Acta Math. Univ. Comenianae
Vol. LXXXV, 2 (2016), pp. 347–356

RELATIVE RANK OF THE FINITE FULL TRANSFORMATION SEMIGROUP WITH RESTRICTED RANGE

K. TINPUN AND J. KOPPITZ

ABSTRACT. In this paper, we determine the relative rank of the semigroup $T(X,Y)$ of all transformations on a finite set X with restricted range Y modulo the semigroup of all extensions of the bijections on Y, modulo the idempotent order-preserving transformations in $T(X,Y)$, and modulo the semigroup of all order-preserving transformations in $T(X,Y)$.

1. Introduction and Preliminaries

The rank of a semigroup S, denoted $\operatorname{rank}(S)$, is the minimum size of a generating set for S [**8**]. The ranks of certain finite semigroups were studied in [**4**]. This concept was generalized in [**9**]. The authors introduced a 'new' rank property, the relative rank of S modulo a subset A of S. For a semigroup S, if $A \subseteq S$, then we call the minimum size of a set B such that $\langle A \cup B\rangle = S$ the relative rank of S modulo A, denoted $\operatorname{rank}(S : A)$. In [**5**], the authors considered the relative rank of $T(X)$ modulo the semigroup $O(X)$ of all order-preserving maps on a finite linearly ordered set X, i.e., $\operatorname{rank}(T(X) : O(X)) = 2$. The relative rank of $T(X)$ modulo the symmetric group $S(X)$ on a finite set X is 1 [**10**].

Recall that for a finite linearly ordered set $(X;\leq)$, a map $\alpha \in T(X)$ is order preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X$. For a finite set X of size n, the semigroup $O(X)$ has been studied extensively. Its order is $\binom{2n-1}{n-1}$, its rank is n, it is idempotent generated, and the minimum size of a generating set of $O(X)$ consisting of idempotents, the idempotent rank, is $2n-2$, see [**4**] or [**7**].

The present paper deals with only finite transformation semigroups, i.e., X is finite. The rank of the semigroup $T(X,Y)$ was determined in [**2**]. It is a 'large size'. Hence, we consider the relative rank of $T(X,Y)$ modulo the semigroup $O(X,Y)$ of all order-preserving maps from X in Y on a linearly ordered set X as well as the semigroup $S(X,Y)$ of all maps α from X in Y whose restriction to Y (denoted $\alpha|_Y$) is a bijection on Y, i.e., $\alpha|_Y \in S(Y)$. The semigroup $T(X,Y)$ was introduced by J. S. V. Symons in 1975 and called semigroup with restricted range [**12**].

Received August 7, 2015; revised October 9, 2015.
2010 *Mathematics Subject Classification.* Primary 20M20, 54H15.
Key words and phrases. relative rank, bijection, order-preserving, transformations with restricted range.

Let X be a finite set of size n and consider a subset $Y \subseteq X$ of size m. The semigroup $T(X,Y)$ was studied in [**1, 12, 6, 10, 11, 12**]. When X is linearly ordered, $O(X,Y)$ can be written as

$$O(X,Y) := O(X) \cap T(X,Y)$$

and has order $\binom{n+m-1}{m-1}$ by the same calculations as in the proof of the order of $O(X)$ in [**7**]. The semigroup $O(X,Y)$ is not idempotent generated. It has rank $\binom{n-1}{m-1}+\left|Y^{\#}\right|$, where $Y^{\#}$ is the set of so-called captive elements [**1**].

The semigroup $S(X,Y)$ was firstly mentioned in [**11**]. The authors of this paper consider it as $\mathcal{J}$-class of the semigroup $F(X,Y) := \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$, namely $J(F,m) := \{\alpha \in F(X,Y) : |X\alpha| = m\}$. Notice that $J(F,m) = S(X,Y)$, $\operatorname{rank}(J(F,m)) = m^{n-m}$ if Y is a proper subset of X, and $\operatorname{rank}(J(F,m)) = 2$ if $X = Y$.

The cardinality of the image of α, $\operatorname{im}\alpha := X\alpha$ is called rank of α, denoted $\operatorname{rank}\alpha = |\operatorname{im}\alpha|$. The kernel of α, denoted $\ker\alpha := \{(x,y) \in X \times X : x\alpha = y\alpha\}$, is an equivalence relation on X, which corresponds uniquely to a decomposition of X into blocks, called $\ker\alpha$-classes. This justifies the notation $B \in \ker\alpha$ in case B is $\ker\alpha$-class. Moreover, a set $T \subseteq X$ with $|B \cap T| = 1$ for all $B \in \ker\alpha$, is called transversal of $\ker\alpha$. If we restrict α to T, we obtain a map $\alpha|_T$ from T in Y defined by $x(\alpha|_T) := x\alpha$ for all $x \in T$. In particular, it is easy to verify that $\alpha \in S(X,Y)$ if and only if Y is a transversal of $\ker\alpha$.

The rest of this paper is organized in three sections. In the next section, we determine the relative rank of $T(X,Y)$ modulo $S(X,Y)$. As a consequence, we obtain the already known rank of $T(X,Y)$, but here as the sum of the rank of $S(X,Y)$ and the relative rank $T(X,Y)$ modulo $S(X,Y)$. In the second section, we determine the number of idempotents in $O(X,Y)$ and the relative rank of $O(X,Y)$ modulo the set of its idempotents. In the last section, we give the relative rank of $T(X,Y)$ modulo $O(X,Y)$. If $Y = \{y\}$ is a singleton set, then $T(X,Y)$ contains only one element, namely the constant map c_Y mapping all elements of X to y. Hence, we drop the case $m = 1$ and assume without loss of generality that

$$X = \{x_1, \ldots, x_n\} \qquad \text{and} \qquad Y = \{x_{i_1}, \ldots, x_{i_m}\} \text{ with } m \geq 2.$$

2. The relative rank of $T(X,Y)$ modulo $S(X,Y)$

In this section, we determine the relative rank of $T(X,Y)$ modulo $S(X,Y)$. Since $T(X,Y) \setminus S(X,Y)$ is an ideal [**10**], the rank of $T(X,Y)$ is the sum of the relative rank of $T(X,Y)$ modulo $S(X,Y)$ and the rank of $S(X,Y)$.

Recall that for $1 \leq k \leq n$, the Stirling number of the second kind (or Sterling partition number) is the number of ways to decompose an n-element set into k non-empty subsets, denoted $S(n,k)$, where

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

Proposition 2.1. *Let X and Y be subsets as in Section 1 with cardinality n and m, respectively. The relative rank of $T(X,Y)$ modulo $S(X,Y)$ is $S(n,m)-m^{n-m}$.*

Proof. Let $\mathcal{D} := \{\ker\beta : \beta \in T(X,Y) \setminus S(X,Y), \operatorname{rank}\beta = m\}$. For each $D \in \mathcal{D}$, we choose a transformation $\alpha_D \in T(X,Y)$ with $\operatorname{im}\alpha_D = Y$ and $\ker\alpha_D = D$. It is easy to see that $\mathcal{D}$ is the set of all decompositions D of X into m non-empty sets such that Y is not a transversal of D. In order to calculate the cardinality of D, we need to count the ways to decompose the n-element set X into m non-empty subsets having Y as transversal. This is a simple combinatorial problem: We have to distribute the elements of the set $X \setminus Y$ to the elements of Y. There are exactly m^{n-m} ways to do this. Hence, $|\mathcal{D}| = S(n,m) - m^{n-m}$.

We will show that $S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}$ generates $T(X,Y)$. If this is the case, then $\operatorname{rank}(T(X,Y) : S(X,Y)) \le S(n,m) - m^{n-m}$.

Let $\gamma \in T(X,Y) \setminus S(X,Y)$ with $\operatorname{rank}\gamma = m$. Then there is $D \in \mathcal{D}$ with $\ker\gamma = \ker\alpha_D = D$. Let $x_D \in x\alpha_D^{-1}$ for $x \in Y$. Then define θ from X to Y by

$$x\theta := \begin{cases} x_D\gamma & \text{if } x \in Y, \\ x_{i_1} & \text{otherwise.} \end{cases}$$

It is easy to see that Y is a transversal of $\ker\theta$, i.e., $\theta \in S(X,Y)$. For $x \in X$. we have $x\alpha_D\theta = (x\alpha_D)_D\gamma = x\gamma$ since $(x\alpha_D)_D \in x\alpha_D\alpha_D^{-1}$ is in the $\ker\alpha_D$-class of x, which is also a $\ker\gamma$-class (since $\ker\gamma = \ker\alpha_D$). Therefore, $\gamma = \alpha_D\theta \in \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$.

Suppose now that $\{\beta \in T(X,Y) : \operatorname{rank}\beta = p\} \subseteq \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$ for some $p \le m$ and take $\gamma \in T(X,Y)$ with $\operatorname{rank}\gamma = p-1$. We can assume without loss of generality that

$$\operatorname{im}\gamma = \{x_{i_1}, \dots, x_{i_{p-1}}\}.$$

Then there is $k \in \{1, \dots, p-1\}$ with $\left|x_{i_k}\gamma^{-1}\right| \ge 2$ and decompose $x_{i_k}\gamma^{-1}$ into two non-empty sets (C_1 and C_2, say). Define α from X to Y by

$$x\alpha := \begin{cases} x\gamma & \text{if } x \notin x_{i_k}\gamma^{-1}, \\ x_{i_k} & \text{if } x \in C_1, \\ x_{i_p} & \text{if } x \in C_2. \end{cases}$$

Clearly, $\operatorname{rank}\alpha = p$ and thus $\alpha \in \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$ by assumption. Now define δ from X to Y by

$$x\delta := \begin{cases} x & \text{if } x \in \operatorname{im}\gamma \setminus \{x_{i_k}\}, \\ x_{i_k} & \text{if } x \in \{x_{i_k}, x_{i_p}\}, \\ x_{i_p} & \text{if } x \notin (\operatorname{im}\gamma \cup \{x_{i_p}\}). \end{cases}$$

Clearly, $\operatorname{rank}\delta = p$ and thus $\delta \in \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$ by assumption. For $x \in X \setminus x_{i_k}\gamma^{-1}$, we have $x\alpha = x\gamma \in \operatorname{im}\gamma \setminus \{x_{i_k}\}$ and thus $x\alpha\delta = x\gamma$. For $x \in x_{i_k}\gamma^{-1}$, we have $x\alpha \in \{x_{i_k}, x_{i_p}\}$ and thus $x\alpha\delta = x_{i_k} = x\gamma$. This shows that $\gamma = \alpha\delta \in \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$. Thus, we have $T(X,Y) \subseteq \langle S(X,Y) \cup \{\alpha_D : D \in \mathcal{D}\}\rangle$.

Next, we will show if $B \subseteq T(X,Y)$ with $\langle S(X,Y) \cup B\rangle = T(X,Y)$, then $\mathcal{D} \subseteq \{\ker\beta : \beta \in B\}$. If it is the case, then $\operatorname{rank}(T(X,Y) : S(X,Y)) \ge |\mathcal{D}| = S(m,n) - m^{n-m}$ and altogether, we have proved the equality as required. Assume

that $\mathcal{D} \not\subseteq \{\ker\beta : \beta \in B\}$. Then there is $D \in \mathcal{D}$ such that $\ker\beta \neq D$ for all $\beta \in B$. From $\alpha_D \in \langle S(X,Y) \cup B\rangle$, it follows the existence of $\theta_1 \in S(X,Y) \cup B$ and $\theta_2 \in T(X,Y)$ with $\alpha_D = \theta_1\theta_2$. Thus $D = \ker\alpha_D = \ker\theta_1\theta_2 = \ker\theta_1$ since $\operatorname{im}\alpha_D = Y$. But $\ker\theta_1 = D$ implies $\theta_1 \notin B$. Hence, $\theta_1 \in S(X,Y)$ and Y is a transversal of $\ker\theta_1$. This contradicts $\ker\theta_1 = D \in \mathcal{D}$. □

Corollary 2.2. $\operatorname{rank}(T(X,Y)) = S(n,m)$.

Proof. It follows from Proposition 2.1 and the fact that the rank of $S(X,Y)$ equals m^{n-m}. □

3. The relative rank of $O(X,Y)$ modulo its idempotents

In this section, we assume that X is linearly ordered. Set

$$X = \{x_1 < \cdots < x_n\}$$

and Y a non-empty subchain with m elements, say

$$Y = \{x_{i_1} < \cdots < x_{i_m}\}.$$

Let $EO(X,Y)$ be the set of all idempotent order-preserving transformations in $T(X,Y)$. For a subchain $P = \{x_{p_1} < x_{p_2} < \cdots < x_{p_k}\}$ of Y of size k, we define g_P from $\{0,1,\ldots,k-1\}$ in $\{1,\ldots,n\}$ by $g_P(l) := p_{l+1} - p_l$ for $l \in \{1,\ldots,k-1\}$ and $g_P(0) := 1$.

Next proposition determines how many idempotents with rank m are in the semigroup $O(X,Y)$.

Proposition 3.1. $|EO(X,Y)| = \displaystyle\sum_{\emptyset\neq P\subseteq Y} \prod_{l=0}^{|P|-1} g_P(l)$.

Proof. Let P be a non-empty subchain of Y. If $|P| = 1$, then there is exactly one idempotent with image P, where $\prod_{l=0}^{0} g_P(l) = g_P(0) = 1$. Admit $|P| \geq 2$ and let $l \in \{1,2,\ldots,|P|-1\}$. Then there are $p_{l+1} - p_l$ ways to decompose $\{x_{p_l}, x_{p_l+1}, \ldots, x_{p_{l+1}}\}$ into two non-empty sets $C_1 < C_2$ ($x_1 \in C_1$ and $x_2 \in C_2$ implies $x_1 < x_2$). From the definition of g_P, we have $|\{\beta \in EO(X,Y) : \operatorname{rank}\beta = |P|\}| = g_P(0)g_P(1)\ldots g_P(|P|-1) = \prod_{l=0}^{|P|-1} g_P(l)$. Considering all non-empty subsets $P \subseteq Y$, we obtain the assertion. □

We consider the set

$$A(X,Y) := \{\beta \in O(X,Y) : \beta \notin EO(X,Y),\ \operatorname{rank}\beta = m\}.$$

Lemma 3.2. $\langle EO(X,Y) \cup A(X,Y)\rangle = O(X,Y)$.

Proof. We reason by induction on the rank of any $\gamma \in O(X,Y)$. Let $\operatorname{rank}\gamma = m$. Then $\gamma \in EO(X,Y) \cup A(X,Y)$, i.e., $\gamma \in \langle EO(X,Y) \cup A(X,Y)\rangle$.

Suppose that $\gamma \in \langle EO(X,Y) \cup A(X,Y)\rangle$ whenever $\gamma \in O(X,Y)$ with $\operatorname{rank}\gamma = k$ for some $k \leq m$, and let $\gamma \in O(X,Y)$ with $\operatorname{rank}\gamma = k-1$. We put $P := \operatorname{im}\gamma = \{x_{p_1} < x_{p_2} < \cdots < x_{p_{k-1}}\}$. Then there is $i \in \{1,\ldots,k-1\}$ with $\left|x_{p_i}\gamma^{-1}\right| \geq 2$. Moreover, there is $j \in \{1,\ldots,k-1\}$ with $x_{p_j+1} \notin P$ or $x_{p_j-1} \notin P$. Suppose

without loss of generality $x_{p_j+1} \notin P$. We decompose $x_{p_i}\gamma^{-1}$ into two non-empty sets $C_1 < C_2$ and put

$$\begin{aligned} D_l &:= x_{p_l}\gamma^{-1} && \text{for } 1 \le l < i, \\ D_i &:= C_1 \\ D_{i+1} &:= C_2 \\ D_l &:= x_{p_{l-1}}\gamma^{-1} && \text{for } i+2 \le l \le k, \text{ as well as} \\ y_l &:= x_{p_l} && \text{for } 1 \le l \le j, \\ y_{j+1} &:= x_{p_j+1} \\ y_l &:= x_{p_{l-1}} && \text{for } j+1 < l \le k. \end{aligned}$$

We define θ from X to Y by

$$x\theta := y_l \text{ if } x \in D_l,\ 1 \le l \le k.$$

It is easy to verify that θ is order-preserving with rank k. Hence, $\theta \in \langle EO(X,Y) \cup A(X,Y)\rangle$.
Suppose that $i = j$. Then we define ε from X to Y by

$$x\varepsilon := \begin{cases} y_1 & \text{if } x < y_1, \\ y_l & \text{if } y_l \le x < y_{l+1};\ 1 \le l < i, \\ y_i & \text{if } y_i \le x \le y_{i+1}, \\ y_l & \text{if } y_{l-1} < x \le y_l;\ i+1 < l \le k, \\ y_k & \text{if } x > y_k. \end{cases}$$

It is easy to see that ε is order-preserving as well as idempotent and thus $\varepsilon \in \langle EO(X,Y) \cup A(X,Y)\rangle$. Let $x \in D_l$ for some $l \in \{1,\dots,k\}$. If $l < i$, then $x\theta\varepsilon = y_l = x_l = x\gamma$. If $l \in \{i, i+1\}$, then $x\theta\varepsilon = y_l\varepsilon = y_i = x_i = x\gamma$. And if $l > i+1$, then $x\theta\varepsilon = y_l\varepsilon = y_l = x_{l-1} = x\gamma$. This shows that $\gamma = \theta\varepsilon \in \langle EO(X,Y) \cup A(X,Y)\rangle$.

We consider now $j < i$. The case $j > i$ works analogously. For $0 \le r \le i-j-1$, we define ε_r from X in Y by

$$x\varepsilon_r := \begin{cases} y_1 & \text{if } x < y_1, \\ y_l & \text{if } y_l \le x < y_{l+1}; 1 \le l < i-r, \\ y_{i-r+1} & \text{if } y_{i-r} \le x \le y_{i-r+1}, \\ y_l & \text{if } y_{l-1} < x \le y_l; i-r+1 < l \le k, \\ y_k & \text{if } x > y_k. \end{cases}$$

It is easy to see that ε_r is order-preserving as well as idempotent and thus $\varepsilon_r \in \langle EO(X,Y) \cup A(X,Y)\rangle$. Let $x \in X$. If $x \in D_l$, $1 \le l \le j$, then $x\theta\varepsilon_0\varepsilon_1\dots\varepsilon_{i-j-1} = y_l\varepsilon_0\varepsilon_1\dots\varepsilon_{i-j-1} = y_l = x_l = x\gamma$. If $x \in D_l$, $i+2 \le l \le k$, then $x\theta\varepsilon_0\varepsilon_1\dots\varepsilon_{i-j-1} = y_l\varepsilon_0\varepsilon_1\dots\varepsilon_{i-j-1} = y_l = x_{l-1} = x\gamma$. If $x \in D_l$, $j+1 \le l \le i$, then

$$\begin{aligned} x\theta\varepsilon_0\varepsilon_1\dots\varepsilon_{i-j-1} &= y_l(\varepsilon_0\varepsilon_1\dots\varepsilon_{i-l})\varepsilon_{i-l+1}\dots\varepsilon_{i-j-1} = y_l\varepsilon_{i-l}(\varepsilon_{i-l+1}\dots\varepsilon_{i-j-1}) \\ &= y_{l+1}(\varepsilon_{i-l+1}\dots\varepsilon_{i-j-1}) = y_{l+1} = x_l = x\gamma. \end{aligned}$$

If $x \in D_{l+1}$, then $x\theta\varepsilon_0\varepsilon_1 \dots \varepsilon_{i-j-1} = y_{i+1}\varepsilon_0\varepsilon_1 \dots \varepsilon_{i-j-1} = y_{i+1} = x_i = x\gamma$.
As a consequence of the above reasoning, we get that $\gamma = \theta\varepsilon_0\varepsilon_1 \dots \varepsilon_{i-j-1} \in \langle EO(X,Y) \cup A(X,Y)\rangle$. □

Next lemma shows the size of $A(X,Y)$.

Lemma 3.3. $|A(X,Y)| = \binom{n-1}{m-1} - \prod_{l=0}^{m-1} g_P(l)$.

Proof. Any $\alpha \in O(X,Y)$ with $\operatorname{rank}\alpha = m$ has image Y and it is uniquely determined by its kernel $\ker\alpha$. But this kernel is uniquely determined by the least elements in each $\ker\alpha$-class. Note that x_1 is mapped to x_{i_1}. Hence, there are $\binom{n-1}{m-1}$ ways to create this kernel $\ker\alpha$. Any $\alpha \in O(X,Y)$ with $\operatorname{rank}\alpha = m$ is idempotent if Y is a transversal of $\ker\alpha$, then there are $\prod_{l=0}^{m-1} g_P(l)$ ways to create this $\ker\alpha$ as already shown in the proof of Proposition 3.1. Hence, $|A(X,Y)| = |\{\beta \in O(X,Y) : \beta \notin EO(X,Y),\ \operatorname{rank}\beta = m\}| = \binom{n-1}{m-1} - \prod_{l=0}^{m-1} g_P(l)$. □

We are now in the position to present the main result of this section.

Theorem 3.4. *With the above notations, the relative rank of the semigroup of order-preserving transformations $O(X,Y)$ modulo its idempotent elements satisfies*

$$\operatorname{rank}(O(X,Y) : EO(X,Y)) = \binom{n-1}{m-1} - \prod_{l=0}^{m-1} g_P(l).$$

Proof. By the Lemma 3.2 and 3.3, we have

$$\operatorname{rank}(O(X,Y) : EO(X,Y)) \le |A(X,Y)| = \binom{n-1}{m-1} - \prod_{l=0}^{m-1} g_P(l).$$

For the equality, we show that each set $B \subseteq O(X,Y)$ with $\langle EO(X,Y) \cup B\rangle = O(X,Y)$ contains $A(X,Y)$.

Assume there is a set B as above but $A(X,Y) \not\subseteq B$, i.e., there is $\gamma \in A(X,Y) \setminus B$. Since $\gamma \in O(X,Y) = \langle EO(X,Y) \cup B\rangle$, there are $\theta_1 \in EO(X,Y) \cup B$ and $\theta_2 \in O(X,Y)$ with $\gamma = \theta_1\theta_2$. We have $\ker\gamma = \ker\theta_1$ because $\operatorname{rank}\gamma = m$. Note that the elements in $A(X,Y)$ are uniquely, determined by their kernels. So we can conclude $\theta_1 = \gamma \in A(X,Y)$, i.e., $\theta_1 = \gamma \notin B$ and $\gamma = \theta_1 \in EO(X,Y)$, a contradiction. □

4. Relative rank of $T(X,Y)$ modulo $O(X,Y)$

This section is devoted to state and prove the main result of this paper: the computation the relative rank of $T(X,Y)$ modulo the semigroup of all order-preserving transformations in $T(X,Y)$. In the case $X = Y$, it is two. Set X and Y as in Section 3. We define

$$\mathcal{M} := \{\ker\beta : \beta \in T(X,Y),\ \operatorname{rank}\beta = m\} \setminus \{\ker\beta : \beta \in O(X,Y),\ \operatorname{rank}\beta = m\}.$$

Lemma 4.1. $|\mathcal{M}| = S(n,m) - \binom{n-1}{m-1}$.

Proof. It follows from $|\{\ker\beta : \beta \in O(X,Y),\ \operatorname{rank}\beta = m\}| = \binom{n-1}{m-1}$ (see proof of Lemma 3.3) and the fact that $|\{\ker\beta : \beta \in T(X,Y),\ \operatorname{rank}\beta = m\}| = S(n,m)$. □

For each $M \in \mathcal{M}$, we choose $\alpha_M \in T(X,Y)$ with $\operatorname{im}\alpha_M = Y$ and $\ker\alpha_M = M$. Clearly, $|\{\alpha_M : M \in \mathcal{M}\}| = S(n,m) - \binom{n-1}{m-1}$. Note for all $s \in S(Y)$ there is $\mu_S \in T(X,Y) \setminus O(X,Y)$ with $\mu_S|_Y = s$ whenever s is not the identity map on Y.

Lemma 4.2. *If $S \subseteq S(Y)$ with $\langle S\rangle = S(Y)$, then*

$$T(X,Y) = \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle .$$

Proof. Let $\gamma \in T(X,Y)$. Suppose that $\operatorname{rank}\gamma = m$. Then there is $\delta \in O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$ with $\ker\delta = \ker\gamma$. For $x \in Y$, we choose $x_\delta \in x\delta^{-1}$ and define map θ from X to Y by

$$x\theta := \begin{cases} x_\delta\gamma & \text{if } x \in Y \\ x_{i_1} & \text{otherwise.} \end{cases}$$

If $a, b \in Y$ with $a\theta = b\theta$, then $a_\delta\gamma = b_\delta\gamma$, $a\delta^{-1}\gamma = b\delta^{-1}\gamma$, and $a = b$ since $\ker\delta = \ker\gamma$. Hence, Y is a transversal of $\ker\theta$, i.e., $\theta|_Y \in S(Y) = \langle S\rangle$. That means that there is $\mu \in \langle\{\mu_S : s \in S\}\rangle$ with $\theta|_Y = \mu|_Y$.

Let $x \in X$. Then we have $x\delta\mu = x\delta\theta = (x\delta)_\delta\gamma = x\gamma$. This shows $\gamma = \delta\mu \in \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$.

Suppose now that $\gamma \in \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$ whenever $\operatorname{rank}\gamma = p$ for some $p \le m$ and let $\operatorname{rank}\gamma = p - 1$ with

$$\operatorname{im}\gamma = \{x_{j_1} < \cdots < x_{j_{p-1}}\} \subseteq Y.$$

Then there is $i \in \{1, 2, \ldots, p-1\}$ with $\left|x_{j_i}\gamma^{-1}\right| \ge 2$ and we decompose $x_{j_i}\gamma^{-1}$ into two sets $M_i < M_{i+1}$. So, there is $z \in Y \setminus \operatorname{im}\gamma$ and set

$$\begin{aligned} M_l &:= x_{j_l}\gamma^{-1} && \text{and } y_l := x_{j_l} \text{ for } 1 \le l < i, \\ M_l &:= x_{j_{(l-1)}}\gamma^{-1} && \text{and } y_l := x_{j_{(l-1)}} \text{ for } i+1 < l \le p, \\ y_i &:= x_{j_i} && \text{and } y_{i+1} := z. \end{aligned}$$

Now we define map α from X to Y by

$$x\alpha := y_l \qquad \text{if } x \in M_l,\ 1 \le l \le p.$$

For $x \in \operatorname{im}\gamma \cup \{z\}$, we choose $x^\alpha \in x\alpha^{-1}$ and define η from X to Y by

$$x\eta := \begin{cases} x^\alpha\gamma & \text{if } x \in \{y_1, \ldots, y_{i-1}, y_{i+2}, \ldots, y_p\} \\ y_i & \text{if } x \in \{y_i, y_{i+1}\} \\ z & \text{if } x \in X \setminus \{y_1, \ldots y_p\}. \end{cases}$$

Notice that $X \setminus \{y_1, \ldots y_p\} \ne \emptyset$ since $p \le m < n$. Hence, it is easy to verify that both α and η have rank p and thus

$$\alpha, \eta \in \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle .$$

Let $x \in M_l$ for some $1 \le l \le p$. If $l \ne i$ and $l \ne i+1$, then $x\alpha\eta = (x\alpha)^\alpha\gamma = x\gamma$. If $l = i$ or $l = i+1$, then $x\alpha\eta = y_i = x_{j_i} = x\gamma$. This shows that

$\gamma = \alpha\eta \in \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$. The above reasoning proves the assertion $T(X,Y) \subseteq \langle O(X,Y) \cup \{\mu_S : s \in S\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$. □

Lemma 4.3. *If $A \subseteq T(X,Y) \setminus O(X,Y)$ with $\langle O(X,Y) \cup A\rangle = T(X,Y)$ then $\mathcal{M} \subseteq \{\ker\beta : \beta \in A\}$.*

Proof. Let $A \subseteq T(X,Y) \setminus O(X,Y)$ with $\langle O(X,Y) \cup A\rangle = T(X,Y)$. Assume that there is $M \in \mathcal{M}$ with $M \notin \{\ker\beta : \beta \in A\}$. Since $\alpha_M \in T(X,Y) = \langle O(X,Y) \cup A\rangle$, there is an element $\theta_1 \in O(X,Y) \cup A$ and $\theta_2 \in T(X,Y)$ such that $\alpha_M = \theta_1\theta_2$. Because $\operatorname{rank}\alpha_M = m$, we obtain $\ker\alpha_M = \ker\theta_1$, i.e., $\ker\theta_1 = M$. Hence, $\theta_1 \notin A$ (by assumption) and $\theta_1 \notin O(X,Y)$ (since $M \notin \{\ker\beta : \beta \in O(X,Y)\}$), a contradiction. □

We define the following subset $P^*(X)$ of the power set $P(X)$ of X:
If $|X| \geq 5$, then $P^*(X) := P(X) \setminus (\{\emptyset, X\} \cup \{\{x\} : x \in X\})$,
if $|X| = 4$, then $P^*(X) := \{Y \subseteq X : |Y| \geq 2,\ |X \setminus Y| = 2 \text{ or } \{x_2, x_3\} \subseteq Y\}$ and
if $|X| = 3$, then $P^*(X) := \{Y \subseteq X : |Y| = 2,\ x_2 \in Y\}$.
We call two elements $a, b \in X$ to be neighbors if a is immediate successor or predecessor of b.

Theorem 4.4. *With the previous notations, assume that $Y \in P^*(X)$. Then*

$$\operatorname{rank}(T(X,Y) : O(X,Y)) = S(n,m) - \binom{n-1}{m-1}.$$

Proof. If $|X| \geq 5$ or $|X| = 4$ and $\{x_2, x_3\} \subseteq Y$, then there are $x \in X \setminus Y$ and $y_1, y_2 \in Y$ such that neither y_1 nor y_2 is neighbor of x. So we can put $M_i := \{\{r\} : r \in Y \setminus \{y_i\}\} \cup \{X \setminus (Y \setminus \{y_i\})\}$, $i = 1, 2$. It is easy to verify that Y is transversal of M_1 as well as of M_2. Moreover, $M_1, M_2 \notin \{\ker\beta : \beta \in O(X,Y)\}$, i.e., $M_1, M_2 \in \mathcal{M}$. It is well known that the symmetric group $S(Y)$ on Y is generated by two bijections (s_1 and s_2, say). We can assume without loss of generality that $\alpha_{M_1}|_Y = s_1$ and $\alpha_{M_2}|_Y = s_2$, i.e., $\mu_{s_1} = \alpha_{M_1}$ and $\mu_{s_2} = \alpha_{M_2}$.

If $|X| = 4$ and $|X \setminus Y| = 2$ or $|X| = 3$ and $x_2 \in Y_2$ then $|Y| = 2$. Here there are $x \in X \setminus Y$ and $y \in Y$ such that x is not neighbor of y. Then we put $M_3 := \{\{r\} : r \in Y \setminus \{y\}\} \cup \{X \setminus (Y \setminus \{y\})\}$. It is easy to verify that Y is transversal of M_3 and $M_3 \notin \{\ker\beta : \beta \in O(X,Y)\}$. Thus $M_3 \in \mathcal{M}$. The symmetric group $S(Y)$ on the two-element set Y is generated by one bijection, say s. We can assume without loss of generality that $\alpha_{M_3}|_Y = s$, i.e., $\mu_s = \alpha_{M_3}$.

The above fact shows that there is $S \subseteq S(Y)$ with $\langle S\rangle = S(Y)$ such that $\{\mu_S : s \in S\} \subseteq \{\alpha_M : M \in \mathcal{M}\}$. Now we can use Lemma 4.2. It provides that $T(X,Y) = \langle O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$ and thus $\operatorname{rank}(T(X,Y) : O(X,Y)) \leq |\{\alpha_M : M \in \mathcal{M}\}| = S(n,m) - \binom{n-1}{m-1}$ by Lemma 4.1. On the other hand, Lemma 4.3 shows that the minimum size of a relative generating set modulo $O(X,Y)$ is $|\mathcal{M}| = S(n,m) - \binom{n-1}{m-1}$ and altogether, we obtain the assertion $\operatorname{rank}(T(X,Y) : O(X,Y)) = S(n,m) - \binom{n-1}{m-1}$. □

Theorem 4.5. *If $Y \notin P^*(X)$, then*

$$\operatorname{rank}(T(X,Y) : O(X,Y)) = S(n,m) - \binom{n-1}{m-1} + 1.$$

Proof. If $|X| = 4$, $|X \setminus Y| = 1$, and $\{x_2, x_3\} \not\subseteq Y$, then there is exactly one pair $(x,y) \in (X \setminus Y) \times Y$ such that x and y are not neighbors. Then $M_1 := \{\{r\} : r \in Y \setminus \{y\}\} \cup \{\{x,y\}\}$ is the only element $M \in \mathcal{M}$ with Y is transversal of M and $M \notin \{\ker\beta : \beta \in O(X,Y)\}$. Note that the symmetric group $S(Y)$ is generated by two bijections (s_1 and s_2, say). We can assume without loss of generality that $\alpha_{M_1}|_Y = s_1$, i.e., $\mu_{s_1} = \alpha_{M_1}$. Then $\mu_{s_2} \notin O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$.

If $|X| = 3$ and $x_2 \notin Y$, then $|Y| = 2$ and there is no $M \in \mathcal{M}$ with Y is transversal of M and $M \notin \{\ker\beta : \beta \in O(X,Y)\}$. The 2-element symmetric group $S(Y)$ is generated by one $s \in S(Y)$, where $\mu_s \notin O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$.

We put $\alpha := \mu_{s_2}$ and $\alpha := \mu_s$, respectively. We can apply Lemma 4.2 and obtain $T(X,Y) = \langle O(X,Y) \cup \{\alpha\} \cup \{\alpha_M : M \in \mathcal{M}\}\rangle$ and thus

$$\operatorname{rank}(T(X,Y) : O(X,Y)) \leq |\{\alpha_M : M \in \mathcal{M}\} \cup \{\alpha\}| = S(n,m) - \binom{n-1}{m-1} + 1.$$

Let $A \subseteq T(X,Y) \setminus O(X,Y)$ with $\langle O(X,Y) \cup A\rangle = T(X,Y)$, then $\mathcal{M} \subseteq \{\ker\beta : \beta \in A\}$ by Lemma 4.3. If $\gamma \in O(X,Y)$ then Y is not a transversal of $\ker\gamma$ or $\gamma|_Y$ is the identity map on Y. Hence, there is $S \subseteq A$ such that $\{s|_Y : s \in S\} = S(Y)$.

If $|X| = 4$ and $|X \setminus Y| = 1$ and $\{x_2, x_3\} \not\subseteq Y$ then S contains at least two elements (μ_1 and μ_2, say). Note that Y is a transversal of $\ker\mu_1$ as well as of $\ker\mu_2$ and $\ker\mu_1, \ker\mu_2 \notin \{\ker\beta : \beta \in O(X,Y)\}$. But we have only one $M \in \mathcal{M}$ with Y is a transversal of M and $M \notin \{\ker\beta : \beta \in O(X,Y)\}$. Hence, we need one additional element in A which is not in $O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$. Hence,

$$|A| \geq |\mathcal{D}| + 1 = S(n,m) - \binom{n-1}{m-1} + 1.$$

If $|X| = 3$ and $x_2 \notin Y$, then $|Y| = 2$ and $S(Y)$ is a cyclic group with one generator, i.e., S has to contain one element, say s. But in this case $\mu_S \notin O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$, we need one additional element in A which is not in $O(X,Y) \cup \{\alpha_M : M \in \mathcal{M}\}$. Hence,

$$|A| \geq |\mathcal{M}| + 1 = S(n,m) - \binom{n-1}{m-1} + 1.$$

□

Acknowledgment. This work was supported by the Ministry of Science and Technology of Thailand, the Royal Thai Government Scholarship of Thailand, and Prince of Songkla University, Pattani Campus, Pattani.

References

1. Fernandes V. H., Honyam P., Quinteiro, T. M. and Singha B., *On semigroups of endomorphisms of a chain with restriced range*, Semigroup Forum **89** (2013), 77–104.

2. Fernandes V. H. and Sanwong J., *On the ranks of semigroups of transformations on a finite set with restricted range*, Algebra Colleg. **21(3)** (2014), 497–510.
3. Gomes G. M. S. and Howie J. M., *On the ranks of certain finite semigroups of transformations*, Math. Proc. Cambridge Philos. Soc. **101** (1987), 395–403.
4. Gomes G. M. S.and Howie J. M., *On the ranks of certain semigroups of order-preserving transformations*, Semigroup Forum **45** (1992), 272–282.
5. Higgins P. M., Mitchell J. D. and Ruškuc N., *Generating the full transformation semigroup using order preserving mappings*, Glasg. Math. J. **45(3)** (2003), 557–566.
6. Honyam P. and Sanwong J., *Semigroups of transformations with invariant set*, J. Korean Math. Soc. **48** (2008), 289-300.
7. Howie J. M., *Products of idempotents in certain semigroups of transformations*, Proc. Edinb. Math. Soc. **17(2)** (1971), 223–236.
8. Howie J.M., *Fundamental of semigroup Theory*, Oxford University Press 1995.
9. Howie J. M., Ruškuc N. and Higgins, P. M., *On relative ranks of full transformation semigroups*, Comm. Algebra **26(3)** (1998), 733–748.
10. Sanwong J., *The regular part of a semigroup of transformations with restricted range*, Semigroup Forum **83** (2011), 134–146.
11. Sommanee W. and Sanwong J., *Rank and idempotent rank of finite full transformation semigroups with restricted range*, Semigroup Forum **87** (2013), 230–242.
12. Symons J. S. V., *Some results concerning a transformation semigroup*, J. Aust. Math. Soc. A **19** (1975), 413–425.

K. Tinpun, Institute of Mathematics, Faculty of Science, Potsdam University, Karl-Liebknecht--Strasse 24-25, D-14476 Potsdam, Germany, *e-mail*: `tinpun@uni-potsdam.de`

J. Koppitz, Institute of Mathematics, Faculty of Science, Potsdam University, Karl-Liebknecht--Strasse 24-25, D-14476 Potsdam, Germany, *e-mail*: `koppitz@uni-potsdam.de`