Acta Math. Univ. Comenianae
Vol. LXXXVI, 2 (2017), pp. 187–204

ON FRACTIONAL DELAY INTEGRODIFFERENTIAL EQUATIONS WITH FOUR-POINT MULTITERM FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

K. SHRI AKILADEVI AND K. BALACHANDRAN

Abstract. In this paper, we study the existence and uniqueness of solutions for the fractional delay integrodifferential equations with four-point multiterm fractional integral boundary conditions by using fixed point theorems. The fractional derivative considered here is in the Caputo sense. Examples are provided to illustrate the results.

1. Introduction

Fractional differential equations have been receiving greater attention during the past few decades due to their varied applications to various fields of science and engineering. For a detailed study, one can refer to the books [**18, 25, 27, 33**]. In recent years, fractional differential equations involving a variety of boundary conditions have been investigated by several researchers. In particular, fractional boundary value problems with integral boundary conditions form a very important class of problems which includes two, three, multi-point and nonlocal boundary conditions as special cases. Multi-point boundary conditions arise in problems related to heat conduction, nonlinear elasticity, electric power networks, electric railway systems, telecommunication lines, and so on. For some recent contributions to fractional boundary value problems, see [**1, 3, 5, 12, 24, 35, 40, 41**].

On the other hand, delay differential equations are often used as tools in several areas of applied mathematics including the study of epidemics, population dynamics, automation, control theory, industrial robotics, traffic flow and so on. The literature related to the existence of solutions of integer order delay differential equations is very extensive; see, for instance, [**2, 6, 7, 8, 9, 21, 29, 34, 36**] and the references therein. For fractional order initial value problems with delay, one can refer [**11, 13, 15, 22, 26, 28, 39**]. But for fractional boundary value problems with delay, the theory is relatively less developed and many aspects of these problems are yet to be explored. Some recent works on fractional boundary

Received July 28, 2015; revised April 5, 2017.
2010 *Mathematics Subject Classification.* Primary 26A33, 34A08, 34B10, 34B15.
Key words and phrases. nonlinear fractional integrodifferential equation; delay; Caputo fractional derivative; fractional integral boundary conditions; fixed point theorems.

value problems with delay can be found in [**14, 16, 17, 23, 31, 32, 37, 38**] and the references therein.

Balachandran [**10**] studied the existence of mild solutions for a class of abstract fractional integrodifferential equation with nonlocal condition of the form

$$
\begin{aligned}
{}^{C}D^{q}\big(u(t)+e(t,u(t))\big) &= Au(t)+f\Big(t,u(t),u(\alpha(t)),\int_0^t k(t,s,u(s),u(\beta(s)))\mathrm{d}s\Big),\\
u(0)+g(u) &= u_0,
\end{aligned}
$$

where ${}^{C}D^{q}$ is the Caputo fractional derivative of order $0<q<1, t\in J=[0,a]$, A is a closed linear unbounded operator in a Banach space X with dense domain $D(A)$, $u_0\in X$ and $f\colon J\times X^3\to X$, $e\colon J\times X\to X$, $k\colon \Delta\times X^2\to X$, $g\colon C(J;X)\to X$, $\alpha,\beta\colon J\to J$ are continuous with $\Delta=\{(t,s):0\le s\le t\le a\}$. The result is obtained using Krasnoselskii's fixed point theorem.

Ntouyas [**30**] studied the existence results for the following fractional differential equation with fractional integral boundary condition

$$
\begin{aligned}
&{}^{C}D^{q}x(t)=f\left(t,x(t)\right), && 0<t<1,\ 1<q\le 2,\\
&x(0)=0,\quad x(1)=\alpha I^{p}x(\eta), && 0<\eta<1,
\end{aligned}
$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function, $\alpha\in\mathbb{R}$ is such that $\alpha\neq\Gamma(p+2)/\eta^{p+1}$ and I^{p} is the Riemann-Liouville fractional integral of order $0<p<1$. The existence results are obtained by using Krasnoselskii's fixed point theorem and Leray-Schauder degree theory.

Guezane-Lakoud and Khaldi [**20**] discussed the existence and uniqueness of solutions to the fractional differential equation with integral boundary condition

$$
\begin{aligned}
&{}^{C}D^{q}_{0+}u(t)=f\left(t,u(t),{}^{C}D^{\sigma}_{0+}u(t)\right), && 0<t<1,\\
&u(0)=0,\quad u^{'}(1)=I^{\sigma}_{0+}u(1),
\end{aligned}
$$

where $1<q<2$, $0<\sigma<1$ and $f\colon[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. The results are proved using Banach contraction principle and Leray-Schauder nonlinear alternative.

The existence and uniqueness of solutions to the fractional differential equation with four-point nonlocal Riemann-Liouville fractional integral boundary conditions of the form

$$
\begin{aligned}
{}^{C}D^{q}x(t)&=f\left(t,x(t)\right), && 1<q\le 2,\ t\in[0,1],\\
x(0)&=aI^{\beta}x(\eta), && 0<\beta\le 1,\\
x(1)&=bI^{\alpha}x(\sigma), && 0<\alpha\le 1,\ 0<\eta,\sigma<1.
\end{aligned}
$$

was investigated by Ahmad in [**4**]. Here $f\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function and a,b are real constants. The results are established using fixed point theorems.

Motivated by the above works, in this paper, we study the existence and uniqueness of solutions for the following nonlinear fractional integrodifferential boundary value problem with delay of the form

$$
\begin{aligned}
&{}^{C}D_{0+}^{q}x(t) = f\Big(t, x(t), x(\lambda(t)), \int_0^t k(t,s,x(s),x(\sigma(s)))\mathrm{d}s\Big),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 < q \le 2,\ t \in J = [0,1],\\
&x(0) = \sum_{i=1}^{n} a_i (I_{0+}^{\gamma_i} x)(\zeta),\\
&x(1) = \sum_{i=1}^{n} b_i (I_{0+}^{\delta_i} x)(\eta),\ 0 < \zeta < \eta < 1,
\end{aligned}
\tag{1}
$$

where the functions $f\colon J\times X^3 \to X$, $k\colon \Omega\times X^2 \to X$, $\lambda, \sigma\colon J \to J$ are continuous with $0 \le \lambda(t),\ \sigma(t) \le t,\ t \in J$. I_{0+}^{μ} is the Riemann-Liouville fractional integral of order $\mu > 0$ for $\mu = \gamma_i$ or δ_i, and a_i, b_i are suitably chosen real constants for $i = 1, 2, \ldots, n$. Here $\Omega = \{(t,s) : 0 \le s \le t \le 1\}$. $(X, \|\cdot\|)$ is a Banach space and $Z = C(J,X)$ denotes the Banach space of all continuous functions from $J \to X$ endowed with the topology of uniform convergence with the norm denoted by $\|\cdot\|_C$.

The paper is organized as follows: In Section 2, we introduce definitions, notations and some preliminary notions. In Section 3, we present our main results on existence and uniqueness of solutions using Krasnoselskii's fixed point theorem, Leray-Schauder nonlinear alternative and Banach contraction principle, respectively. Examples are presented in Section 4 illustrating the applicability of the imposed conditions. To the best of the authors' knowledge, no paper has considered the existence of solutions to the fractional delay integrodifferential equations with four-point multiterm fractional integral boundary conditions in Banach spaces.

2. Preliminaries

In this section, we give some of the basic definitions, notations and lemmas [**25**] which will be used throughout the work.

Definition 2.1. The Riemann-Liouville fractional integral of a function $f \in L^1(\mathbb{R}^+)$ of order $q > 0$ is defined by

$$I_{0+}^{q} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\mathrm{d}s,$$

provided the integral exists.

Definition 2.2. The Caputo fractional derivative of order $n-1 < q \le n$ is defined by

$${}^{C}D_{0+}^{q} f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)\mathrm{d}s,$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$. In particular, if $0 < q \le 1$,

$${}^{C}D_{0+}^{q} f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q}\mathrm{d}s,$$

where $f'(s) = Df(s) = \frac{\mathrm{d}f(s)}{\mathrm{d}s}$.

For brevity of notation, $Kx(t) = \int_0^t k(t,s,x(s),x(\sigma(s)))\mathrm{d}s$, I^q_{0+} is taken as I^q and ${}^C D^q_{0+}$ is taken as ${}^C D^q$.

Lemma 2.1 ([**1**]). *Let $p, q \geq 0$, $f \in L^1[a,b]$. Then $I^p I^q f(t) = I^{p+q} f(t) = I^q I^p f(t)$ and ${}^c D^q I^q f(t) = f(t)$ for all $t \in [a,b]$.*

Definition 2.3. A function $x(t) \in C(J,X)$ is said to be a solution of (1) if it satisfies the equation

$$ {}^C D^q x(t) = f\left(t, x(t), x(\lambda(t)), Kx(t)\right), \qquad t \in J, $$

and the boundary conditions

$$ \begin{aligned} x(0) &= \sum_{i=1}^n a_i (I^{\gamma_i} x)(\zeta), \\ x(1) &= \sum_{i=1}^n b_i (I^{\delta_i} x)(\eta), \qquad 0 < \zeta < \eta < 1. \end{aligned} $$

To study the nonlinear problem (1), we first consider the linear problem and obtain its solution.

Lemma 2.2. *For $f(t) \in C(J,X)$, the unique solution of the fractional boundary value problem*

$$ \begin{aligned} &{}^C D^q x(t) = f(t),\ 1 < q \leq 2, && t \in J, \\ &x(0) = \sum_{i=1}^n a_i (I^{\gamma_i} x)(\zeta), \quad x(1) = \sum_{i=1}^n b_i (I^{\delta_i} x)(\eta), && 0 < \zeta < \eta < 1, \end{aligned} \tag{2} $$

is given by

$$ \begin{aligned} x(t) = I^q f(t) &+ (A_4 - A_3 t) \sum_{i=1}^n a_i I^{\gamma_i + q} f(\zeta) \\ &+ (A_2 + A_1 t) \Big\{ \sum_{i=1}^n b_i I^{\delta_i + q} f(\eta) - I^q f(1) \Big\}, \end{aligned} \tag{3} $$

where

$$ A_1 = \frac{1}{A}\Big(1 - \sum_{i=1}^n a_i \frac{\zeta^{\gamma_i}}{\Gamma(\gamma_i + 1)}\Big), \qquad A_2 = \frac{1}{A} \sum_{i=1}^n a_i \frac{\zeta^{\gamma_i + 1}}{\Gamma(\gamma_i + 2)}, $$

$$ A_3 = \frac{1}{A}\Big(1 - \sum_{i=1}^n b_i \frac{\eta^{\delta_i}}{\Gamma(\delta_i + 1)}\Big), \qquad A_4 = \frac{1}{A}\Big(1 - \sum_{i=1}^n b_i \frac{\eta^{\delta_i + 1}}{\Gamma(\delta_i + 2)}\Big), $$

$$ \begin{aligned} A = &\Big(1 - \sum_{i=1}^n a_i \frac{\zeta^{\gamma_i}}{\Gamma(\gamma_i + 1)}\Big)\Big(1 - \sum_{i=1}^n b_i \frac{\eta^{\delta_i + 1}}{\Gamma(\delta_i + 2)}\Big) \\ &+ \Big(\sum_{i=1}^n a_i \frac{\zeta^{\gamma_i + 1}}{\Gamma(\gamma_i + 2)}\Big)\Big(1 - \sum_{i=1}^n b_i \frac{\eta^{\delta_i}}{\Gamma(\delta_i + 1)}\Big). \end{aligned} $$

Proof. For some vector constants $c_0, c_1 \in X$, the general solution of (2) can be written as [**25**]

$$x(t) = I^q f(t) + c_0 + c_1 t. \tag{4}$$

Using the boundary condition $x(0) = \sum_{i=1}^{n} a_i (I^{\gamma_i} x)(\zeta)$ and Lemma 2.1 in (4), we have

$$\Big(1 - \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i}}{\Gamma(\gamma_i + 1)}\Big) c_0 - \Big(\sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i+1}}{\Gamma(\gamma_i + 2)}\Big) c_1 = \sum_{i=1}^{n} a_i I^{\gamma_i+q} f(\zeta). \tag{5}$$

Next, using the boundary conditiotn $x(1) = \sum_{i=1}^{n} b_i (I^{\delta_i} x)(\eta)$ and Lemma 2.1 in (4), we have

$$\Big(1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i}}{\Gamma(\delta_i + 1)}\Big) c_0 + \Big(1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i+1}}{\Gamma(\delta_i + 2)}\Big) c_1 = \sum_{i=1}^{n} b_i I^{\delta_i+q} f(\eta) - I^q f(1). \tag{6}$$

Solving (5) and (6) for c_0 and c_1, we have

$$\begin{aligned}
c_0 = \frac{1}{A}\Big[&\Big(1 - \sum_{i=1}^{n} b_i \frac{\eta^{\delta_i+1}}{\Gamma(\delta_i + 2)}\Big) \sum_{i=1}^{n} a_i I^{\gamma_i+q} f(\zeta) \\
&+ \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i+1}}{\Gamma(\gamma_i + 2)} \Big\{ \sum_{i=1}^{n} b_i I^{\delta_i+q} f(\eta) - I^q f(1) \Big\} \Big], \\
c_1 = \frac{1}{A}\Big[&\Big(\sum_{i=1}^{n} b_i \frac{\eta^{\delta_i}}{\Gamma(\delta_i + 1)} - 1\Big) \sum_{i=1}^{n} a_i I^{\gamma_i+q} f(\zeta) \\
&+ \Big(1 - \sum_{i=1}^{n} a_i \frac{\zeta^{\gamma_i}}{\Gamma(\gamma_i + 1)}\Big) \Big\{ \sum_{i=1}^{n} b_i I^{\delta_i+q} f(\eta) - I^q f(1) \Big\} \Big].
\end{aligned}$$

Substituting the above values of c_0 and c_1 in (4), we get

$$x(t) = I^q f(t) + (A_4 - A_3 t) \sum_{i=1}^{n} a_i I^{\gamma_i+q} f(\zeta) + (A_2 + A_1 t) \Big\{ \sum_{i=1}^{n} b_i I^{\delta_i+q} f(\eta) - I^q f(1) \Big\}.$$

□

3. Main Results

In view of Lemma 2.2, we transform (1) as

$$x = F(x), \tag{7}$$

where $F\colon Z \to Z$ is given by

(8)

$$
\begin{aligned}
(Fx)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), x(\lambda(s)), Kx(s)\right) \mathrm{d}s \\
& + (A_4 - A_3 t) \sum_{i=1}^{n} a_i \int_0^{\zeta} \frac{(\zeta - s)^{\gamma_i + q - 1}}{\Gamma(\gamma_i + q)} f\left(s, x(s), x(\lambda(s)), Kx(s)\right) \mathrm{d}s \\
& + (A_2 + A_1 t) \Big\{ \sum_{i=1}^{n} b_i \int_0^{\eta} \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} f\left(s, x(s), x(\lambda(s)), Kx(s)\right) \mathrm{d}s \\
& - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), x(\lambda(s)), Kx(s)\right) \mathrm{d}s \Big\}
\end{aligned}
$$

for $t \in J$. Observe that the problem (1) has solutions if the operator equation (7) has fixed points.

Assume that the following conditions hold:

(A1) There exist positive constants L_f and L_k such that
  (i) $\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \le L_f(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|)$, $t \in J$, $x_1, x_2, y_1, y_2, z_1, z_2 \in X$,
  (ii) $\|k(t, s, x_1, y_1) - k(t, s, x_2, y_2)\| \le L_k(\|x_1 - x_2\| + \|y_1 - y_2\|)$, $t, s \in J$, $x_1, x_2, y_1, y_2 \in X$.

(A2) $\|f(t, x, y, z)\| \le l(t)\phi(\|x\|)$, $(t, x, y, z) \in J \times X^3$, where $l \in L^1(J, \mathbb{R}^+)$ and $\phi : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(A3) Let $\Delta_1 = 2L_f\big(\theta_1 + L_k\theta_2\big) < 1$, where

$$\theta_1 = \frac{1 + |A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_3 \text{ and}$$

$$\theta_2 = \frac{1 + |A_2| + |A_1|}{\Gamma(q+2)} + (|A_4| + |A_3|)\rho_2 + (|A_2| + |A_1|)\rho_4 \text{ with}$$

$$\rho_1 = \sum_{i=1}^{n} |a_i| \frac{\zeta^{\gamma_i + q}}{\Gamma(\gamma_i + q + 1)}, \qquad \rho_2 = \sum_{i=1}^{n} |a_i| \frac{\zeta^{\gamma_i + q + 1}}{\Gamma(\gamma_i + q + 2)},$$

$$\rho_3 = \sum_{i=1}^{n} |b_i| \frac{\eta^{\delta_i + q}}{\Gamma(\delta_i + q + 1)} \qquad \rho_4 = \sum_{i=1}^{n} |b_i| \frac{\eta^{\delta_i + q + 1}}{\Gamma(\delta_i + q + 2)}.$$

We prove the existence of solutions to (1) by applying Krasnoselskii's fixed point theorem.

Lemma 3.1 ([**19**, Krasnoselskii Theorem])**.** *Let S be a closed, convex, nonempty subset of a Banach space X. Let $\mathcal{P}, \mathcal{Q}$ be two operators such that*

(i) *$\mathcal{P}x + \mathcal{Q}y \in S$, whenever $x, y \in S$,*

(ii) *$\mathcal{P}$ is compact and continuous,*

(iii) *$\mathcal{Q}$ is a contraction mapping.*

Then there exists $z \in S$ such that $z = \mathcal{P}z + \mathcal{Q}z$.

Theorem 3.1. *Suppose that the assumptions* (A1) *and* (A2) *hold with*

$$L = 2L_f\bigg[\bigg\{\frac{|A_2|+|A_1|}{\Gamma(q+1)} + (|A_4|+|A_3|)\rho_1 + (|A_2|+|A_1|)\rho_3\bigg\} + L_k\bigg\{\frac{|A_2|+|A_1|}{\Gamma(q+2)} + (|A_4|+|A_3|)\rho_2 + (|A_2|+|A_1|)\rho_4\bigg\}\bigg] < 1. \tag{9}$$

Then the boundary value problem (1) *has at least one solution on* J.

Proof. Consider $B_r = \{x \in Z : \|x\| \le r\}$. Now, for $t \in J$, we decompose F as $F_1 + F_2$ on B_r, where

$$\begin{aligned}
(F_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), x(\lambda(s)), Kx(s)\right) \mathrm{d}s,\\
(F_2x)(t) &= (A_4 - A_3t)\sum_{i=1}^n a_i \int_0^\zeta \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)} f\left(s, x(s), x(\lambda(s)), Kx(s)\right) \mathrm{d}s\\
&\quad + (A_2 + A_1t)\bigg\{\sum_{i=1}^n b_i \int_0^\eta \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} f\left(s, x(s), x(\lambda(s)), Kx(s)\right) \mathrm{d}s\\
&\quad - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), x(\lambda(s)), Kx(s)\right) \mathrm{d}s\bigg\}.
\end{aligned}$$

Choose $r > \|l\|_{L^1}\phi(r)\theta_1$. For $x, y \in B_r$, we find that

$$\begin{aligned}
&\|F_1x + F_2y\|\\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f\left(s, x(s), x(\lambda(s)), Kx(s)\right)\| \mathrm{d}s\\
&\qquad + (|A_4|+|A_3|)\sum_{i=1}^n |a_i| \int_0^\zeta \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)} \|f\left(s, y(s), y(\lambda(s)), Ky(s)\right)\| \mathrm{d}s\\
&\qquad + (|A_2|+|A_1|)\bigg\{\sum_{i=1}^n |b_i| \int_0^\eta \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} \|f\left(s, y(s), y(\lambda(s)), Ky(s)\right)\| \mathrm{d}s\\
&\qquad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \|f\left(s, y(s), y(\lambda(s)), Ky(s)\right)\| \mathrm{d}s\bigg\}\\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} l(s)\phi(\|x\|)\mathrm{d}s + (|A_4|+|A_3|)\sum_{i=1}^n |a_i| \int_0^\zeta \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}\\
&\qquad \times l(s)\phi(\|y\|)\mathrm{d}s + (|A_2|+|A_1|)\bigg\{\sum_{i=1}^n |b_i| \int_0^\eta \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} l(s)\phi(\|y\|)\mathrm{d}s\\
&\qquad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} l(s)\phi(\|y\|)\mathrm{d}s\bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\le \|l\|_{L^1}\phi(r)\Big[\frac{1+|A_2|+|A_1|}{\Gamma(q+1)}+(|A_4|+|A_3|)\sum_{i=1}^{n}|a_i|\frac{\zeta^{\gamma_i+q}}{\Gamma(\gamma_i+q+1)}\\
&\quad+(|A_2|+|A_1|)\sum_{i=1}^{n}|b_i|\frac{\eta^{\delta_i+q}}{\Gamma(\delta_i+q+1)}\Big]\\
&\le \|l\|_{L^1}\phi(r)\theta_1\le r.
\end{aligned}
$$

Thus $F_1x+F_2y\in B_r$. Next we prove that F_2 is a contraction.

$$
\begin{aligned}
&\|(F_2x)(t)-(F_2y)(t)\|\\
&\quad\le (|A_4|+|A_3|)\sum_{i=1}^{n}|a_i|\int_0^{\zeta}\frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)\\
&\qquad-f\left(s,y(s),y(\lambda(s)),Ky(s)\right)\|\mathrm{d}s+(|A_2|+|A_1|)\Big\{\sum_{i=1}^{n}|b_i|\int_0^{\eta}\frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)}\\
&\qquad\times\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)-f\left(s,y(s),y(\lambda(s)),Ky(s)\right)\|\mathrm{d}s\\
&\qquad+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)-f\left(s,y(s),y(\lambda(s)),Ky(s)\right)\|\mathrm{d}s\Big\}\\
&\quad\le (|A_4|+|A_3|)\sum_{i=1}^{n}|a_i|\int_0^{\zeta}\frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}L_f\Big(\|x(s)-y(s)\|+\|x(\lambda(s))-y(\lambda(s))\|\\
&\qquad+\|Kx(s)-Ky(s)\|\Big)\mathrm{d}s+(|A_2|+|A_1|)\Big\{\sum_{i=1}^{n}|b_i|\int_0^{\eta}\frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)}\\
&\qquad\times L_f\Big(\|x(s)-y(s)\|+\|x(\lambda(s))-y(\lambda(s))\|+\|Kx(s)-Ky(s)\|\Big)\mathrm{d}s\\
&\qquad+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}L_f\Big(\|x(s)-y(s)\|+\|x(\lambda(s))-y(\lambda(s))\|\\
&\qquad+\|Kx(s)-Ky(s)\|\Big)\mathrm{d}s\Big\}\\
&\quad\le (|A_4|+|A_3|)\sum_{i=1}^{n}|a_i|\int_0^{\zeta}\frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}L_f\Big(2\|x-y\|\\
&\qquad+\|\int_0^s k(s,\tau,x(\tau),x(\sigma(\tau)))\mathrm{d}\tau-\int_0^s k(s,\tau,y(\tau),y(\sigma(\tau)))\mathrm{d}\tau\|\Big)\mathrm{d}s\\
&\qquad+(|A_2|+|A_1|)\Big\{\sum_{i=1}^{n}|b_i|\int_0^{\eta}\frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)}L_f\Big(2\|x-y\|\\
&\qquad+\|\int_0^s k(s,\tau,x(\tau),x(\sigma(\tau)))\mathrm{d}\tau-\int_0^s k(s,\tau,y(\tau),y(\sigma(\tau)))\mathrm{d}\tau\|\Big)\mathrm{d}s\\
&\qquad+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}L_f\Big(2\|x-y\|+\|\int_0^s k(s,\tau,x(\tau),x(\sigma(\tau)))\mathrm{d}\tau\\
&\qquad-\int_0^s k(s,\tau,y(\tau),y(\sigma(\tau)))\mathrm{d}\tau\|\Big)\mathrm{d}s\Big\}
\end{aligned}
$$

$$\leq 2L_f\left[\left\{\frac{|A_2|+|A_1|}{\Gamma(q+1)}+(|A_4|+|A_3|)\rho_1+(|A_2|+|A_1|)\rho_3\right\}\right.$$
$$\left.+L_k\left\{\frac{|A_2|+|A_1|}{\Gamma(q+2)}+(|A_4|+|A_3|)\rho_2+(|A_2|+|A_1|)\rho_4\right\}\right]\|x-y\|\leq L\|x-y\|.$$

Hence F_2 is a contraction. Continuity of f and k implies that the operator F_1 is continuous. Also F_1 is uniformly bounded on B_r as

$$\begin{aligned}\|(F_1x)(t)\| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)\|\mathrm{d}s\\ &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}l(s)\phi(\|x\|)\mathrm{d}s \leq \frac{\|l\|_{L^1}\phi(r)}{\Gamma(q+1)}.\end{aligned}$$

To prove that the operator F_1 is compact, it remains to show that F_1 is equicontinuous. Now, for any $t_1,t_2\in J$ with $t_1<t_2$ and $x\in B_r$, we have

$$\begin{aligned}&\|(F_1x)(t_2)-(F_1x)(t_1)\|\\ &\quad\leq \int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)\|\mathrm{d}s\\ &\qquad+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)\|\mathrm{d}s\\ &\quad\leq \int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}l(s)\phi(\|x\|)\mathrm{d}s\\ &\qquad+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}l(s)\phi(\|x\|)\mathrm{d}s\\ &\quad\leq \phi(r)\left[\int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}l(s)\mathrm{d}s+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}l(s)\mathrm{d}s\right].\end{aligned}$$

As $t_2\to t_1$, the right hand side of the above inequality tends to zero independent of $x\in B_r$. Thus F_1 is equicontinuous. By Arzela-Ascoli's Theorem, F_1 is compact on B_r. Hence, by the Krasnoselskii fixed point theorem, there exists a fixed point $x\in Z$ such that $Fx=x$ which is a solution to the fractional boundary value problem (1). □

The next existence result is based on Leray-Schauder nonlinear alternative.

Theorem 3.2 ([**19**, Leray-Schauder nonlinear alternative]). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F\colon \bar{U}\to C$ is a continuous, compact (that is, $F(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\bar{U}$ or*

(ii) *there is a $u\in\partial U$ (the boundary of U in C) and $\lambda_0\in(0,1)$ with $u=\lambda_0F(u)$.*

Theorem 3.3. *Assume that the following hypotheses hold:*

(A4) *There exist a continuous nondecreasing function* $\psi\colon [0,\infty) \to (0,\infty)$ *and functions* $n_1, n_2 \in L^1(J, \mathbb{R}^+)$ *such that for each* $(t,x,y,z) \in J \times X^3$, $\|f(t,x,y,z)\| \le n_1(t)\psi(\|x\|) + n_2(t)$.

(A5) *There exists a constant* $M > 0$ *such that* $M\Lambda^{-1} > 1$, *where*

$$\Lambda = (\psi(M)\|n_1\|_{L^1} + \|n_2\|_{L^1})\theta_1.$$

Then the boundary value problem (1) *has at least one solution on* J.

Proof. Observe that the operator $F\colon Z \to Z$ defined by (8) is continuous. Next we show that F maps bounded sets into bounded sets in Z.

For a positive number r, let $B_r = \{x \in Z : \|x\| \le r\}$ be a bounded ball in Z. Then, for $x \in B_r$, we have

$$\begin{aligned}
&\|(Fx)(t)\| \\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f\,(s, x(s), x(\lambda(s)), Kx(s))\,\|\mathrm{d}s \\
&\qquad + (|A_4| + |A_3|) \sum_{i=1}^n |a_i| \int_0^\zeta \frac{(\zeta - s)^{\gamma_i + q - 1}}{\Gamma(\gamma_i + q)} \|f\,(s, x(s), x(\lambda(s)), Kx(s))\,\|\mathrm{d}s \\
&\qquad + (|A_2| + |A_1|) \Bigg\{ \sum_{i=1}^n |b_i| \int_0^\eta \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} \|f\,(s, x(s), x(\lambda(s)), Kx(s))\,\|\mathrm{d}s \\
&\qquad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \|f\,(s, x(s), x(\lambda(s)), Kx(s))\,\|\mathrm{d}s \Bigg\} \\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (n_1(s)\psi(\|x\|) + n_2(s))\mathrm{d}s + (|A_4| + |A_3|) \\
&\qquad \times \sum_{i=1}^n |a_i| \int_0^\zeta \frac{(\zeta - s)^{\gamma_i + q - 1}}{\Gamma(\gamma_i + q)} (n_1(s)\psi(\|x\|) + n_2(s))\mathrm{d}s + (|A_2| + |A_1|) \\
&\qquad \times \Bigg\{ \sum_{i=1}^n |b_i| \int_0^\eta \frac{(\eta - s)^{\delta_i + q - 1}}{\Gamma(\delta_i + q)} (n_1(s)\psi(\|x\|) + n_2(s))\mathrm{d}s \\
&\qquad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} (n_1(s)\psi(\|x\|) + n_2(s))\mathrm{d}s \Bigg\} \\
&\quad\le \psi(\|x\|)\|n_1\|_{L^1} \left[\frac{1 + |A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_3 \right] \\
&\qquad + \|n_2\|_{L^1} \left[\frac{1 + |A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_3 \right] \\
&\quad\le (\psi(r)\|n_1\|_{L^1} + \|n_2\|_{L^1})\theta_1.
\end{aligned}$$

Thus

$$\|(Fx)(t)\| \le (\psi(r)\|n_1\|_{L^1} + \|n_2\|_{L^1})\theta_1.$$

Now we show that F maps bounded sets into equicontinuous sets in B_r. For that, let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for $x \in B_r$,

$$
\begin{aligned}
&\|(Fx)(t_2) - (Fx)(t_1)\| \\
&\quad \le \int_0^{t_1} \frac{\left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right]}{\Gamma(q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| \mathrm{d}s \\
&\qquad + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| \mathrm{d}s + |A_3|(t_2-t_1) \\
&\qquad \times \sum_{i=1}^{n} |a_i| \int_0^{\zeta} \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| \mathrm{d}s + |A_1| \\
&\qquad \times (t_2-t_1) \Bigg\{ \sum_{i=1}^{n} |b_i| \int_0^{\eta} \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| \mathrm{d}s \\
&\qquad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \| f(s, x(s), x(\lambda(s)), Kx(s)) \| \mathrm{d}s \Bigg\} \\
&\quad \le \psi(r) \Bigg[\int_0^{t_1} \frac{\left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right]}{\Gamma(q)} n_1(s) \mathrm{d}s + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} n_1(s) \mathrm{d}s \\
&\qquad + |A_3|(t_2-t_1) \sum_{i=1}^{n} |a_i| \int_0^{\zeta} \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)} n_1(s) \mathrm{d}s + |A_1|(t_2-t_1) \\
&\qquad \times \Bigg(\sum_{i=1}^{n} |b_i| \int_0^{\eta} \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} n_1(s) \mathrm{d}s + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} n_1(s) \mathrm{d}s \Bigg) \Bigg] \\
&\qquad + \Bigg[\int_0^{t_1} \frac{\left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right]}{\Gamma(q)} n_2(s) \mathrm{d}s + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} n_2(s) \mathrm{d}s \\
&\qquad + |A_3|(t_2-t_1) \sum_{i=1}^{n} |a_i| \int_0^{\zeta} \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)} n_2(s) \mathrm{d}s + |A_1|(t_2-t_1) \\
&\qquad \times \Bigg(\sum_{i=1}^{n} |b_i| \int_0^{\eta} \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} n_2(s) \mathrm{d}s + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} n_2(s) \mathrm{d}s \Bigg) \Bigg].
\end{aligned}
$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero independent of $x \in B_r$. Thus F maps bounded sets into equicontinuous sets in B_r. By Arzela-Ascoli's Theorem, F is completely continuous.

Now let $x = \lambda_0 Fx$, where $\lambda_0 \in (0,1)$. Then, for $t \in J$, we have

$$
\begin{aligned}
x(t) = {} & \lambda_0 \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, \mathrm{d}s \\
& + \lambda_0 (A_4 - A_3 t) \sum_{i=1}^{n} a_i \int_0^{\zeta} \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, \mathrm{d}s \\
& + \lambda_0 (A_2 + A_1 t) \Bigg\{ \sum_{i=1}^{n} b_i \int_0^{\eta} \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, \mathrm{d}s \\
& - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s), x(\lambda(s)), Kx(s)) \, \mathrm{d}s \Bigg\}.
\end{aligned}
$$

Then, using the computations of the first step, we have

$$\|x(t)\| \le (\psi(\|x\|)\|n_1\|_{L^1} + \|n_2\|_{L^1})\theta_1$$

which can be written as

$$\|x\|\Big\{(\psi(\|x\|)\|n_1\|_{L^1} + \|n_2\|_{L^1})\theta_1\Big\}^{-1} \le 1.$$

In view of $(A5)$, there exists M such that $\|x\| \neq M$. We set

$$U = \big\{x \in Z : \|x\| < M\big\}.$$

Note that the operator $F\colon \bar{U} \to Z$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda_0 Fx$ for some $\lambda_0 \in (0,1)$. Consequently, by the Leray-Schauder nonlinear alternative, we deduce that F has a fixed point $x \in \bar{U}$ which is a solution to the problem (1). □

The next result is based on Banach contraction principle.

Theorem 3.4. *Assume that the hypotheses* $(A1)$ *and* $(A3)$ *hold. Then the boundary value problem* (1) *has a unique solution on* J.

Proof. Let $M_1 = \sup_{t\in J}\|f(t,0,0,0)\|$ and $M_2 = \sup_{t,s\in J}\|k(t,s,0,0)\|$. Consider $B_r = \{x \in Z : \|x\| \le r\}$, where $r \ge \frac{\Delta_2}{1-\Delta_1}$ with $\Delta_2 = L_f M_2\theta_2 + M_1\theta_1$ and Δ_1 is given by the assumption $(A3)$. Now we show that $FB_r \subset B_r$, where $F\colon Z \to Z$ is defined by (8). For $x \in B_r$, we have

$$\begin{aligned}
&\|(Fx)(t)\| \\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)\|\mathrm{d}s \\
&\qquad + (|A_4|+|A_3|)\sum_{i=1}^n |a_i|\int_0^\zeta \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)\|\mathrm{d}s \\
&\qquad + (|A_2|+|A_1|)\bigg\{\sum_{i=1}^n |b_i|\int_0^\eta \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)\|\mathrm{d}s \\
&\qquad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right)\|\mathrm{d}s\bigg\} \\
&\quad\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big[\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right) - f(s,0,0,0)\| + \|f(s,0,0,0)\|\big]\mathrm{d}s \\
&\qquad + (|A_4|+|A_3|)\sum_{i=1}^n |a_i|\int_0^\zeta \frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}\big[\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right) \\
&\qquad - f(s,0,0,0)\| + \|f(s,0,0,0)\|\big]\mathrm{d}s + (|A_2|+|A_1|)\bigg\{\sum_{i=1}^n |b_i|\int_0^\eta \frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)} \\
&\qquad \times \big[\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right) - f(s,0,0,0)\| + \|f(s,0,0,0)\|\big]\mathrm{d}s \\
&\qquad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\big[\|f\left(s,x(s),x(\lambda(s)),Kx(s)\right) - f(s,0,0,0)\| + \|f(s,0,0,0)\|\big]\mathrm{d}s\bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big[L_f(\|x(s)\|+\|x(\lambda(s))\|+\|Kx(s)\|)+M_1\big]\mathrm{d}s+(|A_4|+|A_3|)\\
&\quad\times\sum_{i=1}^n|a_i|\int_0^\zeta\frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}\big[L_f(\|x(s)\|+\|x(\lambda(s))\|+\|Kx(s)\|)+M_1\big]\mathrm{d}s\\
&\quad+(|A_2|+|A_1|)\bigg\{\sum_{i=1}^n|b_i|\int_0^\eta\frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)}\big[L_f(\|x(s)\|+\|x(\lambda(s))\|+\|Kx(s)\|)\\
&\quad+M_1\big]\mathrm{d}s+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\big[L_f(\|x(s)\|+\|x(\lambda(s))\|+\|Kx(s)\|)+M_1\big]\mathrm{d}s\bigg\}\\
&\leq\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\bigg[L_f\bigg(\|x(s)\|+\|x(\lambda(s))\|+\int_0^s\big[\|k(s,\tau,x(\tau),x(\sigma(\tau)))-k(s,\tau,0,0)\|\\
&\quad+\|k(s,\tau,0,0)\|\big]d\tau\bigg)+M_1\bigg]\mathrm{d}s+(|A_4|+|A_3|)\sum_{i=1}^n|a_i|\int_0^\zeta\frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}\\
&\quad\times\bigg[L_f\bigg(\|x(s)\|+\|x(\lambda(s))\|+\int_0^s\big[\|k(s,\tau,x(\tau),x(\sigma(\tau)))-k(s,\tau,0,0)\|\\
&\quad+\|k(s,\tau,0,0)\|\big]d\tau\bigg)+M_1\bigg]\mathrm{d}s+(|A_2|+|A_1|)\bigg\{\sum_{i=1}^n|b_i|\int_0^\eta\frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)}\\
&\quad\times\bigg[L_f\bigg(\|x(s)\|+\|x(\lambda(s))\|+\int_0^s\big[\|k(s,\tau,x(\tau),x(\sigma(\tau)))-k(s,\tau,0,0)\|\\
&\quad+\|k(s,\tau,0,0)\|\big]d\tau\bigg)+M_1\bigg]\mathrm{d}s+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\bigg[L_f\bigg(\|x(s)\|+\|x(\lambda(s))\|\\
&\quad+\int_0^s\big[\|k(s,\tau,x(\tau),x(\sigma(\tau)))-k(s,\tau,0,0)\|+\|k(s,\tau,0,0)\|\big]d\tau\bigg)+M_1\bigg]\mathrm{d}s\bigg\}\\
&\leq 2rL_f\big(\theta_1+L_k\theta_2\big)+\big(L_fM_2\theta_2+M_1\theta_1\big)\\
&\leq\Delta_1r+\Delta_2\leq r.
\end{aligned}
$$

This shows that $FB_r\subset B_r$. Next, for $x,y\in Z$ and $t\in J$, we obtain

$$
\begin{aligned}
&\|(Fx)(t)-(Fy)(t)\|\\
&\quad\leq\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),x(\lambda(s)),Kx(s))-f(s,y(s),y(\lambda(s)),Ky(s))\|\mathrm{d}s\\
&\qquad+(|A_4|+|A_3|)\sum_{i=1}^n|a_i|\int_0^\zeta\frac{(\zeta-s)^{\gamma_i+q-1}}{\Gamma(\gamma_i+q)}\|f(s,x(s),x(\lambda(s)),Kx(s))\\
&\qquad-f(s,y(s),y(\lambda(s)),Ky(s))\|\mathrm{d}s+(|A_2|+|A_1|)\bigg\{\sum_{i=1}^n|b_i|\int_0^\eta\frac{(\eta-s)^{\delta_i+q-1}}{\Gamma(\delta_i+q)}\\
&\qquad\times\|f(s,x(s),x(\lambda(s)),Kx(s))-f(s,y(s),y(\lambda(s)),Ky(s))\|\mathrm{d}s\\
&\qquad+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),x(\lambda(s)),Kx(s))-f(s,y(s),y(\lambda(s)),Ky(s))\|\mathrm{d}s\bigg\}\\
&\quad\leq 2L_f\big(\theta_1+L_k\theta_2\big)\|x-y\|
\end{aligned}
$$

$$\|Fx-Fy\|\leq\Delta_1\|x-y\|.$$

Here Δ_1 depends only on the parameters involved in the problem. By assumption $(A3)$, $\Delta_1<1$ and therefore, F is a contraction. Hence, by the Banach contraction principle, the problem (1) has a unique solution on J. □

Remark 3.1. By fixing the parameters in the given problem, several special cases can be obtained.

(a) For $f = f(t, x(t))$ and taking a_i, $b_i = 0$, $i = 2, \dots, n$ in the problem (1),
  (i) the results of [**3**] are got by taking $a_1 = 0$, $\delta_1 = 1$.
  (ii) the results of [**4**] appear as a special case with $X = \mathbb{R}$.
  (iii) the results of [**30**] are obtained as a special case by taking $a_1 = 0$ and $X = \mathbb{R}$.

(b) For a_i, $b_i = 0$, $i = 1, 2, \dots, n$, the problem (1) reduces to a Dirichlet problem.

(c) Taking a_i, $b_i = 0$, $i = 2, \dots, n$, and $\gamma_1, \delta_1 = 1$, the problem (1) reduces to a fractional delay integrodifferential equation with integral boundary conditions.

(d) For $q = 2$, taking a_i, $b_i = 0$, $i = 2, \dots, n$, and $\gamma_1, \delta_1 = 1$, we obtain new results for a second order delay differential equation with integral boundary conditions.

(e) The problem (1) can be generalized to fractional integrodifferential boundary value problem with multiple delay of the form

$$\begin{aligned}
{}^{C}D^{q}x(t) &= f\Big(t, x(t), x(\lambda_1(t)), x(\lambda_2(t)), \dots, x(\lambda_{m_1}(t)),\\
&\qquad \int_0^t k(t, s, x(s), x(\sigma_1(s)), x(\sigma_2(s)), \dots, x(\sigma_{m_2}(s)))\mathrm{d}s\Big), \quad t \in J = [0, 1],\\
x(0) &= \sum_{i=1}^{n} a_i (I^{\gamma_i}x)(\zeta),\\
x(1) &= \sum_{i=1}^{n} b_i (I^{\delta_i}x)(\eta), \qquad 0 < \zeta < \eta < 1,
\end{aligned}$$

where $1 < q \leq 2$, the functions $\lambda_i, \sigma_j \colon J \to J$, $i = 1, 2, \dots, m_1$, $j = 1, 2, \dots, m_2$, are continuous such that $0 \leq \lambda_i(t) \leq t$, $0 \leq \sigma_j(t) \leq t$, $t \in J$, and under suitable assumptions, we can establish the existence results.

4. Example

Consider the following fractional boundary value problem

$$\begin{aligned}
{}^{C}D^{3/2}x(t) &= \frac{1}{(t+63)}\frac{|x(t)|}{1+|x(t)|} + \frac{\mathrm{e}^{-t}}{62+\mathrm{e}^{t}}\frac{|x(t/2)|}{1+|x(t/2)|}\\
&\quad + \frac{1}{63}\int_0^t \frac{\mathrm{e}^{-s}}{7}\frac{|x(s^2)|}{1+|x(s^2)|}\mathrm{d}s, \qquad t \in [0, 1],\\
x(0) &= \sum_{i=1}^{5} a_i (I^{\gamma_i}x)(\zeta),\\
x(1) &= \sum_{i=1}^{5} b_i (I^{\delta_i}x)(\eta), \qquad 0 < \zeta < \eta < 1.
\end{aligned} \tag{10}$$

Here $X=\mathbb{R}$, $\quad q=\dfrac{3}{2}, \quad n=5, \quad \zeta=\frac{1}{2}, \quad \eta=\dfrac{2}{3}$,

$$\begin{array}{lllll}
a_1=3, & a_2=4, & a_3=7, & a_4=11, & a_5=17,\\
\gamma_1=\dfrac{1}{3}, & \gamma_2=\dfrac{2}{3}, & \gamma_3=\dfrac{4}{3}, & \gamma_4=\dfrac{5}{3}, & \gamma_5=\dfrac{8}{3},\\
b_1=2, & b_2=5, & b_3=16, & b_4=11, & b_5=10,\\
\delta_1=\dfrac{1}{2}, & \delta_2=\dfrac{3}{4}, & \delta_3=\dfrac{5}{4}, & \delta_4=\dfrac{3}{2}, & \delta_5=\dfrac{7}{4}.
\end{array}$$

From the above given data, we see that

$A=-3.126434$, $A_1=3.122104$, $A_2=-0.914796$, $A_3=6.693181$, $A_4=1.858697$,

$\rho_1=1.264617$, $\rho_2=0.195105$, $\rho_3=3.284678$, $\rho_4=0.612445$,

$\theta_1=27.86379$, $\quad \theta_2=5.656502$.

(i) From (10), we have

$$\begin{aligned}
f\left(t, x(t), x(\lambda(t)), K x(t)\right) = {} & \frac{1}{(t+63)} \frac{|x(t)|}{1+|x(t)|}+\frac{\mathrm{e}^{-t}}{62+\mathrm{e}^{t}} \frac{|x(t/2)|}{1+|x(t/2)|} \\
& +\frac{1}{63} \int_0^t \frac{\mathrm{e}^{-s}}{7} \frac{\left|x(s^2)\right|}{1+\left|x(s^2)\right|} \mathrm{d}s
\end{aligned}$$

where $Kx(t)=\int_0^t \frac{\mathrm{e}^{-s}}{7} \frac{|x(s^2)|}{1+|x(s^2)|} \mathrm{d}s$, $\lambda(t)=t/2$, $\sigma(t)=t^2$. The condition $(A1)$ is satisfied with $L_f=1/63$ and $L_k=1/7$. The condition $(A2)$ is satisfied with $l(t)=1$ and $\phi(\|x\|)=5/147$. Computing the value of L, we have $L=0.884972<1$, thereby satisfying the condition (9). Thus all the assumptions of the Theorem 3.1 are satisfied. Hence the problem (10) with the given function f has at least one solution on J.

(ii) Now we take

$$f\left(t, x(t), x(\lambda(t)), K x(t)\right)=\frac{\sin 2\pi x}{72\pi}+\frac{1+\mathrm{e}^{-t}}{71+\mathrm{e}^{t}} \frac{x(t^3)}{1+x(t^3)}+\frac{1}{36} \int_0^t \frac{\mathrm{e}^{-s}}{11} \frac{x(\sin s)}{1+x(\sin s)} \mathrm{d}s$$

in (10), where $Kx(t)=1/36 \int_0^t \frac{\mathrm{e}^{-s}}{11} \frac{x(\sin s)}{1+x(\sin s)} \mathrm{d}s$, $\lambda(t)=t^3$, $\sigma(t)=\sin t$. Clearly

$$\left\|f\left(t, x(t), x(\lambda(t)), K x(t)\right)\right\| \leq \frac{\|x\|}{36}+\frac{1}{33}.$$

Here $n_1(t)=1/36$, $\psi(\|x\|)=\|x\|$ and $n_2(t)=1/33$. From $(A5)$, we have

$$\frac{M}{\left(\psi(M)\|n_1\|_{L^1}+\|n_2\|_{L^1}\right)\theta_1}>1$$

from which we find that $M>M_1$, where $M_1 \backsimeq 3.735994$ thereby satisfying the condition $(A5)$. Thus all the assumptions of the Theorem 3.3 are satisfied. Hence the problem (10) with the given function f has at least one solution on J.

(iii) Taking

$$f\left(t,x(t),x(\lambda(t)),Kx(t)\right)=\frac{1}{(2t+8)^2}\frac{x(t)}{1+x(t)}+\frac{1+\mathrm{e}^{-t}}{127+\mathrm{e}^{t}}\frac{x(\sin t)}{1+x(\sin t)}$$
$$+\frac{1}{64}\int_0^t \mathrm{e}^{-\frac{x(\sin s)}{15}}\,\mathrm{d}s$$

in (10), we have $Kx(t)=\int_0^t e^{-\frac{x(\sin s)}{15}}\,\mathrm{d}s$, $\lambda(t)=\sigma(t)=\sin t$. The condition $(A1)$ is satisfied with $L_f = 1/64$ and $L_k = 1/15$. Computing the value of Δ_1, we have $\Delta_1 = 0.882528 < 1$, thereby satisfying the condition $(A3)$. Thus all the assumptions of the Theorem 3.4 are satisfied. Hence the problem (10) with the given function f has a unique solution on J.

References

1. Agarwal R. P., Benchohra M. and Hamani S., *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109** (2010), 973–1033.

2. Ahmad B. and Nieto J. J., *Existence and approximation of solutions for a class of nonlinear impulsive functional differential equations with anti-periodic boundary conditions*, Nonlinear Anal., **69** (2008), 3291–3298.

3. Ahmad B., Ntouyas S. K. and Alsaedi A., *New existence results for nonlinear fractional differential equations with three-point integral boundary conditions*, Adv. Difference Equ., **2011**, Art. ID 107384, 11 pp.

4. Ahmad B., Ntouyas S. K. and Assolami A., *Caputo type fractional differential equations with nonlocal Riemann-Liouville integral boundary conditions*, J. Appl. Math. Comput., **41** (2013), 339–350.

5. Alsaedi A., Ntouyas S. K. and Ahmad B., *Existence results for Langevin fractional differential inclusions involving two fractional orders with four-point multiterm fractional integral boundary conditions*, Abstr. Appl. Anal., **2013**, Art. ID 869837, 17 pp.

6. Ardjouni A. and Djoudi A., *Existence of periodic solutions for a second order nonlinear neutral functional differential equation*, Electron J. of Math. Anal. and Appl., **2** (2014), 117–126.

7. Balachandran K. and Samuel F. P., *Existence of solutions for quasilinear delay integrodifferential equations with nonlocal conditions*, Electron. J. Differential Equations, **6** (2009), 1–7.

8. Balachandran K., Kim J. H. and Leelamani A., *Existence results for nonlinear abstract neutral differential equations with time varying delays*, Applied Mathematics E-Notes, **6** (2006), 186–193.

9. Balachandran K., Park J. Y. and Chandrasekaran M., *Nonlocal Cauchy problem for delay integrodifferential equations of Sobolov type in Banach spaces*, Appl. Math. Lett., **15** (2002), 845–854.

10. Balachandran K., Kiruthika S., Rivero M. and Trujillo J. J., *Existence of solutions for fractional delay integrodifferential equations*, Journal of Applied Nonlinear Dynamics, **1** (2012), 309–319.

11. Benchohra M., Henderson J., Ntouyas S. K. and Ouahab A., *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl., **338** (2008), 1340–1350.

12. Cabada A. and Wang G., *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl., **389** (2012), 403–411.

13. Chauhan A. and Dabas J., *Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition*, Commun. Nonlinear Sci. Numer. Simul., **19** (2014), 821-829.

14. Dabas J. and Gautam G. R., *Impulsive neutral fractional integro-differential equation with state-dependent delays and integral condition*, Electron. J. Differential Equations, **273** (2013), 1–13.

15. Darwish M. A. and. Ntouyas S. K., *Semilinear functional differential equations of fractional order with state-dependent delay*, Electron. J. Differential Equations, **38** (2009), 1–10.

16. Darwish M. A. and Ntouyas S. K., *Boundary value problems for fractional functional differential equations of mixed type*, Commun. Appl. Anal., **13** (2009), 31–38.

17. Darwish M. A. and Ntouyas S. K., *On initial and boundary value problems for fractional order mixed type functional differential inclusions*, Comput. Math. Appl., **59** (2010), 1253–1265.

18. Diethelm K., *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.

19. Granas A. and Dugundji J., *Fixed Point Theory*, Springer-Verlag, New York, 2003.

20. Guezane-Lakoud A. and Khaldi R., *Solvability of a fractional boundary value problem with fractional integral condition*, Nonlinear Anal., **75** (2012), 2692–2700.

21. Han J., Liu Y. and Zhao J., *Integral boundary value problems for first order nonlinear impulsive functional integro-differential differential equations*, Appl. Math. Comput., **218** (2012), 5002–5009.

22. Jalilian Y. and Jalilian R., *Existence of solution for delay fractional differential equations*, Mediterr. J. Math., **10** (2013), 1731–1747.

23. Jankowski T., *Existence results to delay fractional differential equations with nonlinear boundary conditions*, Appl. Math. Comput., **219** (2013), 9155–9164.

24. Khan R. A., Rehman M. U. and Henderson J., *Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions*, Fractional Diff. Calc., **1** (2011), 29–43.

25. Kilbas A. A., Srivastava H. M. and Trujillo J. J., *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam, 2006.

26. Liao J., Chen F. and Hu S., *Existence of solutions for fractional impulsive neutral functional differential equations with infinite delay*, Neurocomputing, **122** (2013), 156–162.

27. Miller K. S. and Ross B., *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New York, 1993.

28. Mophou G. M. and N'Guérékata G. M., *Existence of mild solutions of some semilinear neutral fractional functional evolution equations with infinite delay*, Appl. Math. Comput., **216** (2010), 61–69.

29. Ntouyas S. K., *Initial and boundary value problems for functional differential equations via the topological transversality method: A survey*, Bull. Greek Math. Soc., 40 (1998), 3–41.

30. Ntouyas S. K., *Existence results for nonlocal boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions*, Discuss. Math. Differ. Incl. Control Optim., **33** (2013), 17–39.

31. Nyamoradi N., *A six-point nonlocal integral boundary value problem for fractional differential equations*, Indian J. Pure Appl. Math., **43** (2012), 429–454.

32. Ouyang Z., Chen Y. and Zou S., *Existence of positive solutions to a boundary value problem for a delayed nonlinear fractional differential system*, Bound. Value Probl., **2011**, Art. ID 475126, 17 pages.

33. Podlubny I., *Fractional Differential Equations*, Academic Press, New York, 1999.

34. Shri Akiladevi K., Balachandran K. and Kim J. K., *Existence of solutions of nonlinear neutral integrodifferential equations of Sobolev type in Banach spaces*, Nonlinear Funct. Anal. Appl., **18** (2013), 359–381.

35. Shri Akiladevi K., Balachandran K. and Kim J. K., *Existence results for neutral fractional integrodifferential equations with fractional integral boundary conditions*, Nonlinear Funct. Anal. Appl., **19** (2014), 251–270.

36. Tsamatos P. Ch. and Ntouyas S. K., *Existence and uniqueness of solutions of a general type boundary value problem for second order functional differential equations*, J. Math. Anal. Appl., **173** (1993), 3–17.

37. Wang H., *Existence results for fractional functional differential equations with impulses*, J. Appl. Math. Comput., **38** (2012), 85–101.

38. Yuan Q. and Yang W., Positive solutions of nonlinear boundary value problems for delayed fractional q-difference systems, Adv. Difference Equ., **2014**, Art. ID 51.

39. Zhang X., Huang X. and Liu Z., *The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay*, Nonlinear Analysis: Hybrid Systems, **4** (2010), 775–781.

40. Zhang X., *Existence and iteration of positive solutions for high-order fractional differential equations with integral conditions on a half-line*, J. Appl. Math. Comput., **45** (2014), 137–150.

41. Zhou W. and Chu Y., *Existence of solutions for fractional differential equations with multi-point boundary conditions*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 1142–1148.

K. Shri Akiladevi, Department of Mathematics, Bharathiar University, Coimbatore, India,
e-mail: `shriakiladevi@gmail.com`

K. Balachandran, Department of Mathematics, Bharathiar University, Coimbatore, India,
e-mail: `kb.maths.bu@gmail.com`