(t) = 1/36 \int_0^t \frac{e^{-s}}{1} \frac{x(s^3)}{1 + x(s^3)} ds, \lambda(t) = t^3, \sigma(t) = \sin t \). Clearly

\[
\|f(t, x(t), x(\lambda(t)), Kx(t))\| \leq \frac{||x||}{36} + \frac{1}{33}.
\]

Here \( n_1(t) = 1/36, \psi(||x||) = ||x|| \) and \( n_2(t) = 1/33. \) From (A5), we have

\[
M \left( \psi(M) ||n_1||_{L^1} + ||n_2||_{L^1} \right) > 1
\]

from which we find that \( M > M_1 \), where \( M_1 < 3.735994 \) thereby satisfying the condition (A5). Thus all the assumptions of the Theorem 3.3 are satisfied. Hence the problem (10) with the given function \( f \) has at least one solution on \( J \).
(iii) Taking
\[ f(t, x(t), x(\lambda(t))), Kx(t) = \frac{1}{(2t+8)^2} \frac{x(t)}{1+x(t)} + \frac{1+e^{-t}}{127 + e^t} \frac{x(\sin t)}{1+x(\sin t)} \]
\[ + \frac{1}{64} \int_0^t e^{-\frac{x(\sin s)}{15}} ds \]
in (10), we have \( Kx(t) = \int_0^t e^{-\frac{x(\sin s)}{15}} ds \), \( \lambda(t) = \sigma(t) = \sin t \). The condition (A1) is satisfied with \( L_f = 1/64 \) and \( L_k = 1/15 \). Computing the value of \( \Delta_1 \), we have \( \Delta_1 = 0.882528 < 1 \), thereby satisfying the condition (A3). Thus all the assumptions of the Theorem 3.4 are satisfied. Hence the problem (10) with the given function \( f \) has a unique solution on \( J \).

REFERENCES


K. Shri Akiladevi, Department of Mathematics, Bharathiar University, Coimbatore, India, e-mail: shriakiladevi@gmail.com

K. Balachandran, Department of Mathematics, Bharathiar University, Coimbatore, India, e-mail: kb.maths.bu@gmail.com