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DATKO-PERRON'S PROBLEM FOR DICHOTOMY OF DIFFERENTIAL EQUATIONS

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ABSTRACT. The purpose of this paper is to show that in the study of dichotomy for evolutionary family generated by differential systems, using the Perron's method, it is enough to consider space L^∞ as an output, because when the output space is L^q, $q<\infty$, it can be used the Datko's theorem (see P. Preda, M. Megan [**17**]).

1. PRELIMINARIES

In 1930, O. Perron ([**16**])proved that the differential system

$$\dot{x}(t) = A(t)x(t) \tag{A}$$

is exponentially dichotomus if and only if for each $f\colon \mathbb{R}_+ \to X$, f continuous and bounded ($f \in \mathcal{C}$), the inhomogeneous differential equation $(A,f)\colon \dot{x}(t) = A(t)x(t) + f(t)$ has a bounded solutions on $\mathbb{R}_+$. This fact was proved in finite dimensional spaces.

In 1948, R. Bellman [**1**] and D. L. Kucer ([**8**]) studied that in conditions mentioned above, there is a connection between the norm from $\mathcal{C}$ of function x (also called output) and the norm of function f(called input).

In 1958, J. L. Massera and J. J. Schăffer ([**11**]) studied in Perron's problem for evolutionary family generated by differential equations, the input space with L^p and the output space with L^∞.

If the output space is $\mathcal{C}$ or L^∞, we can use the inequality $\|x(t)\| \le \|x\|_\infty$ a.e., where $\|x(t)\|$ is the norm in the Banach space X and $\|x\|_\infty$ is the norm in L^∞.

In 1966, J. L. Massera and J. J. Schăffer([**12**]) studied the asymptotic behaviour for differential systems (A) using input-output spaces, spaces that are translation-invariant.

J. L. Massera and J. J. Schăffer in ([**11**, Example 5.1, p. 536]) showed that the hypothesis that the evolutionary family generated by differential system (A) has exponentially growth (there exist $M, \omega > 0$ such that $\|\Phi(t,t_0)\| \le M\,\mathrm{e}^{\omega(t-t_0)}$ for all $t \ge t_0$), is required in the study of asymptotic uniform behaviour of this family.

In 1970, R. Datko ([**5**]) proved that in Hilbert space a C_0-semigroup with the infinitesimal generator A is exponentially stable if and only if there is a linear

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operator W that is bounded, positive and such that

$$\langle Ax, Wx\rangle + \langle Wx, Ax\rangle \leq \|x\|^2, \qquad \text{for all } x \in D(A).$$

To prove this R. Datko highlights a result, proved to be extremely useful in the literature dedicated to this issue, according to which the semigroup $\{T(t)\}_{t\geq 0}$ is exponentially stable if and only if

$$\int_0^\infty \|T(t)x\|^2 \mathrm{d}t < \infty \qquad \text{for all } x \in X.$$

This result was extended by A. Pazy ([**15**]) from $p = 2$ to $p \geq 1$, and by W. Littman ([**9**]) to $p > 0$.

In 1972, R. Datko ([**6**]) extended this result to evolutionary family with exponential growth and in 1985, P. Preda with M. Megan ([**17**]) extended Datko's result from stability of evolutionary family to dichotomy of evolutionary family.

In 1974, D. L. Lovelady ([**10**]) showed that if U is the fundamental solution of the differential system (A):

$$\begin{cases} \dot{U}(t) = A(t)U(t) \\ U(0) = I, \end{cases}$$

$X_1 = \{x \in X : U(\cdot)x \in L^\infty\}$ is complemented, X_2 is one of it's complements and P_1, P_2 are the projectors associated to this decomposition, then the pair (L^p, L^∞) is admissible to the differential system (A) (for every $f \in L^p$, there is $x \in L^\infty$ such that $\dot{x}(t) = A(t)x(t) + f(t)$ for all $t \geq 0$) with $p > 1$ if and only if

$$\Big(\int_0^t \|U(t)P_1U^1(\tau)\|^p d\tau\Big)^{\frac{1}{p}} + \Big(\int_t^\infty \|U(t)P_2U^1(\tau)\|^p d\tau\Big)^{\frac{1}{p}} \leq k \quad \text{for each } t \geq 0.$$

Making a link between the admissibility of a pair of function spaces and the dichotomy of the differential system (A), see W. A. Coppel [**3**].

It is easy to see that when you study the stability of evolutionary family with exponential growth, using the admissibility of (L^p, L^q) and considering the function $f(t) = \varphi_{[t_0,t_0+1]}\Phi(t, t_0)x$ as an input, where φ_E is the characteristic function of the set E, we get $x(t) = \Phi(t, t_0)x$ for all $t \geq t_0 + 1$.

In this way we can see the importance of studying the case where L^∞ is the output space, because the case when $L^q, q < \infty$ is the output space has beenalready solved by Datko's theorem.

The purpose of this article is to show that for evolutionary family with exponential growth generated by differential systems, using pairs of spaces (L^p, L^q) for $q < \infty$, Datko's theorem can be used to study the dichotomy (see P. Preda, M. Megan([**17**]).) Thus the comment made above for stability can be extended for the dichotomy of the evolutionary family. In this way we have a Datko-Perron's method to the dichotomy of the evolutionary family generated by the differential system in infinite dimensional spaces.

Let X be a Banach space and $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators acting on X (the norm on both X and $\mathcal{B}(X)$ denoted by $|| \cdot ||$).

Notations. For an upper unbounded interval I, we denote by $\mathcal{M}(I,X)$ the spaces of all Bochner measurable functions from I to X.

$$L^p(I,X)=\Big\{f\in\mathcal{M}(I,X):\int_I\|f(t)\|^p\mathrm{d}t<\infty\Big\},\qquad\text{where } p\in[1,\infty),$$
$$L^\infty(I,X)=\Big\{f\in\mathcal{M}(I,X):ess\sup_{t\in I}\|f(t)\|\mathrm{d}t<\infty\Big\},$$
$$M_1(I,X)=\Big\{f\in\mathcal{M}(I,X):\sup_{t\in I}\int_t^{t+1}\|f(t)\|\mathrm{d}t<\infty\Big\}.$$

We note that $L^p(I,X)$, $L^\infty(I,X)$, $M_1(I,X)$ are Banach spaces endowed with the norms:

$$\|f\|_p=\Big(\int_I\|f(t)\|^p\mathrm{d}t\Big)^{\frac{1}{p}},\quad \|f\|_\infty=\operatorname{ess}\sup_{t\in I}\|f(t)\|,\quad \|f\|_{M_1}=\sup_{t\in I}\int_t^{t+1}\|f(s)\|\mathrm{d}s.$$

We denote $M_1(\mathbb{R}_+,\mathcal{B}(X))$ with $M_1(\widetilde{X})$.

$$L^1_{loc}(\mathcal{B}(X))=\Big\{f\in\mathcal{M}(\mathbb{R}_+,\mathcal{B}(X)):\int_K\|f(t)\|\mathrm{d}t<\infty \text{ for each compact } K \text{ in } \mathbb{R}_+\Big\}.$$

For $A\in L^1_{loc}(\mathcal{B}(X))$, $t_0\geq 0$ and $x_0\in X$, we consider the homogeneous Cauchy problem

$$(A,t_0,x_0):\begin{cases}\dot{x}(t)=A(t)x(t)\\ x(t_0)=x_0.\end{cases}\tag{1}$$

Theorem 1.1. *The homogeneous Cauchy's problem* (1) *has a unique solution.*

Proof. See [**4**]. □

We consider now the Cauchy operatorial problem

$$(A,0,I):\begin{cases}\dot{X}(t)=A(t)X(t)\\ X(0)=I,\end{cases}\tag{2}$$

where $A\colon\mathbb{R}_+\to\mathcal{B}(X)$, $A\in L^1_{loc}(\mathbb{R}_+,\mathcal{B}(X))$.

It was proved that the problem (2) has a unique solution, which is denoted by U, U is invertible and U^{-1} is a solution for

$$(\widetilde{A},0,I):\begin{cases}\dot{X}(t)=-X(t)A(t)\\ X(0)=I.\end{cases}\tag{3}$$

We denote $\Phi(t,t_0)=U(t)U^{-1}(t_0)$.

It is known that Φ is the evolution family generated by the equation

$$\dot{x}(t)=A(t)x(t).\tag{A}$$

We list now some properties of the evolution family generated by the equation (A).

Proposition 1.1.
a) $\Phi(t,t) = I$ *for all* $t \geq 0$.
b) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$ *for all* $t,s,t_0 \geq 0$.
c) $\dfrac{\mathrm{d}}{\mathrm{d}t}\Phi(t,s) = A(t)\Phi(t,s)$ *for all* $t,s \geq 0$.
d) $\dfrac{\mathrm{d}}{\mathrm{d}s}\Phi(t,s) = -\Phi(t,s)A(s)$ *for all* $t,s \geq 0$.
e) *If* $A \in M_1(\widetilde{X})$, *then there exist* $M > 0$ *and* $\omega > 0$ *such that* $\|\Phi(t,t_0)\| \leq M\,\mathrm{e}^{\omega(t-t_0)}$ *for all* $t \geq t_0 \geq 0$.

Proof. See [**4**]. □

If $f\colon \mathbb{R}_+ \to X$, $f \in L^1_{loc}(\mathbb{R}_+, X)$, $t_0 \geq 0, x_0 \in X$, we consider the Cauchy problem

$$(A, t_0, x_0; f) = \begin{cases} \dot{x}(t) = A(t)x(t) + f(t) \\ x(t_0) = x_0. \end{cases} \tag{4}$$

Theorem 1.2. *The Cauchy problem* (4) *has a unique solution*

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)f(t)\mathrm{d}\tau.$$

Proof. See [**12**]. □

We denote by $X_1 = \{x \in X : U(\cdot)x \in L^\infty\}$. Assume that X_1 is closed subspace and that there exists a closed subspace X_2 such that $X = X_1 \bigoplus X_2$ and P_1, P_2 the projectors associated to this decomposition.

Definition 1.1. The system (A) is called ordinary dichotomic if and only if there exists $N > 0$ such that:
i) $\|U(t)P_1U^{-1}(s)\| \leq N$ for all $t \geq s \geq 0$,
ii) $\|U(t)P_2U^{-1}(s)\| \leq N$ for all $s \geq t \geq 0$.

Definition 1.2. The system (A) is called exponentially dichotomic if and only if there exist $N, \nu > 0$ such that:
i) $\|U(t)P_1U^{-1}(s)\| \leq N\,\mathrm{e}^{-\nu(t-s)}$ for all $t \geq s \geq 0$,
ii) $\|U(t)P_2U^{-1}(s)\| \leq N\,\mathrm{e}^{-\nu(s-t)}$ for all $s \geq t \geq 0$.

Remark 1.1. If the system (A) is exponentially dichotomic, then (A) is ordinary dichotomic.

Theorem 1.3 (Datko's Theorem)**.** *Let* $A \in M_1(\widetilde{X})$. *The differential system* (A) *is exponentially dichotomic if and only if there exist* $p > 0$ *and* $M > 0$ *such that*

$$\Big(\int_t^\infty \|U(\tau)P_1U^{-1}(t)x\|^p\mathrm{d}\tau\Big)^{\frac{1}{p}} + \Big(\int_0^t \|U(\tau)P_2U^{-1}(t)x\|^p\mathrm{d}\tau\Big)^{\frac{1}{p}} \leq M\|x\|$$

for all $t \geq 0$, $x \in X$.

Proof. See [**17**] □

Lemma 1.1. *Let* $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be a function with the property that there exist* $H, \delta > 0$ *and* $\eta \in (0,1)$ *such that:*

(i) $f(t) \le Hf(t_0)$ *for all* $t \in [t_0, t_0+\delta]$ *and all* $t_0 \ge 0$*;*

(ii) $f(t_0+\delta) \le \eta f(t_0)$ *for all* $t_0 \ge 0$.

Then there exist $N, \nu > 0$ *such that*

$$f(t) \le N\,\mathrm{e}^{-\nu(t-t_0)}\,f(t_0) \qquad \text{for all } t \ge t_0 \ge 0.$$

Proof. See [**18**]. □

2. The main result

We consider $A \in M_1(\widetilde{X})$.

Definition 2.1. The differential system (A) satisfies the (p,q) Perron's condition for dichotomy if and only if for every $f \in L^p(X)$, there exists $x \in L^q(X)$ such that $\dot{x}(t) = A(t)x(t) + f(t)$.

We set now $X_1^q = \{x \in X : U(\cdot)x \in L^q\}$. Assume that X_1^q is closed and that there exists X_2^q closed and such that $X = X_1^q \bigoplus X_2^q$. We denote P_1^q, P_2^q the projectors associated to this decomposition.

Remark 2.1. $X_1^\infty = X_1$ and $X_2^\infty = X_2$.

Proposition 2.1. *If* (A) *satisfies the* (p,q) *Perron's condition for dichotomy, then for every* $f \in L^p(X)$*, there is a unique* $x \in L^q(X)$ *such that* $\dot{x}(t) = A(t)x(t) + f(t)$ *and* $x(0) \in X_2^q$.

Proof. Let $f \in L^p(X)$. We have that there exists $x \in L^q(X)$ such that $\dot{x}(t) = A(t)x(t) + f(t)$.

We consider $y(t) = x(t) - U(t)P_1^q x(0)$, it results $\dot{y}(t) = A(t)y(t) + f(t)$, $y(0) = x(0) - P_1^q x(0) = P_2^q x(0) \in X_2^q$ and $y \in L^q(X)$.

To prove the uniqueness, we suppose that there exist $x_1, x_2 \in L^q(X)$ such that $\dot{x_i}(t) = A(t)x_i(t) + f(t)$ and $x_i(0) \in X_2^q$ for $i \in \{1,2\}$.

Let $w(t) = x_1(t) - x_2(t)$, it implies $\dot{w}(t) = A(t)w(t), w(0) \in X_2^q, w \in L^q(X)$ and from here we have $w(0) \in X_1^q \cap X_2^q = \{0\}$. It results $w = 0$ and then $x_1 = x_2$. □

Let $f \in L^p(X)$ and $\dot{x}(t) = A(t)x(t) + f(t), x(0) \in X_2^q$. Throughout this paper we will denote this x by x_f.

Theorem 2.1. *If* (A) *satisfies the* (p,q) *Perron's condition for dichotomy then there exists* $K > 0$ *such that* $\|x_f\|_q \le K\|f\|_p$ *and* $\|x_f(0)\| \le K\|f\|_p$ *for all* $f \in L^p(X)$.

Proof. Let $\mathcal{U}\colon L^p(X) \to X_2^q \bigoplus L^q(X)$, defined by $\mathcal{U}f = (x_f(0), x_f)$. It is obvious that $\mathcal{U}$ is a linear operator. We will show that $\mathcal{U}$ is also closed.

Let $(f_n)_{n\in\mathbb{N}^*}$ be a sequence such that $f_n \in L^p(X)$, $f \in L^p(X)$, $g \in L^q(X)$, $y \in X$ such that $f_n \to f$ in $L^p(X)$, $x_{f_n}(0) \to y$ in X and $x_{f_n} \to g$ in $L^q(X)$, which means that $\mathcal{U}f_n \to (y,g)$ in $X_2^q \bigoplus L^q(X)$.

But

$$\Big\| \int_0^t \Phi(t,\tau) f_n(\tau) \mathrm{d}\tau - \int_0^t \Phi(t,\tau) f(\tau) \mathrm{d}\tau \Big\|$$
$$= \Big\| \int_0^t \Phi(t,\tau)(f_n(\tau) - f(\tau)) \mathrm{d}\tau \Big\| \leq \int_0^t \|\Phi(t,\tau)\| \cdot \|f_n(\tau) - f(\tau)\| \mathrm{d}\tau.$$

Since $\Phi(t,\cdot)x\colon [0,t] \to X$ is continuous for every $x \in X$, there exists $M_{t,x}$ such that $\|\Phi(t,\tau)x\| \leq M_{t,x}$, for every $\tau \in [0,t]$. From the uniform boundedness principle we have that there exist $M(t) > 0$ such that $\|\Phi(t,\tau)\| \leq M(t)$ for every $\tau \in [0,t]$.

Therefore,

$$\Big\| \int_0^t \Phi(t,\tau) f_n(\tau) \mathrm{d}\tau - \int_0^t \Phi(t,\tau) f(\tau) \mathrm{d}\tau \Big\| \leq M(t) \int_0^t \|f_n(\tau) - f(\tau)\| \mathrm{d}\tau$$
$$\leq M(t) t^{1-\frac{1}{p}} \Big(\int_0^t \|f_n(\tau) - f(\tau)\|^p \mathrm{d}\tau \Big)^{\frac{1}{p}} = M(t) t^{1-\frac{1}{p}} \|f_n - f\|_p \to 0, \quad \text{for } n \to \infty.$$

$$x_{f_n}(t) = \Phi(t,0) x_{f_n}(0) + \int_0^t \Phi(t,\tau) f_n(\tau) \mathrm{d}\tau.$$

It implies $g(t) = \Phi(t,0)y + \int_0^t \Phi(t,\tau) f(\tau) \mathrm{d}\tau$.

From here we get $\dot{g}(t) = A(t)g(t) + f(t), g(0) = y \in X_2^q, g \in L^q(X)$, which proves that $g = x_f$ and implies $\mathcal{U}f = (y,g)$ showing that $\mathcal{U}$ is a closed operator, and by the Closed Graph Theorem, it results that there exists $K > 0$ such that

$$\|(x_f(0), x_f)\| \leq K \|f\|_p$$

is equivalent to$\|x_f(0)\| + \|x_f\|_q \leq K \|f\|_p$ for all $f \in L^p(X)$. □

Theorem 2.2. *If* (A) *satisfies the* (p,q) *Perron's condition for dichotomy, then* $x_f \in L^\infty(X)$ *and* $\|x_f(t)\| \leq M\,\mathrm{e}^{\omega}(K+1)\|f\|_p$ *for all* $t \geq 0$. *(*$M > 0, \omega > 0$ *such that* $\|\Phi(t,t_0)\| \leq M\,\mathrm{e}^{\omega(t-t_0)}$ *for every* $t \geq t_0 \geq 0$.*)*

Proof. Let $t \geq 1$, $s \in [t-1,t]$, then

$$\begin{aligned} x_f(t) &= \Phi(t,0)x_f(0) + \int_0^t \Phi(t,\tau) f(\tau) \mathrm{d}\tau \\ &= \Phi(t,s)\Phi(s,0)x_f(0) + \int_0^s \Phi(t,s)\Phi(s,\tau) f(\tau) \mathrm{d}\tau + \int_s^t \Phi(t,\tau) f(\tau) \mathrm{d}\tau \\ &= \Phi(t,s)x_f(s) + \int_s^t \Phi(t,\tau) f(\tau) \mathrm{d}\tau. \end{aligned}$$

It results

$$\begin{aligned} \|x_f(t)\| &\leq \|\Phi(t,s)\| \cdot \|x_f(s)\| + \int_s^t \|\Phi(t,\tau)\| \cdot \|f(\tau)\| \mathrm{d}\tau \\ &\leq M\,\mathrm{e}^{\omega(t-s)}\,\|x_f(s)\| + \int_{t-1}^t M\,\mathrm{e}^{\omega(t-\tau)}\,\|f(\tau)\| \mathrm{d}\tau \\ &\leq M\,\mathrm{e}^{\omega} \left(\|x_f(s)\| + \int_{t-1}^t \|f(\tau)\| \mathrm{d}\tau \right). \end{aligned}$$

By using the Hőlder's inequality, we get

$$\|x_f(t)\| \le M\,\mathrm{e}^{\omega}\Big(\|x_f(s)\| + \Big(\int_{t-1}^{t}\|f(\tau)\|^p\mathrm{d}\tau\Big)^{\frac{1}{p}}\Big) \le M\,\mathrm{e}^{\omega}(\|x_f(s)\| + \|f\|_p)$$

for all $s \in [t-1,t]$. Integrating on $[t-1,t]$, we obtain

$$\begin{aligned}\|x_f(t)\| &\le M\,\mathrm{e}^{\omega}\Big(\int_{t-1}^{t}\|x_f(s)\|\mathrm{d}s + \|f\|_p\Big) \le M\,\mathrm{e}^{\omega}\Big[\Big(\int_{t-1}^{t}\|x_f(s)\|^q ds\Big)^{\frac{1}{q}} + \|f\|_p\Big]\\ &\le M\,\mathrm{e}^{\omega}(\|x_f\|_q + \|f\|_p) \le M\,\mathrm{e}^{\omega}(K\|f\|_p + \|f\|_p) = M\,\mathrm{e}^{\omega}(K+1)\|f\|_p\end{aligned}$$

for all $t \ge 1$. For $t \in [0,1]$, we have

$$x_f(t) = \Phi(t,0)x_f(0) + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau.$$

That implies

$$\begin{aligned}&\|x_f(t)\| \le \|\Phi(t,0)\|\cdot\|x_f(0)\| + \int_0^1\|\Phi(t,\tau)\|\cdot\|f(\tau)\|\mathrm{d}\tau\\ &\le M\,\mathrm{e}^{\omega t}\,\|x_f(0)\| + \int_0^t M\,\mathrm{e}^{\omega(t-\tau)}\,\|f(\tau)\|\mathrm{d}\tau \le M\,\mathrm{e}^{\omega}\Big(\|x_f(0)\| + \int_0^1\|f(\tau)\|\mathrm{d}\tau\Big)\\ &\le M\,\mathrm{e}^{\omega}\Big[K\|f\|_p + \Big(\int_0^1\|f(\tau)\|^p\Big)^{\frac{1}{p}}\Big] \le M\,\mathrm{e}^{\omega}(K+1)\|f\|_p.\end{aligned}$$

We have proved that $\|x_f(t)\| \le M\,\mathrm{e}^{w}(K+1)\|f\|_p$ for all $t \ge 0$, and from this relation we get also that $x_f \in L^\infty$. ☐

Theorem 2.3. (A) *satisfies the* $(1,\infty)$ *Perron's condition for dichotomy if and only if* (A) *is ordinary dichotomic.*

Proof. If (A) is ordinary dichotomic and $f \in L^1(X)$, we denote

$$\begin{aligned}x_f(t) = &\int_0^t U(t)P_1U^{-1}(\tau)f(\tau)\mathrm{d}\tau - \int_t^\infty U(t)P_2U^{-1}(\tau)f(\tau)\mathrm{d}\tau\\ &= U(t)P_1\int_0^t U^{-1}(\tau)f(\tau)\mathrm{d}\tau - U(t)P_2\int_t^\infty U^{-1}(\tau)f(\tau)\mathrm{d}\tau.\\ \dot{x}_f(t) = &\,A(t)x_f(t) + (U(t)P_1U^{-1}(t) + U(t)P_2U^{-1}(t))f(t) = A(t)x_f(t) + f(t).\\ x_f(0) = &\,-P_2\int_0^\infty U^{-1}(\tau)f(\tau)\mathrm{d}\tau\end{aligned}$$

and that proves that $x_f(0) \in X_2$.

We have also that

$$\begin{aligned}\|x_f(t)\| \le &\int_0^t\|U(t)P_1U^{-1}(\tau)\|\cdot\|f(\tau)\|\mathrm{d}\tau + \int_t^\infty\|U(t)P_2U^{-1}(\tau)\|\cdot\|f(\tau)\|\mathrm{d}\tau\\ &\le N\int_0^\infty\|f(\tau)\|\mathrm{d}\tau = N\|f\|_1\end{aligned}$$

and from here we get that $x_f \in L^\infty(X)$, therefore, (A) satisfies the $(1,\infty)$ Perron's condition for dichotomy.

If (A) satisfies the $(1,\infty)$ Perron's condition for dichotomy, we consider $\delta>0$ and $f(t)=\varphi_{[t_0,t_0+\delta]}(t)\frac{U(t)x}{\|U(t)x\|}$, $x\neq 0$, where $\varphi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$.

$\displaystyle\int_0^\infty \|f(t)\|\mathrm{d}t=\int_{t_0}^{t_0+\delta}\varphi_{[t_0,t_0+\delta]}(t)\mathrm{d}t=\delta$, thus $f\in L^1(X)$ and $\|f\|_1=\delta$.

We have

$$\begin{aligned}
x_f(t)&=\int_0^t U(t)P_1U^{-1}(\tau)f(\tau)\mathrm{d}\tau-\int_t^\infty U(t)P_2U^{-1}(\tau)f(\tau)\mathrm{d}\tau\\
&=\int_0^t \varphi_{[t_0,t_0+\delta]}(\tau)\frac{\mathrm{d}\tau}{\|U(\tau)x\|}U(t)P_1x-\int_t^\infty \varphi_{[t_0,t_0+\delta]}(\tau)\frac{\mathrm{d}\tau}{\|U(\tau)x\|}U(t)P_2x\\
&=\begin{cases}\displaystyle\int_{t_0}^{t_0+\delta}\frac{\mathrm{d}\tau}{\|U(\tau)x\|}U(t)P_1x, & t\geq t_0+\delta,\\ \displaystyle-\int_{t_0}^{t_0+\delta}\frac{\mathrm{d}\tau}{\|U(\tau)x\|}U(t)P_2x, & t\leq t_0.\end{cases}
\end{aligned}$$

Using the Theorem 2.1 we obtain $\|x_f\|_\infty\leq K\cdot\|f\|_1=K\delta$, but $\|x_f(t)\|\leq\|x_f\|_\infty$, a.e., thus $\|x_f(t)\|\leq K\delta$ for all $t\geq 0$.

Let $t>t_0$ then there exists $\delta_0>0$ such that $t>t_0+\delta_0$, which implies $t>t_0+\delta$ for all $\delta\in(0,\delta_0]$. It results

$$\int_{t_0}^{t_0+\delta}\frac{\mathrm{d}\tau}{\|U(\tau)x\|}\|U(t)P_1x\|\leq K\delta,$$

which is equivalent to

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\frac{\mathrm{d}\tau}{\|U(\tau)x\|}\|U(t)P_1x\|\leq K\qquad\text{for all }\delta\in(0,\delta_0].$$

For $\delta\to 0_+$, we obtain

$$\frac{1}{\|U(t_0)x\|}\|U(t)P_1x\|\leq K\qquad\text{for all }t\geq t_0,$$

equivalent to

$$\|U(t)P_1x\|\leq K\|U(t_0)x\|.$$

If we take $x=U^{-1}(t_0)y$, $y\in X$, we get

$$\|U(t)P_1U^{-1}(t_0)y\|\leq K\|y\|\qquad\text{for all }y\in X\text{ and }t\geq t_0.$$

We consider now $t\leq t_0$, from $\|x_f(t)\|\leq K\delta$, we get

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\frac{\mathrm{d}\tau}{\|U(\tau)x\|}\|U(t)P_2x\|\leq K$$

and for $\delta\to 0_+$, we obtain

$$\frac{1}{\|U(t_0)x\|}\|U(t)P_2x\|\leq K,$$

and if we take $U(t_0)x = y, y \in X$, we get

$$\|U(t)P_2U^{-1}(t_0)y\| \leq K \qquad \text{for all } t \leq t_0.$$

That proves that (A) is ordinary dichotomic. □

Theorem 2.4. *If* (A) *satisfy the* (p,q) *Perron's condition for dichotomy and* $(p,q) \neq (1,\infty)$*, then* (A) *is exponentially dichotomic and* $X_1^q = X_1$*.*

Proof. Let $q < \infty$ and $f(t) = \varphi_{[t_0,t_0+1]}(t)\frac{U(t)x}{\|U(t)x\|}$, $x \neq 0$ and $t_0 \geq 0$. Then $f \in L^p(X)$ and $\|f\|_p = 1$.

$$x_f(t) = \int_0^t U(t)P_1^qU^{-1}(\tau)f(\tau)\mathrm{d}\tau - \int_t^\infty U(t)P_2^qU^{-1}(\tau)f(\tau)\mathrm{d}\tau.$$

We observe that

$$\dot{x}_f(t) = A(t)x_f(t) + f(t) \quad \text{and} \quad x_f(0) = -P_2^q\int_0^\infty U^{-1}(\tau)f(\tau)\mathrm{d}\tau \in X_2^q.$$

Since

$$x_f(t) = \begin{cases} \displaystyle\int_{t_0}^{t_0+1} \frac{\mathrm{d}\tau}{\|U(\tau)x\|}U(t)P_1^qx, & t \geq t_0+1 \\ \displaystyle -\int_{t_0}^{t_0+1} \frac{\mathrm{d}\tau}{\|U(\tau)x\|}U(t)P_2^qx, & t \leq t_0, \end{cases}$$

we get that $x_f \in L^q(X)$.

From $U(t)x = U(t_0)x + \int_{t_0}^t A(\tau)U(\tau)x\mathrm{d}\tau$, we obtain

$$\|U(t)x\| \leq \|U(t_0)x\| + \int_{t_0}^t \|A(\tau)\| \cdot \|U(\tau)x\|\mathrm{d}\tau$$

and using the Gronwall's inequality, we get

$$\|U(\tau)x\| \leq \|U(t_0)x\| \cdot \mathrm{e}^{\int_{t_0}^{t_0+1}\|A(\tau)\|\mathrm{d}\tau} \leq \|U(t_0)x\| \cdot \mathrm{e}^\alpha \qquad \text{for all } \tau \in [t_0, t_0+1],$$

where $\alpha = \sup_{t_0 \geq 0}\int_{t_0}^{t_0+1}\|A(\tau)\|\mathrm{d}\tau < \infty$.

Thus we obtain

$$\frac{1}{\|U(t_0)x\|\,\mathrm{e}^\alpha} \leq \frac{1}{\|U(\tau)x\|} \qquad \text{for all } \tau \in [t_0, t_0+1].$$

Integrating on $[t_0, t_0+1]$, we get

$$\frac{1}{\|U(t_0)x\|\,\mathrm{e}^\alpha} \leq \int_{t_0}^{t_0+1}\frac{\mathrm{d}\tau}{\|U(\tau)x\|}, \tag{5}$$

which implies

$$\frac{1}{\|U(t_0)x\|e^\alpha} \cdot \|U(t)P_1^qx\| \leq \|x_f(t)\| \qquad \text{for all } t \geq t_0+1,$$

equivalent to

$$\|U(t)P_1^qx\| \leq \mathrm{e}^\alpha\,\|U(t_0)x\| \cdot \|x_f(t)\|.$$

Denoting $U(t_0)x = y$, we have

$$\|U(t)P_1^q U^{-1}(t_0)y\| \le \mathrm{e}^{\alpha}\,\|y\| \cdot \|x_f(t)\| \qquad \text{for all } t \ge t_0 + 1.$$

We have

$$\begin{aligned}
&\int_{t_0}^{\infty} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau \\
&= \int_{t_0}^{t_0+1} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau + \int_{t_0+1}^{\infty} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau \\
&\le \int_{t_0}^{t_0+1} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau + (\mathrm{e}^{\alpha}\,\|y\|)^q \int_{t_0+1}^{\infty} \|x_f(\tau)\|^q \mathrm{d}\tau \\
&\le \int_{t_0}^{t_0+1} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau + (\mathrm{e}^{\alpha}\,\|y\|)^q \|x_f\|_q^q.
\end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}U(t)P_1^q U^{-1}(t_0)y = A(t)U(t)P_1^q U^{-1}(t_0)y$, we obtain

$$U(t)P_1^q U^{-1}(t_0)y = U(t_0)P_1^q U^{-1}(t_0)y + \int_{t_0}^{t} A(\tau)U(\tau)P_1^q U^{-1}(t_0)y \mathrm{d}\tau$$

which implies

$$\|U(t)P_1^q U^{-1}(t_0)y\| \le \|U(t_0)P_1^q U^{-1}(t_0)y\| + \int_{t_0}^{t} \|A(\tau)U(\tau)P_1^q U^{-1}(t_0)y\| \mathrm{d}\tau.$$

Applying the Gronwall's inequality, we get

$$\begin{aligned}
\|U(t)P_1^q U^{-1}(t_0)y\| &\le \|U(t_0)P_1^q U^{-1}(t_0)y\|\,\mathrm{e}^{\int_{t_0}^{t}\|A(\tau)\|\mathrm{d}\tau} \\
&\le \|U(t_0)P_1^q U^{-1}(t_0)y\|\,\mathrm{e}^{\alpha} \qquad \text{for all } t \in [t_0, t_0+1].
\end{aligned}$$

Then $\int_{t_0}^{t_0+1} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau \le \|U(t_0)P_1^q U^{-1}(t_0)y\|^q \cdot \mathrm{e}^{\alpha q}$. Therefore,

$$\int_{t_0}^{\infty} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau \le \|U(t_0)P_1^q U^{-1}(t_0)y\|^q \cdot \mathrm{e}^{\alpha q} + (\mathrm{e}^{\alpha}\,\|y\|)^q \|x_f\|_q^q. \tag{6}$$

For $t \le t_0$, we have $x_f(t) = -\int_{t_0}^{t_0+1} \frac{\mathrm{d}\tau}{\|U(\tau)x\|} U(t)P_2^q x$. We obtain

$$\|x_f(t)\| = \int_{t_0}^{t_0+1} \frac{\mathrm{d}\tau}{\|U(\tau)x\|} \|U(t)P_2^q x\| \qquad \text{for all } t \le t_0.$$

From (5), we have $\frac{1}{\mathrm{e}^{\alpha}\,\|U(t_0)x\|} \le \int_{t_0}^{t_0+1} \frac{\mathrm{d}\tau}{\|U(\tau)x\|}$, therefore,

$$\frac{1}{e^{\alpha}\|U(t_0)x\|} \cdot \|U(t)P_2^q x\| \le \|x_f(t)\| \qquad \text{for all } t \le t_0.$$

We obtain

$$\|U(t)P_2^q U^{-1}(t_0)y\| \le \mathrm{e}^{\alpha}\,\|y\| \cdot \|x_f(t)\| \qquad \text{for all } t \le t_0 \text{ and } y \in X, \tag{7}$$

where we denote $y = U(t_0)x$. Using the Theorem 2.2, we have

$$\|U(t)P_2^q U^{-1}(t_0)y\| \le M\,\mathrm{e}^{\omega}(K+1)\|y\|\,\mathrm{e}^{\alpha} \qquad \text{for all } t \in [0, t_0].$$

For $t = t_0$, we obtain

$$\|U(t_0)P_2^q U^{-1}(t_0)\| \le M\,\mathrm{e}^{\omega}(K+1)\,\mathrm{e}^{\alpha}\,.$$

From here, we get

$$\|U(t_0)P_1^q U^{-1}(t_0)\| = \|I - U(t_0)P_2^q U^{-1}(t_0)\| \le 1 + M\,\mathrm{e}^{\omega}(K+1)\,\mathrm{e}^{\alpha}\,.$$

Denoting $\mathrm{e}^{-\alpha} + M\,\mathrm{e}^{\omega}(K+1) = L$ we have

$$\|U(t_0)P_1^q U^{-1}(t_0)\| \le L\,\mathrm{e}^{\alpha} \qquad \text{for all } t_0 \ge 0.$$

Using now relation (6), we get

$$\int_{t_0}^{\infty} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau \le \mathrm{e}^{\alpha q}\,\|y\|^q (L^q + \|x_f\|^q) \le \mathrm{e}^{\alpha q}\,\|y\|^q (L^q + K^q),$$

thus

$$\Big(\int_{t_0}^{\infty} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau\Big)^{\frac{1}{q}} \le \mathrm{e}^{\alpha}\,\|y\|(L^q + K^q)^{\frac{1}{q}} \quad \text{for all } y \in X \text{ and } t_0 \ge 0.$$

Integrating relation (7) on $[0, t_0]$, we get

$$\begin{aligned}\int_0^{t_0} \|U(\tau)P_2^q U^{-1}(t_0)y\|^q \mathrm{d}\tau &\le \mathrm{e}^{\alpha q}\,\|y\|^q \int_0^{t_0} \|x_f(\tau)\|^q \mathrm{d}\tau \\ &\le \mathrm{e}^{\alpha q}\,\|y\|^q \cdot \|x_f\|_q^q \le \mathrm{e}^{\alpha q}\,\|y\|^q K^q.\end{aligned}$$

Since

$$\begin{aligned}&\Big(\int_{t_0}^{\infty} \|U(\tau)P_1^q U^{-1}(t_0)y\|^q \mathrm{d}\tau\Big)^{\frac{1}{q}} + \Big(\int_0^{t_0} \|U(\tau)P_2^q U^{-1}(t_0)y\|^q \mathrm{d}\tau\Big)^{\frac{1}{q}} \\ &\le \mathrm{e}^{\alpha}\,\|y\|\big[(L^q + K^q)^{\frac{1}{q}} + K\big] \qquad \text{for all } t_0 \ge 0,\ y \in X,\end{aligned}$$

we use now the Theorem 1.3 and obtain that (A) is exponential dichotomic on X_1^q, which shows that $X_1^q \subset X_1$.

Let $x \in X_1$, $x = u + v$, $x \in X_1^q$, $v \in X_2^q$. If we assume $v \ne 0$, it implies $\|\Phi(t,t_0)v\| \ge N\,\mathrm{e}^{\nu(t-t_0)}\,\|v\|$. From here we get $\lim_{t\to\infty} \|\Phi(t,t_0)v\| = \infty$.

As $\Phi(t,t_0)x = \Phi(t,t_0)v - (-\Phi(t,t_0)u)$, we get

$$\|\Phi(t,t_0)x\| \ge \big\|\|\Phi(t,t_0)v\| - \|\Phi(t,t_0)u\|\big\| \to \infty \qquad \text{for } n \to \infty,$$

which is a contradiction because $x \in X_1$. We have $X_1 \subset X_1^q$, therefore, $X_1 = X_1^q$ and $P_1^q = P_1$, $P_2^q = P_2$.

If $q = \infty$, then $p > 1$. If we consider $t_0 \ge 0$, $x \in X$ such that $P_i x \ne 0$ for $i \in \{1,2\}$ and $f(t) = \varphi_{[t_0,t_0+1]}(t)\frac{U(t)x}{\|U(t)x\|}$, we get

$$x_f(t) = \begin{cases} \displaystyle\int_{t_0}^{t_0+1} \frac{\mathrm{d}\tau}{\|U(\tau)x\|} U(t)P_1 x, & t \ge t_0 + 1 \\ \displaystyle -\int_{t_0}^{t_0+1} \frac{\mathrm{d}\tau}{\|U(\tau)x\|} U(t)P_2 x, & t \le t_0, \end{cases}.$$

Since $x_f \in L^{\infty}(X)$ from Theorem 2.1, there is an $K > 0$ such that $\|x_f(t)\| \le K$ for all $t \ge 0$.

For $t \geq t_0 + 1$ and $x \in X$, we get $\|U(t)P_1x\| \leq K\,\mathrm{e}^{\alpha}\,\|U(t_0)P_1x\|$ for all $t \geq t_0 + 1$. Since $\frac{d}{\mathrm{d}t}U(t)P_1x = A(t)U(t)P_1x$, we have

$$U(t)P_1x = U(t_0)P_1x + \int_{t_0}^{t_0+1} A(\tau)U(\tau)P_1x\mathrm{d}\tau.$$

Then $\|U(t)P_1x\| \leq \|U(t_0)P_1x\| + \int_{t_0}^{t_0+1} \|A(\tau)\| \cdot \|U(\tau)P_1x\|\mathrm{d}\tau$, and using the Gronawall's inequality, we obtain

$$\|U(t)P_1x\| \leq \|U(t_0)P_1x\| \cdot \mathrm{e}^{\alpha} \qquad \text{for all } t \in [t_0, t_0+1].$$

If we denote $L = \max\{1, K\} \cdot \mathrm{e}^{\alpha}$, we have

$$\|U(t)P_1x\| \leq L\|U(t_0)P_1x\| \qquad \text{for all } t \geq t_0.$$

For $t \leq t_0$, we have $\int_{t_0}^{t_0+1} \frac{\mathrm{d}\tau}{\|U(\tau)x\|} \cdot \|U(t)P_2x\| \leq K$, which implies

$$\frac{1}{\mathrm{e}^{\alpha}\,\|U(t_0)x\|} \cdot \|U(t)P_2x\| \leq K \qquad \text{for all } t \leq t_0.$$

Replacing x by P_2x and t_0 with t, we get $\|U(t_0)P_2x\| \leq K\,\mathrm{e}^{\alpha}\,\|U(t)P_2x\| \leq L\|U(t)P_2x\|$ for all $t \leq t_0$.

We consider now $g(t) = \varphi_{[t_0,t_0+\delta]}(t)\frac{U(t)P_1x}{\|U(t)P_1x\|}$ for $\delta > 0$, $t_0 \geq 0$, $x \in X$, $P_ix \neq 0$, with $i \in \{1, 2\}$. Then $g \in L^p(X)$, $\|g\|_p = \delta^{\frac{1}{p}}$ and

$$\begin{aligned} x_g(t) &= \int_0^t U(t)P_1U^{-1}(\tau)g(\tau)\mathrm{d}\tau - \int_t^{\infty} U(t)P_2U^{-1}(\tau)g(\tau)\mathrm{d}\tau \\ &= \int_0^t \varphi_{[t_0,t_0+\delta]}(\tau)\frac{\mathrm{d}\tau}{\|U(\tau)P_1x\|} \cdot U(t)P_1x \qquad \text{for all } t \geq 0,\ x_g \in L^{\infty}, \end{aligned}$$

thus

$$x_g(t_0+\delta) = \int_{t_0}^{t_0+\delta} \frac{\mathrm{d}\tau}{\|U(\tau)P_1x\|}U(t_0+\delta)P_1x.$$

Using the Theorem 2.1, we have $\|x_g(t_0+\delta)\| \leq K \cdot \delta^{\frac{1}{p}}$, which is equivalent to

$$\int_{t_0}^{t_0+\delta} \frac{\mathrm{d}\tau}{\|U(\tau)P_1x\|}\|U(t_0+\delta)P_1x\| \leq K\delta^{\frac{1}{p}}.$$

Since $\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)P_1x = A(\tau)U(\tau)P_1x$ and $\|U(\tau)P_1x\| \leq L\|U(t_0)P_1x\|$ for all $\tau \geq t_0$, we get

$$\frac{1}{L\|U(t_0)P_1x\|} \leq \frac{1}{\|U(\tau)P_1x\|} \qquad \text{for all } \tau \in [t_0, t_0+\delta].$$

Integrating on $[t_0, t_0+\delta]$, we obtain

$$\frac{\delta}{L\|U(t_0)P_1x\|} \leq \int_{t_0}^{t_0+\delta} \frac{\mathrm{d}\tau}{\|U(\tau)P_1x\|},$$

which implies $\frac{\delta}{L\|U(t_0)P_1x\|} \cdot \|U(t_0+\delta)P_1x\| \leq K\delta^{\frac{1}{p}}$ and from here we get

$$\|U(t_0+\delta)P_1x\| \leq K \cdot L \cdot \delta^{\frac{1}{p}-1} \cdot \|U(t_0)P_1x\|.$$

Since $\lim_{\delta\to\infty} K \cdot L \cdot \delta^{\frac{1}{p}-1} = 0$, it exists $\delta_0 \geq 0$ such that

$$\|U(t_0+\delta_0)P_1x\| \leq \frac{1}{2}\|U(t_0)P_1x\| \qquad \text{for all } t_0 \geq 0.$$

It results from Lemma 1.1 that there exist $N_1, \nu_1 > 0$ such that

$$\|U(t)P_1x\| \leq N_1\, \mathrm{e}^{-\nu_1(t-t_0)}\, \|U(t_0)P_1x\| \qquad \text{for all } t \geq t_0 \geq 0.$$

We consider now $h(t) = \frac{U(t)P_2x}{\|U(t_0+\delta)P_2x\|} \cdot \varphi_{[t_0,t_0+\delta]}(t)$, $x \in X$ with $P_2x \neq 0$.

$$\frac{1}{L}\|U(t)P_2x\| \geq \|U(t_0)P_2x\| \qquad \text{for all } t \geq t_0,$$

which implies

$$\frac{1}{L}\|U(t_0+\delta)P_2x\| \geq \|U(t)P_2x\| \qquad \text{for all } t \in [t_0, t_0+\delta].$$

From here we obtain $h \in L^p$ and $\|h\|_p \leq \frac{1}{L}\delta^{\frac{1}{p}}$.

$$\begin{aligned} x_h(t) &= \int_0^t U(t)P_1U^{-1}(\tau)h(\tau)\mathrm{d}\tau - \int_t^\infty U(t)P_2U^{-1}(\tau)h(\tau)\mathrm{d}\tau \\ &= -\int_t^\infty \varphi_{[t_0,t_0+\delta]}(\tau)\frac{\mathrm{d}\tau}{\|U(t_0+\delta)P_2x\|}U(t)P_2x \\ &= \begin{cases} \dfrac{-\delta}{\|U(t_0+\delta)P_2x\|}U(t)P_2x, & t \leq t_0 \\ 0, & t \geq t_0+\delta. \end{cases} \end{aligned}$$

We have $x_f \in L^\infty$ and $\|x_h(t)\| \leq K\|h\|_p \leq \frac{K}{L}\delta^{\frac{1}{p}}$ for all $t \geq 0$. Therefore, $\frac{\delta}{\|U(t_0+\delta)P_2x\|} \cdot \|U(t_0)P_2x\| \leq \frac{K}{L} \cdot \delta^{\frac{1}{p}}$ is equivalent to

$$\|U(t_0+\delta)P_2x\| \geq \frac{L}{K}\delta^{1-\frac{1}{p}}\|U(t_0)P_2x\| \qquad \text{for all } \delta > 0 \text{ and } t_0 \geq 0.$$

Since $\lim_{\delta\to\infty} \frac{L}{K}\delta^{1-\frac{1}{p}} = \infty$, there exists $\delta_0 > 0$ such that $\frac{L}{K}\delta_0^{1-\frac{1}{p}} > 2$.
Therefore, $\|U(t_0+\delta_0)P_2x\| \geq 2\|U(t_0)P_2x\|$ for all $t_0 \geq 0$ and $x \in X$ with $P_2x \neq 0$.
It results from Lemma 1.1 that there exist $N_2, \nu_2 > 0$ such that

$$\|U(t)P_2x\| \geq N_2\, \mathrm{e}^{\nu_2(t-t_0)}\, \|U(t_0)P_2x\| \qquad \text{for all } t \geq t_0 \geq 0.$$

We have proved that (A) is exponentially dichtomic. □

Remark 2.2 The converse of the theorem above is true if and only if $1 \leq p \leq q \leq \infty$. For the proof of the remark above, see [**7**, Theorem 6.4, p. 477]. The condition $p \leq q$ is essential as it can be seen from the following example.

Example 2.1. Let $X = \mathbb{R}, U(t) = \mathrm{e}^{-t}$, $\Phi(t, t_0) = \mathrm{e}^{-(t-t_0)}$, be the process generated by the differential system $(A) : \dot{x}(t) = -x(t)$.

It is obvious that $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is exponential stable, but does not satisfy the $(2,1)$ Perron's condition for stability. Indeed if we consider $f(t)=\frac{1}{t+1}$, then $f\in L^2$ and

$$x(t)=\int_0^t \Phi(t,s)f(s)\mathrm{d}s=\int_0^t \frac{\mathrm{e}^{-(t-s)}}{s+1}\mathrm{d}s=\mathrm{e}^{-t}\int_0^t \frac{\mathrm{e}^s}{s+1}\mathrm{d}s\geq \frac{1}{t+1}-\mathrm{e}^{-t},$$

which shows that $x\notin L^1$.

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