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EXISTENCE AND QUALITATIVE BEHAVIOR OF OSCILLATORY SOLUTIONS OF SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

T. KUSANO AND N. YOSHIDA

Dedicated to Professor Jaroslav Jaroš on the occasion of his 60th birthday

Abstract. We consider the second order linear differential equation

(A) $$(p(t)y')' + q(t)y = 0,$$

which is oscillatory under the assumption that $p(t)$ and $q(t)$ are positive, continuously differentiable and monotone functions on $[0, \infty)$. After studying qualitative properties, including amplitudes and slopes, of oscillatory solutions, we establish the existence of three types of solutions of (A) referred to as moderately bounded, small of large oscillatory solutions. Essential use is made of pairs of quadratic forms $P(t)y'(t)^2 + Q(t)y(t)^2$, $R(t)y'(t)^2 + S(t)y(t)^2$, which are monotone for all possible solutions $y(t)$ of (A), but have different monotonicity.

1. Introduction

We consider the second order linear differential equation

(A) $$\big(p(t)y'\big)' + q(t)y = 0$$

under the assumption that $p(t)$ and $q(t)$ are positive, continuously differentiable functions on $[0, \infty)$. The characteristic feature of (A) is that all of its nontrivial solutions are either oscillatory (in which case (A) is called oscillatory) or else nonoscillatory (in which case (A) is called nonoscillatory). This paper is concerned exclusively with the case where equation (A) is oscillatory. It is known [**10**] that (A) is oscillatory if

(1.1) $$\int_0^\infty \frac{\mathrm{d}t}{p(t)} = \infty, \quad \text{and} \quad \int_0^\infty P(t)^\lambda q(t)\mathrm{d}t = \infty \quad \text{for some } \lambda \in [0, 1),$$

where $P(t) = \int_0^t \mathrm{d}s/p(s)$, or if

(1.2) $$\int_0^\infty \frac{\mathrm{d}t}{p(t)} < \infty, \quad \text{and} \quad \int_0^\infty \pi(t)^\mu q(t)\mathrm{d}t = \infty \quad \text{for some } \mu \in (1, 2],$$

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where $\pi(t) = \int_t^\infty \mathrm{d}s/p(s)$. For the special case of (A)

$$(A_0) \qquad y'' + q(t)y = 0,$$

the following (see, e.g., [**11**]) can often be useful oscillation criteria:

$$\liminf_{t\to\infty} t^2 q(t) > \frac{1}{4}, \qquad \text{or} \qquad \liminf_{t\to\infty} t \int_t^\infty q(s)\mathrm{d}s > \frac{1}{4}.$$

We would like to acquire as much and detailed information as possible about the existence and the qualitative properties of oscillatory solutions of equation (A). Let $y(t)$ be an oscillatory solution on $[0,\infty)$ of (A). By $\{\sigma_k\}_{k=1}^\infty$, we denote the sequence of zeros of $y(t)$ and by $\{\tau_k\}_{k=1}^\infty$, the sequence of points at which $y(t)$ takes on extrema (i.e., local maxima or minima). They are arranged as $\sigma_k < \sigma_{k+1}$ and $\tau_k < \tau_{k+1}, k = 1, 2, \dots$. Naturally $y(\sigma_k) = 0$ and $y'(\tau_k) = 0$ for any k. The values $|y'(\sigma_k)|$ and $|y(\tau_k)|$ are referred to as the *slope* and the *amplitude*, respectively, of the k-th wave of $y(t)$. We use the following notations:

$$\mathcal{A}^*[y] = \sup_k |y(\tau_k)|, \qquad \mathcal{A}_*[y] = \inf_k |y(\tau_k)|,$$
$$\mathcal{S}^*[y] = \sup_k |y'(\sigma_k)|, \qquad \mathcal{S}_*[y] = \inf_k |y'(\sigma_k)|.$$

A solution $y(t)$ of (A) satisfying $\mathcal{A}^*[y] = \infty$, i.e., $\limsup_{t\to\infty} |y(t)| = \infty$, is an unbounded solution which may be termed a *large oscillatory solution.* Let $y(t)$ be a bounded solution such that $\mathcal{A}^*[y] < \infty$. Two cases are possible for it: either $\lim_{k\to\infty} |y(\tau_k)| = 0$ which is equivalent to $\lim_{t\to\infty} y(t) = 0$, or $\liminf_{k\to\infty} |y(\tau_k)| > 0$ which amounts to $\mathcal{A}_*[y] > 0$. In the former case it is called a *small oscillatory solution*, while in the latter case it is called a *moderately bounded oscillatory solution* of (A).

With regard to the oscillatory solutions of equation (A), there arise several questions to be answered including the following. Can one detect any law governing the distribution of zeros $\{\sigma_k\}$ and points of extrema $\{\tau_k\}$ of $y(t)$? Is it possible to give estimates, preferably precise, for $\mathcal{A}^*[y]$, $\mathcal{A}_*[y]$ (the upper and lower amplitudes of $y(t)$ on $[0,\infty)$), and for $\mathcal{S}^*[y]$, $\mathcal{S}_*[y]$ (the upper and lower slopes of $y(t)$ on $[0,\infty)$)? Is it possible to specify sufficient conditions for (A) to possess oscillatory solutions which are small, large or moderately bounded in the sense defined above?

In view of the difficulty in handling general oscillatory equations of the form (A), we limit our consideration to the case where both $p(t)$ and $q(t)$ are monotone on $[0,\infty)$, and distinguish the following four possibilities for the combination of their monotonicity:

$$(1.3) \qquad \text{(i)} \quad p'(t) \ge 0, \quad q'(t) \le 0, \qquad \text{(ii)} \quad p'(t) \le 0, \quad q'(t) \ge 0,$$
$$(1.4) \qquad \text{(iii)} \quad p'(t) \ge 0, \quad q'(t) \ge 0, \qquad \text{(iv)} \quad p'(t) \le 0, \quad q'(t) \le 0.$$

Note that in some cases (1.4) may be replaced by

$$(1.5) \qquad \text{(iii)} \quad \big(p(t)q(t)\big)' \ge 0, \qquad \text{(iv)} \quad \big(p(t)q(t)\big)' \le 0.$$

Given an oscillatory equation (A), the two quadratic forms in $y(t)$ and its derivative

$$(1.6)\quad V[y](t) = p(t)y'(t)^2 + q(t)y(t)^2, \qquad W[y](t) = \frac{y'(t)^2}{q(t)} + \frac{y(t)^2}{p(t)},$$

can be formed automatically. As it is easily checked, if $y(t)$ is a solution of (A), then $\big(V[y](t)\big)' \le 0$ (or ≥ 0) if $p(t)$ and $q(t)$ satisfy (i) (or (ii)) of (1.3), and $\big(W[y](t)\big)' \le 0$ (or ≥ 0) if $p(t)$ and $q(t)$ satisfy (iii) (or (iv)) of (1.4), which means that

$V[y](t)$ is decreasing (or increasing) if $p(t)$ is increasing (or decreasing) and $q(t)$ is decreasing (or increasing); and

$W[y](t)$ is decreasing (or increasing) if both $p(t)$ and $q(t)$ are increasing (or decreasing).

This fact was found by Hille [**7**, pp. 380–383] and applied to the qualitative study of oscillatory solutions of (A). It should be noticed here that when specialised to equation (A_0) ($p(t) \equiv 1$), the above-mentioned statements regarding $V[y](t)$ and $W[y](t)$ given by (1.6) are reduced to the following simplified assertion. Let $y(t)$ be a solution of (A_0). Then

$$q'(t) \le 0 \implies \big(V[y](t)\big)' \le 0, \quad \big(W[y](t)\big)' \ge 0,$$
$$q'(t) \ge 0 \implies \big(V[y](t)\big)' \ge 0, \quad \big(W[y](t)\big)' \le 0,$$

which implies that for equation (A_0) with $q(t)$ monotone there exist two quadratic forms $V[y](t)$ and $W[y](t)$, one of which is increasing and the other is decreasing for any of its solutions $y(t)$. This noteworthy result was obtained by Hartman [**5**, pp 510–511] and utilized as a basis for constructing large or small oscillatory solutions of (A_0).

The objective of this paper is to show that for equation (A) with monotone coefficients $p(t)$ and $q(t)$, there always exists a pair of quadratic forms of the type

$$(1.7)\qquad \mathcal{V}[y](t) = P(t)y'(t)^2 + Q(t)y(t)^2, \quad \mathcal{W}[y](t) = R(t)y'(t)^2 + S(t)y(t)^2,$$

having the property that $\big(\mathcal{V}[y](t)\big)' \ge 0$ and $\big(\mathcal{W}[y](t)\big)' \le 0$, so that $\mathcal{V}[y](t)$ is increasing and $\mathcal{W}[y](t)$ is decreasing for any oscillatory solution $y(t)$ of (A), and then to demonstrate the usefulness of these quadratic forms in the qualitative analysis of oscillatory behavior of solutions of (A) as well as in establishing the existence of small or large oscillatory solutions for (A).

The main body of the paper is organized as follows. Section 2 is devoted to the detection of a pair of quadratic forms $\{\mathcal{V}[y], \mathcal{W}[y]\}$ of the type (1.7) which are associated with equation (A) in such a way that $\mathcal{V}[y](t)$ is increasing and $\mathcal{W}[y](t)$ is decreasing for any of its (oscillatory) solutions $y(t)$ on $[0,\infty)$. In Section 3, these pairs $\{\mathcal{V}[y], \mathcal{W}[y]\}$ can be effectively used to obtain explicit upper bounds for $\mathcal{A}^*[y]$ and $\mathcal{S}^*[y]$ as well as explicit lower bounds for $\mathcal{A}_*[y]$ and $\mathcal{S}_*[y]$, for all possible solutions $y(t)$ of (A). The results indicate the existence of a class of equations of the form (A), all solutions of which are moderately bounded oscillatory solutions. Besides, an attempt is made to find laws or rules governing the structure of the sequences of zeros and extrema points of all oscillatory solutions of (A).

In the final Section 4, we focus our attention on the existence of small or large oscillatory solutions for equation (A). Building existence theory of such non-periodic oscillatory solutions seems to be a very difficult task even for second order linear differential equations. To the best of the authors' knowledge Hartman's paper [**4**] on equation (A_0) represents one of the most general and deep results on the subject. See also [**5**, pp. 510–513]. Our aim is to generalize the results of Hartman, given as Theorem 3.1 in [**5**], to equation (A) so as to cover all the four cases for $\big(p(t), q(t)\big)$ as described in (1.3) and (1.4) (or (1.5)). Our main results are designed to provide explicit criteria for (A) to possess desired oscillatory solutions as well as information as to how small or large the obtained solutions are with the help of the quadratic forms $\mathcal{V}[y]$ and $\mathcal{W}[y]$ associated with (A).

Second order linear oscillation theory has a long history, and there is a vast existing literature on the subject. The reader is referred, for example, to the books [**1, 2, 3, 5, 7, 8, 9, 11**] and the papers cited therein for a wide spectrum of studies on oscillation problems for second order linear differential equations. It seems that little in-depth analysis has been made of equations of the form (A), and this observation motivated the present work.

2. Preliminaries

The oscillatory differential equation (A) is under consideration on $[0, \infty)$. It is assumed that $p(t)$ and $q(t)$ are monotone in the sense that one of the conditions in (1.3) and (1.4) (or (1.5)) is satisfied. Our main purpose here is to show that there always exists a pair of positive quadratic forms $\{\mathcal{V}[y], \mathcal{W}[y]\}$ of the type (1.7) such that $\big(\mathcal{V}[y]\big)' \geq 0$ and $\big(\mathcal{W}[y]\big)' \leq 0$ on $[0, \infty)$ for all possible solutions $y(t)$ of (A). Some useful properties of these quadratic forms are also mentioned.

Lemma 2.1. *Let $V[y]$ be the quadratic form defined by*

$$V[y](t) = p(t)y'(t)^2 + q(t)y(t)^2.$$

Then, the following hold for any solution $y(t)$ of (A)*:*

$$\big(V[y](t)\big)' = -p'(t)y'(t)^2 + q'(t)y(t)^2, \tag{2.1}$$

$$\left(\frac{p(t)}{q(t)}V[y](t)\right)' = -\frac{p(t)^2q'(t)}{q(t)^2}y'(t)^2 + p'(t)y(t)^2, \tag{2.2}$$

$$\big(p(t)V[y](t)\big)' = \big(p(t)q(t)\big)'y(t)^2, \tag{2.3}$$

$$\left(\frac{1}{q(t)}V[y](t)\right)' = -\frac{\big(p(t)q(t)\big)'}{q(t)^2}y'(t)^2. \tag{2.4}$$

Proof. In the proof the independent variable t is deleted. Let y be any solution of (A) on $[0, \infty)$. Since $V[y] = (py')^2/p + qy^2$, we see that

$$\big(V[y]\big)' = 2y'(py')' - p'y'^2 + 2qyy' + q'y^2 = -p'y'^2 + q'y^2,$$

which verifies (2.1). Using (2.1), we obtain

$$\begin{aligned}\left(\frac{p}{q}V[y]\right)' &= \frac{p}{q}\big(V[y]\big)' + \left(\frac{p'q-pq'}{q^2}\right)V[y]\\ &= \frac{p}{q}\big(-p'y'^2+q'y^2\big) + \left(\frac{p'q-pq'}{q^2}\right)\big(py'^2+qy^2\big) = -\frac{p^2q'}{q^2}y'^2+p'y^2,\end{aligned}$$

which implies that (2.2) is true. To confirm (2.3) and (2.4), it suffices to proceed as follows:

$$\big(pV[y]\big)' = p'V[y] + p\big(V[y]\big)' \quad = p'\big(py'^2+qy^2\big) + p\big(-p'y'^2+q'y^2\big) = (pq)'y^2,$$

and

$$\begin{aligned}\left(\frac{1}{q}V[y]\right)' &= -\frac{q'}{q^2}V[y] + \frac{1}{q}\big(V[y]\big)' \quad = -\frac{q'}{q^2}\big(py'^2+qy^2\big) + \frac{1}{q}\big(-p'y'^2+q'y^2\big)\\ &= -\frac{(pq)'}{q^2}y'^2.\end{aligned}$$

□

Remark 2.1. In (1.6), there appears another quadratic form $W[y](t) = \frac{y'(t)^2}{q(t)} + \frac{y(t)^2}{p(t)}$. Note that $V[y](t)$ and $W[y](t)$ are not independent. For example, the following identities hold:

$$p(t)q(t)W[y](t) \equiv V[y](t), \qquad p(t)^2W[y](t) \equiv \frac{p(t)}{q(t)}V[y](t),$$

$$p(t)^2q(t)W[y](t) \equiv p(t)V[y](t), \qquad p(t)W[y](t) \equiv \frac{1}{q(t)}V[y](t).$$

This shows that Lemma 2.1 could be formulated in terms of the quadratic form $W[y](t)$.

An important implication of Lemma 2.1 is that in case $p(t)$ and $q(t)$ are monotone, there exist two positive quadratic forms $\mathcal{V}[y]$ and $\mathcal{W}[y]$ with the property that $\big(\mathcal{V}[y]\big)'(t) \ge 0$ and $\big(\mathcal{W}[y]\big)'(t) \le 0$, so that $\mathcal{V}[y](t)$ is increasing and $\mathcal{W}[y](t)$ is decreasing for any solution $y(t)$ of (A). More specifically, such quadratic forms are determined in dependence on the monotonicity of $(p(t), q(t))$ as follows:

$$\mathcal{V}[y](t) = \frac{p(t)^2}{q(t)}y'(t)^2 + p(t)y(t)^2, \quad \mathcal{W}[y](t) = p(t)y'(t)^2 + q(t)y(t)^2 \tag{2.5}$$

if $p'(t) \ge 0$ and $q'(t) \le 0$;

$$\mathcal{V}[y](t) = p(t)y'(t)^2 + q(t)y(t)^2, \quad \mathcal{W}[y](t) = \frac{p(t)^2}{q(t)}y'(t)^2 + p(t)y(t)^2 \tag{2.6}$$

if $p'(t) \le 0$ and $q'(t) \ge 0$;

$$\mathcal{V}[y](t) = p(t)^2y'(t)^2 + p(t)q(t)y(t)^2, \quad \mathcal{W}[y](t) = \frac{p(t)}{q(t)}y'(t)^2 + y(t)^2 \tag{2.7}$$

if $p'(t) \geq 0$ and $q'(t) \geq 0$, or if $(p(t)q(t))' \geq 0$, and

$$\mathcal{V}[y](t) = \frac{p(t)}{q(t)} y'(t)^2 + y(t)^2, \qquad \mathcal{W}[y](t) = p(t)^2 y'(t)^2 + p(t)q(t)y(t)^2 \tag{2.8}$$

if $p'(t) \leq 0$ and $q'(t) \leq 0$, or if $(p(t)q(t))' \leq 0$.

Lemma 2.2. *Let* $\{\mathcal{V}[y], \mathcal{W}[y]\}$ *be any one of the pairs of quadratic forms described in* Lemma 2.1, *and let* $y_0(t)$ *and* $y_1(t)$ *be any linearly independent solutions of* (A) *on* $[0, \infty)$. *Then, there exists a positive constant* C^2 *such that*

$$\mathcal{V}[y_0](t)\mathcal{W}[y_1](t) \geq C^2 \qquad \text{for } t \geq 0. \tag{2.9}$$

Proof. Since $y_0(t)$ and $y_1(t)$ are linearly independent, their Wronskian satisfies $p(t)W[y_0, y_1](t) \equiv C$, $t \geq 0$, for some constant $C \neq 0$, that is,

$$p(t)\big(y_0(t)y_1'(t) - y_0'(t)y_1(t)\big) \equiv C, \qquad t \geq 0. \tag{2.10}$$

We rewrite the Wronskian as

$$y_0(t)y_1'(t) - y_0'(t)y_1(t) = \frac{y_0(t)}{\sqrt{p(t)}}\sqrt{p(t)}y_1'(t) - \frac{y_0'(t)}{\sqrt{q(t)}}\sqrt{q(t)}y_1(t), \tag{2.11}$$

or as

$$y_0(t)y_1'(t) - y_0'(t)y_1(t) = \sqrt{q(t)}y_0(t)\frac{y_1'(t)}{\sqrt{q(t)}} - \sqrt{p(t)}y_0'(t)\frac{y_1(t)}{\sqrt{p(t)}}. \tag{2.12}$$

Combining (2.10) with (2.11) or (2.12) and using the Schwarz inequality, we find that

$$\begin{aligned}
C^2 &= p(t)^2\big(y_0(t)y_1'(t) - y_0'(t)y_1(t)\big)^2 \\
&\leq p(t)^2\left(\frac{|y_0(t)|}{\sqrt{p(t)}}\sqrt{p(t)}|y_1'(t)| + \frac{|y_0'(t)|}{\sqrt{q(t)}}\sqrt{q(t)}|y_1(t)|\right)^2 \\
&\leq p(t)^2\left(\frac{y_0'(t)^2}{q(t)} + \frac{y_0(t)^2}{p(t)}\right)\big(p(t)y_1'(t)^2 + q(t)y_1(t)^2\big) \\
&= \left(\frac{p(t)^2}{q(t)}y_0'(t)^2 + p(t)y_0(t)^2\right)\big(p(t)y_1'(t)^2 + q(t)y_1(t)^2\big),
\end{aligned}$$

or that

$$\begin{aligned}
C^2 &= p(t)^2\big(y_0(t)y_1'(t) - y_0'(t)y_1(t)\big)^2 \\
&\leq p(t)^2\left(\sqrt{q(t)}|y_0(t)|\frac{|y_1'(t)|}{\sqrt{q(t)}} + \sqrt{p(t)}|y_0'(t)|\frac{|y_1(t)|}{\sqrt{p(t)}}\right)^2 \\
&\leq p(t)^2\big(p(t)y_0'(t)^2 + q(t)y_0(t)^2\big)\left(\frac{y_1'(t)^2}{q(t)} + \frac{y_1(t)^2}{p(t)}\right) \\
&= \big(p(t)^2y_0'(t)^2 + p(t)q(t)y_0(t)^2\big)\left(\frac{p(t)}{q(t)}y_1'(t)^2 + y_1(t)^2\right).
\end{aligned}$$

This proves that (2.9) is true. □

Remark 2.2. From (2.9), it follows in particular that

$$\lim_{t\to\infty} \mathcal{W}[y_1](t) = 0 \quad \Longrightarrow \quad \lim_{t\to\infty} \mathcal{V}[y_0](t) = \infty.$$

Sometimes it is useful to consider the following equation together with (A)

$$\text{(B)} \qquad \big(Q(t)z'\big)' + P(t)z = 0, \qquad P(t) = \frac{1}{p(t)}, \qquad Q(t) = \frac{1}{q(t)}.$$

Equations (A) and (B) are interrelated via $z = p(t)y'$ and $y = Q(t)z'$ in the sense that if $y(t)$ (resp., $z(t)$) satisfies (A) (resp., (B)), then $z(t) = p(t)y'(t)$ (resp., $y(t) = Q(t)z'(t)$) satisfies (B) (resp., (A)). It is clear that if (A) is oscillatory, then so is (B), and vice versa.

Consider the quadratic forms (1.6) introduced by Hille for both (A) and (B), and denote them by

$$V_A[y](t) = p(t)y'(t)^2 + q(t)y(t)^2, \qquad W_A[y](t) = \frac{y'(t)^2}{q(t)} + \frac{y(t)^2}{p(t)},$$
$$V_B[z](t) = \frac{z'(t)^2}{q(t)} + \frac{z(t)^2}{p(t)}, \qquad W_B[z](t) = p(t)z'(t)^2 + q(t)z(t)^2.$$

An elementary computation shows that

$$V_A[y](t) = W_B[y](t) = W_A[z](t) = V_B[z](t),$$

if $y(t)$ is a solution of (A) and $z(t) = p(t)y'(t)$, or if $z(t)$ is a solution of (B) and $y(t) = Q(t)z'(t)$.

It was shown above that for equation (A) there always exists a pair of quadratic forms with different monotonicity; see (2.5)–(2.8). These formulas applied to equation (B) have the following representations:

$$\text{(2.13)} \qquad \mathcal{V}_B[z] = \frac{Q(t)^2}{P(t)} z'(t)^2 + Q(t)z(t)^2, \qquad \mathcal{W}_B[z] = Q(t)z'(t)^2 + P(t)z(t)^2$$

if $p'(t) \ge 0$ and $q'(t) \le 0$;

$$\text{(2.14)} \qquad \mathcal{V}_B[z] = Q(t)z'(t)^2 + P(t)z(t)^2, \qquad \mathcal{W}_B[z] = \frac{Q(t)^2}{P(t)} z'(t)^2 + Q(t)z(t)^2$$

if $p'(t) \le 0$ and $q'(t) \ge 0$;

$$\text{(2.15)} \qquad \mathcal{V}_B[z] = \frac{Q(t)}{P(t)} z'(t)^2 + z(t)^2, \qquad \mathcal{W}_B[z] = Q(t)^2 z'(t)^2 + P(t)Q(t)z(t)^2$$

if $p'(t) \ge 0$ and $q'(t) \ge 0$, or if $(p(t)q(t))' \ge 0$, and

$$\text{(2.16)} \qquad \mathcal{V}_B[z] = Q(t)^2 z'(t)^2 + P(t)Q(t)z(t)^2, \qquad \mathcal{W}_B[z] = \frac{Q(t)}{P(t)} z'(t)^2 + z(t)^2$$

if $p'(t) \le 0$ and $q'(t) \le 0$, or if $(p(t)q(t))' \le 0$.

Comparing (2.5)–(2.8) (with the subscript A added to $\mathcal{V}, \mathcal{W}$) with (2.13)–(2.16) suggests the truth of the next lemma.

Lemma 2.3. *Let $y(t)$ be a solution of* (A) *and put $z(t) = p(t)y'(t)$, or let $z(t)$ be a solution of* (B) *and put $y(t) = Q(t)z'(t)$. Then, there hold the following* *formulas:*

$$\begin{aligned}
&\frac{p(t)^2}{q(t)}y'(t)^2 + p(t)y(t)^2 = \frac{Q(t)^2}{P(t)}z'(t)^2 + Q(t)z(t)^2,\\
&p(t)y'(t)^2 + q(t)y(t)^2 = Q(t)z'(t)^2 + P(t)z(t)^2,\\
&p(t)^2y'(t)^2 + p(t)q(t)y(t)^2 = \frac{Q(t)}{P(t)}z'(t)^2 + z(t)^2,\\
&\frac{p(t)}{q(t)}y'(t)^2 + y(t)^2 = Q(t)^2z'(t)^2 + P(t)Q(t)z(t)^2.
\end{aligned}$$

Proof. We omit the independent variable t from $y(t), z(t)$ and their derivatives. Let y be any solution of (A) on $[0,\infty)$ and define $z = p(t)y'$. Then, since $z' = \big(p(t)y'\big)' = -q(t)y$, we obtain

$$\begin{aligned}
\frac{Q(t)^2}{P(t)}z'^2 + Q(t)z^2 &= \frac{p(t)}{q(t)^2}(-q(t)y)^2 + \frac{1}{q(t)}(p(t)y')^2\\
&= \frac{p(t)^2}{q(t)}y'^2 + p(t)y^2,\\
Q(t)z'^2 + P(t)z^2 &= \frac{1}{q(t)}(-q(t)y)^2 + \frac{1}{p(t)}(p(t)y')^2\\
&= p(t)y'^2 + q(t)y^2\\
\frac{Q(t)}{P(t)}z'^2 + z^2 &= \frac{p(t)}{q(t)}(-q(t)y)^2 + (p(t)y')^2\\
&= p(t)^2y'^2 + p(t)q(t)y^2\\
Q(t)^2z'^2 + P(t)Q(t)z^2 &= \frac{1}{q(t)^2}(-q(t)y)^2 + \frac{(p(t)y')^2}{p(t)q(t)}\\
&= \frac{p(t)}{q(t)}y'^2 + y^2.
\end{aligned}$$

The proof starting from a solution z of (B) is essentially the same as above, and so is deleted. □

Remark 2.3. From Lemma 2.2 applied to (B) it follows that for any linearly independent solutions $z_0(t), z_1(t)$ of (B) on $[0,\infty)$, any pair of quadratic forms $\{\mathcal{V}_B[z], \mathcal{W}_B[z]\}$ from (2.13)–(2.16) satisfies

$$\mathcal{V}_B[z_0](t)\mathcal{W}_B[z_1](t) \ge C^2, \qquad t \ge 0,$$

for some constant $C^2 > 0$. This inequality shows that

$$\lim_{t\to\infty} \mathcal{W}_B[z_1](t) = 0 \quad\Longrightarrow\quad \lim_{t\to\infty} \mathcal{V}_B[z_0](t) = \infty.$$

Let us consider equation (B) again and transform it by performing the change of independent variable

$$s = \int_0^t \frac{q(\xi)}{p(\xi)}\mathrm{d}\xi \tag{2.17}$$

under the assumption that

$$\int_0^\infty \frac{q(t)}{p(t)}\mathrm{d}t = \infty. \tag{2.18}$$

As it is easily seen, the transformed equation is expressed as

$$\big(\mathcal{P}(s)\dot{w}(s)\big)^{\cdot} + \mathcal{Q}(s)w(s) = 0, \qquad s \geq 0, \tag{C}$$

where $\cdot = d/\mathrm{d}s$, $w(s) = z(t)$, $\mathcal{P}(s) = P(t) = 1/p(t)$ and $\mathcal{Q}(s) = Q(t) = 1/q(t)$. Recalling the relationship between equations (A) and (B) one recognizes that the direct passage from (A) to (C) is possible by way of the transformation (2.17) combined with $w(s) = p(t)y'(t)$. It is obvious that conversely (C) can be transformed into (A) via the change of variables $y(t) = \mathcal{P}(s)\dot{w}(s)$. Thus there exists a close relationship between the pairs of quadratic forms associated with equations (A) and (C), as it is described in the following lemma which is a generalization of a result of Hartman [**5**, p 512, Lemma 3.1].

Lemma 2.4. *Suppose that* (2.18) *holds. Consider* (C) *which is connected with* (A) *via* (2.17). *Then, any pair* $(y(t), w(s))$ *consisting of a solution* $y(t)$ *of* (A) *and the corresponding solution* $w(s)$ *of* (C) *satisfies the following formulas*:

$$\frac{p(t)}{q(t)}y'(t)^2 + y(t)^2 = \mathcal{P}(s)^2\dot{w}(s)^2 + \mathcal{P}(s)\mathcal{Q}(s)w(s)^2, \tag{2.19}$$

$$p(t)^2y'(t)^2 + p(t)q(t)y(t)^2 = \frac{\mathcal{P}(s)}{\mathcal{Q}(s)}\dot{w}(s)^2 + w(s)^2, \tag{2.20}$$

$$p(t)y'(t)^2 + q(t)y(t)^2 = \frac{\mathcal{P}(s)^2}{\mathcal{Q}(s)}\dot{w}(s)^2 + \mathcal{P}(s)w(s)^2, \tag{2.21}$$

$$\frac{p(t)^2}{q(t)}y'(t)^2 + p(t)y(t)^2 = \mathcal{P}(s)\dot{w}(s)^2 + \mathcal{Q}(s)w(s)^2. \tag{2.22}$$

Proof. Let $y(t)$ be a solution of (A) on $[0,\infty)$ and put $w(s) = p(t)y'(t)$ with s given by (2.17). We then have

$$\dot{w}(s) = \big(p(t)y'(t)\big)'/s' = -q(t)y(t)\frac{p(t)}{q(t)} = -p(t)y(t) \quad \text{or} \quad \mathcal{P}(s)\dot{w}(s) = -y(t),$$

and so

$$\big(\mathcal{P}(s)\dot{w}(s)\big)^{\cdot} = -y'(t)\frac{p(t)}{q(t)} = -\mathcal{Q}(s)w(s),$$

which shows that $w(s)$ is a solution of (C) on $[0,\infty)$. To obtain (2.19), we proceed as follows:

$$\begin{aligned} y(t)^2 + \frac{p(t)}{q(t)}y'(t)^2 &= \big(-\mathcal{P}(s)\dot{w}(s)\big)^2 + \frac{p(t)}{q(t)}\left(\frac{w(s)}{p(t)}\right)^2 \\ &= \mathcal{P}(s)^2\dot{w}(s)^2 + \mathcal{P}(s)\mathcal{Q}(s)w(s)^2. \end{aligned}$$

Multiplying (2.19) by $p(t)q(t) = 1/\mathcal{P}(s)\mathcal{Q}(s)$ gives (2.20). To confirm (2.21) and (2.22), it suffices to multiply (2.19) by $q(t) = 1/\mathcal{Q}(s)$ and $p(t) = 1/\mathcal{P}(s)$, respectively. The proof starting from a solution $w(s)$ of (C) is fundamentally the same as above, and so is deleted. □

3. Moderately bounded oscillatory solutions

We now know that with the oscillatory equation (A) with monotone coefficients one can associate four pairs of quadratic forms $\{\mathcal{V}[y], \mathcal{W}[y]\}$ such that $\big(\mathcal{V}[y](t)\big)' \geq 0$ and $\big(\mathcal{W}[y](t)\big)' \leq 0$ for all solutions $y(t)$ on $[0,\infty)$ of (A). In this section, these quadratic forms are utilized to detect some nontrivial qualitative properties of oscillatory solutions of (A) including their amplitudes and slopes.

Suppose that (A) is oscillatory. Let $y(t)$ be any one of its solutions on $[0,\infty)$, and let $\{\sigma_k\}$ and $\{\tau_k\}$, respectively, denote the sequences of all zeros and all extrema points of $y(t)$. It is clear that $y(\sigma_k) = 0$ and $y'(\tau_k) = 0$ for all $k = 1, 2, \dots$.

We distinguish four cases (1.3) and (1.4) (or (1.5)) for possible combinations of the monotonicity of $p(t)$ and $q(t)$, and recall that in each of these cases the pair of quadratic forms $\{\mathcal{V}[y], \mathcal{W}[y]\}$ with different monotonicity is given explicitly by (2.5)–(2.8). First we are concerned with the amplitudes of solutions of (A). A pioneering study of amplitudes for solutions of (A_0) was attempted by Hartman and Wintner [**6**]. Our discussions are based on the following simple inequalities:

$$\mathcal{V}[y](\tau_k) \leq \mathcal{V}[y](\tau_{k+1}), \qquad \mathcal{W}[y](\tau_k) \geq \mathcal{W}[y](\tau_{k+1}), \qquad k = 1, 2, \dots, \tag{3.1}$$

$$\mathcal{V}[y](\tau_k) \geq \mathcal{V}[y](0), \qquad \mathcal{W}[y](\tau_k) \leq \mathcal{W}[y](0), \qquad k = 1, 2, \dots . \tag{3.2}$$

In what follows use is made of the notation $p(\infty) = \lim_{t\to\infty} p(t)$, $q(\infty) = \lim_{t\to\infty} q(t)$ which exist in the extended positive half-line $[0,\infty]$.

Let $p'(t) \geq 0$ and $q'(t) \leq 0$ for $t \geq 0$. Application of (3.1) to $\{\mathcal{V}[y], \mathcal{W}[y]\}$ given by (2.5) implies that the sequence $\{p(\tau_k)y(\tau_k)^2\}$ is increasing and the sequence $\{q(\tau_k)y(\tau_k)^2\}$ is decreasing, so that

$$\{\sqrt{p(\tau_k)}|y(\tau_k)|\} \text{ is increasing} \qquad \text{and} \qquad \{\sqrt{q(\tau_k)}|y(\tau_k)|\} \text{ is decreasing.}$$

If $p'(t) \leq 0$ and $q'(t) \geq 0$ for $t \geq 0$, then in view of (2.6), it suffices to interchange the role of $\mathcal{V}[y]$ and $\mathcal{W}[y]$ in the above case, and it follows that

$$\{\sqrt{q(\tau_k)}|y(\tau_k)|\} \text{ is increasing} \qquad \text{and} \qquad \{\sqrt{p(\tau_k)}|y(\tau_k)|\} \text{ is decreasing.}$$

Let $p'(t) \geq 0$ and $q'(t) \geq 0$, or more generally $(p(t)q(t))' \geq 0$ for $t \geq 0$. Applying (3.1) to $\{\mathcal{V}[y], \mathcal{W}[y]\}$ given by (2.7), we see that

$$\{\sqrt{p(\tau_k)q(\tau_k)}|y(\tau_k)|\} \text{ is increasing} \qquad \text{and} \qquad \{|y(\tau_k)|\} \text{ is decreasing.} \tag{3.3}$$

If $p'(t) \leq 0$ and $q'(t) \leq 0$, or if $(p(t)q(t))' \leq 0$ for $t \geq 0$, then (3.1) written for (2.8) implies that

$$\{|y(\tau_k)|\} \text{ is increasing} \qquad \text{and} \qquad \{\sqrt{p(\tau_k)q(\tau_k)}|y(\tau_k)|\} \text{ is decreasing.} \tag{3.4}$$

The above statements can be regarded as the description, mostly indirect, of laws governing the variation of the amplitudes of oscillatory solutions of (A).

Direct information involved in (3.3) and (3.4) is that for any oscillatory solution $y(t)$ of (A), the sequence of its extrema or amplitudes $\{|y(\tau_k)|\}$ is decreasing or increasing according to whether $p'(t) \geq 0, q'(t) \geq 0$ or $p'(t) \leq 0, q'(t) \leq 0$ for $t \geq 0$.

Next, let us combine (3.2) with (2.5)–(2.8) to estimate the upper and lower amplitudes $\mathcal{A}^*[y]$ and $\mathcal{A}_*[y]$ on $[0, \infty)$ for the solution $y(t)$ of (A) satisfying the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{3.5}$$

where α and β are any given constants such that $(\alpha, \beta) \neq (0, 0)$. Then, we easily obtain the following four pairs of inequalities:

$$y(\tau_k)^2 \geq \frac{p(0)\big(q(0)\alpha^2 + p(0)\beta^2\big)}{p(\tau_k)q(0)}, \qquad y(\tau_k)^2 \leq \frac{q(0)\alpha^2 + p(0)\beta^2}{q(\tau_k)} \tag{3.6}$$

if $p'(t) \geq 0$ and $q'(t) \leq 0$,

$$y(\tau_k)^2 \geq \frac{q(0)\alpha^2 + p(0)\beta^2}{q(\tau_k)}, \qquad y(\tau_k)^2 \leq \frac{p(0)\big(q(0)\alpha^2 + p(0)\beta^2\big)}{p(\tau_k)q(0)} \tag{3.7}$$

if $p'(t) \leq 0$ and $q'(t) \geq 0$,

$$y(\tau_k)^2 \geq \frac{p(0)\big(q(0)\alpha^2 + p(0)\beta^2\big)}{p(\tau_k)q(\tau_k)}, \qquad y(\tau_k)^2 \leq \frac{q(0)\alpha^2 + p(0)\beta^2}{q(0)} \tag{3.8}$$

if $(p(t)q(t))' \geq 0$, and

$$y(\tau_k)^2 \geq \frac{q(0)\alpha^2 + p(0)\beta^2}{q(0)}, \qquad y(\tau_k)^2 \leq \frac{p(0)\big(q(0)\alpha^2 + p(0)\beta^2\big)}{p(\tau_k)q(\tau_k)} \tag{3.9}$$

if $(p(t)q(t))' \leq 0$. In each of (3.6)–(3.9), take the supremum over k of the right-hand inequality and the infimum of the left-hand inequality. Then, one can indicate the situations in which the upper and lower amplitudes $\mathcal{A}^*[y]$ and $\mathcal{A}_*[y]$ of the solution under consideration have a finite upper bound and a finite nonzero lower bound, respectively, as is shown in the following theorem.

Theorem 3.1. *Let* (A) *be oscillatory and let* $y(t)$ *be a solution of* (A) *on* $[0, \infty)$ *satisfying* (3.5).

(i) *Suppose that* $p'(t) \geq 0$ *and* $q'(t) \leq 0$ *for* $t \geq 0$. *Then,*

$$\mathcal{A}^*[y] \leq \sqrt{\frac{q(0)\alpha^2 + p(0)\beta^2}{q(\infty)}} \qquad \textit{if} \quad q(\infty) > 0, \tag{3.10}$$

$$\mathcal{A}_*[y] \geq \sqrt{\frac{p(0)\big(q(0)\alpha^2 + p(0)\beta^2\big)}{p(\infty)q(0)}} \qquad \textit{if} \quad p(\infty) < \infty. \tag{3.11}$$

(ii) *Suppose that* $p'(t) \leq 0$ *and* $q'(t) \geq 0$ *for* $t \geq 0$. *Then,*

$$\mathcal{A}^*[y] \le \sqrt{\frac{p(0)\big(q(0)\alpha^2 + p(0)\beta^2\big)}{p(\infty)q(0)}} \qquad \textit{if} \quad p(\infty) > 0, \tag{3.12}$$

$$\mathcal{A}_*[y] \ge \sqrt{\frac{q(0)\alpha^2 + p(0)\beta^2}{q(\infty)}} \qquad \textit{if} \quad q(\infty) < \infty. \tag{3.13}$$

(iii) *Suppose that* $(p(t)q(t))' \ge 0$ *for* $t \ge 0$. *Then,*

$$\mathcal{A}^*[y] \le \sqrt{\frac{q(0)\alpha^2 + p(0)\beta^2}{q(0)}}, \tag{3.14}$$

$$\mathcal{A}_*[y] \ge \sqrt{\frac{p(0)\big(q(0)\alpha^2 + p(0)\beta^2\big)}{p(\infty)q(\infty)}} \qquad \textit{if} \quad p(\infty)q(\infty) < \infty. \tag{3.15}$$

(iv) *Suppose that* $(p(t)q(t))' \le 0$ *for* $t \ge 0$. *Then,*

$$\mathcal{A}^*[y] \le \sqrt{\frac{p(0)\big(q(0)\alpha^2 + p(0)\beta^2\big)}{p(\infty)q(\infty)}} \qquad \textit{if} \quad p(\infty)q(\infty) > 0, \tag{3.16}$$

$$\mathcal{A}_*[y] \ge \sqrt{\frac{q(0)\alpha^2 + p(0)\beta^2}{q(0)}}. \tag{3.17}$$

Since the constants α and β in (3.5) are arbitrary, the inequalities (3.10)–(3.17) guarantee under the indicated conditions on $p(\infty)$ and/or $q(\infty)$ that $\mathcal{A}^*[y] < \infty$ and $\mathcal{A}_*[y] > 0$ for all solutions $y(t)$ of (A). Noting that $\mathcal{A}^*[y] < \infty$ imply the boundedness of $y(t)$ on $[0, \infty)$ and that $\mathcal{A}^*[y] < \infty$ plus $\mathcal{A}_*[y] > 0$ implies the moderate boundedness of $y(t)$ on $[0, \infty)$, we recognize that the following propositions are included in Theorem 3.1.

Corollary 3.1. *Suppose that* (A) *is oscillatory. All of its solutions are bounded on* $[0, \infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions*:

(i) $p'(t) \ge 0,\ q'(t) \le 0$ *for* $t \ge 0$ *and* $q(\infty) > 0$;
(ii) $p'(t) \le 0,\ q'(t) \ge 0$ *for* $t \ge 0$ *and* $p(\infty) > 0$;
(iii) $(p(t)q(t))' \ge 0$ *for* $t \ge 0$;
(iv) $(p(t)q(t))' \le 0$ *for* $t \ge 0$ *and* $p(\infty)q(\infty) > 0$.

Corollary 3.2. *Suppose that* (A) *is oscillatory. All of its solutions are moderately bounded on* $[0, \infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions*:

(i) $p'(t) \ge 0,\ q'(t) \le 0$ *for* $t \ge 0$ *and* $p(\infty) < \infty,\ q(\infty) > 0$;
(ii) $p'(t) \le 0,\ q'(t) \ge 0$ *for* $t \ge 0$ *and* $p(\infty) > 0,\ q(\infty) < \infty$;
(iii) $(p(t)q(t))' \ge 0$ *for* $t \ge 0$ *and* $p(\infty)q(\infty) < \infty$;
(iv) $(p(t)q(t))' \le 0$ *for* $t \ge 0$ *and* $p(\infty)q(\infty) > 0$.

Let us turn our attention to the slopes $\{|y'(\sigma_k)|\}$ and the upper and lower slopes $\mathcal{S}^*[y], \mathcal{S}_*[y]$ of oscillatory solutions $y(t)$ of (A) on $[0, \infty)$. As in the study of the amplitudes, we begin by applying to each pair $\{\mathcal{V}[y], \mathcal{W}[y]\}$ in (2.5)–(2.8) the following sets of inequalities

$$\mathcal{V}[y](\sigma_k) \le \mathcal{V}[y](\sigma_{k+1}), \qquad \mathcal{W}[y](\sigma_k) \ge \mathcal{W}[y](\sigma_{k+1}), \qquad k = 1, 2, \dots,$$

and

$$\mathcal{V}[y](\sigma_k) \geq \mathcal{V}[y](0), \qquad \mathcal{W}[y](\sigma_k) \leq \mathcal{W}[y](0), \qquad k = 1, 2, \dots . \tag{3.18}$$

For economy of space we write down only those inequalities which arise from (3.18) for the solutions $y(t)$ of (A) satisfying the initial condition (3.5). They read as follows:

$$y'(\sigma_k)^2 \geq \frac{p(0)q(\sigma_k)}{p(\sigma_k)^2 q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big), \qquad y'(\sigma_k)^2 \leq \frac{q(0)\alpha^2 + p(0)\beta^2}{p(\sigma_k)} \tag{3.19}$$

if $p'(t) \geq 0$ and $q'(t) \leq 0$ for $t \geq 0$,

$$y'(\sigma_k)^2 \geq \frac{q(0)\alpha^2 + p(0)\beta^2}{p(\sigma_k)}, \qquad y'(\sigma_k)^2 \leq \frac{p(0)q(\sigma_k)}{p(\sigma_k)^2 q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big) \tag{3.20}$$

if $p'(t) \leq 0$ and $q'(t) \geq 0$ for $t \geq 0$,

$$\begin{aligned} y'(\sigma_k)^2 &\geq \frac{p(0)}{p(\sigma_k)^2}\big(q(0)\alpha^2 + p(0)\beta^2\big), \\ y'(\sigma_k)^2 &\leq \frac{q(\sigma_k)}{p(\sigma_k)q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big) \end{aligned} \tag{3.21}$$

if $(p(t)q(t))' \geq 0$ for $t \geq 0$,

$$\begin{aligned} y'(\sigma_k)^2 &\geq \frac{q(\sigma_k)}{p(\sigma_k)q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big), \\ y'(\sigma_k)^2 &\leq \frac{p(0)}{p(\sigma_k)^2}\big(q(0)\alpha^2 + p(0)\beta^2\big) \end{aligned} \tag{3.22}$$

if $(p(t)q(t))' \leq 0$ for $t \geq 0$. From these inequalities one can easily find sufficient conditions on $p(t)$ and $q(t)$ which ensure that $\mathcal{S}^*[y] < \infty$ and/or $\mathcal{S}_*[y] > 0$ for all solutions $y(t)$ of (A).

Theorem 3.2. *Let* (A) *be oscillatory and let* $y(t)$ *be the solution of* (A) *on* $[0, \infty)$ *satisfying* (3.5).

(i) *Suppose that* $p'(t) \geq 0$ *and* $q'(t) \leq 0$ *for* $t \geq 0$. *Then,*

$$\mathcal{S}^*[y] \leq \sqrt{\frac{q(0)\alpha^2 + p(0)\beta^2}{p(0)}},$$

$$\mathcal{S}_*[y] \geq \sqrt{\frac{p(0)q(\infty)}{p(\infty)^2 q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \qquad \text{if } p(\infty) < \infty \text{ and } q(\infty) > 0.$$

(ii) *Suppose that* $p'(t) \leq 0$ *and* $q'(t) \geq 0$ *for* $t \geq 0$. *Then,*

$$\mathcal{S}^*[y] \leq \sqrt{\frac{p(0)q(\infty)}{p(\infty)^2 q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \qquad \text{if } p(\infty) > 0 \text{ and } q(\infty) < \infty,$$

$$\mathcal{S}_*[y] \geq \sqrt{\frac{q(0)\alpha^2 + p(0)\beta^2}{p(0)}}.$$

(iii) *Suppose that $p'(t) \ge 0$ and $q'(t) \ge 0$ for $t \ge 0$. Then,*

$$\mathcal{S}^*[y] \le \sqrt{\frac{q(\infty)}{p(0)q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \qquad \text{if } q(\infty) < \infty,$$

$$\mathcal{S}_*[y] \ge \sqrt{\frac{p(0)}{p(\infty)^2}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \qquad \text{if } p(\infty) < \infty.$$

(iv) *Suppose that $p'(t) \le 0$ and $q'(t) \le 0$ for $t \ge 0$. Then,*

$$\mathcal{S}^*[y] \le \sqrt{\frac{p(0)}{p(\infty)^2}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \qquad \text{if } p(\infty) > 0,$$

$$\mathcal{S}_*[y] \ge \sqrt{\frac{q(\infty)}{p(0)q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \qquad \text{if } q(\infty) > 0.$$

Corollary 3.3. *Let* (A) *be oscillatory. If $p(t)$ and $q(t)$ are monotone functions such that $0 < p(\infty) < \infty$ and $0 < q(\infty) < \infty$, then $\mathcal{S}^*[y] < \infty$ and $\mathcal{S}_*[y] > 0$ for all solutions $y(t)$ of* (A).

It may occur that the sequence $\{|y'(\sigma_k)|\}$ tends to 0 or to ∞ as $k \to \infty$. In fact, a closer look at (3.19)–(3.22) will lead to the following finding.

Corollary 3.4. *Let* (A) *be oscillatory and let $y(t)$ be any of its solution having a sequence of zeros $\{\sigma_k\}$. Then, it holds that*

(i) $\lim_{k\to\infty} y'(\sigma_k) = 0$ *if $p'(t) \ge 0$, $q'(t) \le 0$ and $p(\infty) = \infty$, or if $p'(t) \ge 0$, $q'(t) \ge 0$ and $\lim_{t\to\infty} q(t)/p(t) = 0$, and*

(ii) $\lim_{k\to\infty} |y'(\sigma_k)| = \infty$ *if $p'(t) \le 0$, $q'(t) \ge 0$ and $p(\infty) = 0$, or if $p'(t) \le 0$, $q'(t) \le 0$ and $\lim_{t\to\infty} q(t)/p(t) = \infty$.*

We note here that information on the derivatives of oscillatory solutions of (A) can be drawn from the quadratic form $\mathcal{W}[y]$ defined by (2.5)–(2.8).

Theorem 3.3. *Let* (A) *be oscillatory and let $y(t)$ be the solution of it satisfying the initial condition* (3.5).

(i) *Suppose that $p'(t) \ge 0$ and $q'(t) \le 0$ for $t \ge 0$. Then,*

$$\sup_t |y'(t)| \le \sqrt{\frac{q(0)\alpha^2 + p(0)\beta^2}{p(0)}},$$
$$\lim_{t\to\infty} y'(t) = 0 \qquad \text{if } p(\infty) = \infty.$$

(ii) *Suppose that $p'(t) \le 0$ and $q'(t) \ge 0$ for $t \ge 0$. Then,*

$$\sup_t |y'(t)| \le \sqrt{\frac{p(0)q(\infty)}{p(\infty)^2 q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \text{ if } p(\infty) > 0 \text{ and } q(\infty) < \infty.$$

(iii) *Suppose that $p'(t) \ge 0$ and $q'(t) \ge 0$ for $t \ge 0$. Then,*

$$\sup_t |y'(t)| \le \sqrt{\frac{q(\infty)}{p(0)q(0)}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \qquad \text{if } q(\infty) < \infty, \tag{3.23}$$

$$\lim_{t\to\infty} y'(t) = 0 \qquad\qquad \text{if } \lim_{t\to\infty} \frac{q(t)}{p(t)} = 0.$$

(iv) *Suppose that $p'(t) \le 0$ and $q'(t) \le 0$ for $t \ge 0$. Then,*

$$\sup_t |y'(t)| \le \sqrt{\frac{p(0)}{p(\infty)^2}\big(q(0)\alpha^2 + p(0)\beta^2\big)} \qquad \text{if } p(\infty) > 0.$$

Proof. Only the statement (iii) is proved. Since $p'(t) \ge 0$ and $q'(t) \ge 0$, $\mathcal{W}[y]$ is given by (2.7) and we see that

$$\frac{p(t)}{q(t)} y'(t)^2 \le \mathcal{W}[y](t) \le \mathcal{W}[y](0) = M_0, \quad \text{or} \quad |y'(t)| \le \sqrt{\frac{M_0 q(t)}{p(t)}}, \qquad t \ge 0.$$

From this, it follows that $\lim_{t\to\infty} y'(t) = 0$ if $\lim_{t\to\infty} q(t)/p(t) = 0$ and that $\sup_t |y'(t)| \le \sqrt{M_0 q(\infty)/p(0)}$ if $q(\infty) < \infty$. The last inequality coincides with (3.23) since $M_0 = \big(q(0)\alpha^2 + p(0)\beta^2\big)/q(0)$. □

The object of our final study in this section is the sequences of zeros and extrema points of solutions of (A). We are interested in explicit laws or rules, if any, governing the arrangement of these sequences.

Assume that (A) is oscillatory. Let $y(t)$ be any of its solutions on $[0, \infty)$ and let $\{\sigma_k\}$ and $\{\tau_k\}$ represent the sequences of zeros and extrema points of $y(t)$, respectively. The first result concerns the sequence $\{\sigma_k\}$.

Theorem 3.4. (i) *The sequence $\{\sigma_{k+1} - \sigma_k\}$ is decreasing or increasing according to $p'(t) \le 0$ and $q'(t) \ge 0$, or $p'(t) \ge 0$ and $q'(t) \le 0$.*
(ii) *Consider the case where $p'(t) \ge 0$ and $q'(t) \ge 0$ for $t \ge 0$. Suppose that $\int_0^\infty \mathrm{d}t/p(t) < \infty$. Put $\pi(t) = \int_t^\infty \mathrm{d}s/p(s)$ and assume that $\pi(t)^2 p(t)$ and $\pi(t)^4 p(t) q(t)$ are monotone for $t \ge 0$. Then, the sequence $\{\sigma_{k+1} - \sigma_k\}$ is decreasing or increasing according to whether*

$$(\pi(t)^2 p(t))' \le 0 \quad \text{and} \quad (\pi(t)^4 p(t) q(t))' \ge 0 \qquad \text{for } t \ge 0, \tag{3.24}$$

or

$$(\pi(t)^2 p(t))' \ge 0 \quad \text{and} \quad (\pi(t)^4 p(t) q(t))' \le 0 \qquad \text{for } t \ge 0. \tag{3.25}$$

Proof. (i) This was proved by Hille [**7**, Theorem 8.1.7] by using the generalized comparison theorem on the basis of Picone's identity. The special case where $p(t) \equiv 1$ was dealt with independently by Hartman [**5**, p 510, Theorem 3.1].
(ii) The change of variables $s = 1/\pi(t)$, $Y(s) = sy(t)$ transforms (A) into

$$\frac{d^2 Y}{\mathrm{d}s^2} + R(s)Y = 0, \qquad R(s) = \frac{p(t)q(t)}{s^4} = \pi(t)^4 p(t) q(t),$$

where $s \in [1/\pi(0), \infty)$. Clearly, the zeros of $y(t)$ correspond to those of $Y(s)$, which means that the zeros of $Y(s)$ are given by $\{s_k = 1/\pi(\sigma_k)\}$.

Assume that (3.24) holds. Then, $\mathrm{d}R(s)/\mathrm{d}s \ge 0$, and so it follows from (i) that the sequence $\{s_{k+1} - s_k\}$ is decreasing. On the other hand, since $(1/\pi(t))' =$

$1/\pi(t)^2 p(t)$, the function $1/\pi(t)$ is convex and hence, satisfies

$$\frac{1/\pi(t_2) - 1/\pi(t_1)}{t_2 - t_1} \leq \frac{1/\pi(t_3) - 1/\pi(t_2)}{t_3 - t_2}$$

for any t_1, t_2, t_3 such that $0 < t_1 < t_2 < t_3$. Combining the two things mentioned above, we obtain

$$1 \geq \frac{s_{k+2} - s_{k+1}}{s_{k+1} - s_k} = \frac{1/\pi(\sigma_{k+2}) - 1/\pi(\sigma_{k+1})}{1/\pi(\sigma_{k+1}) - 1/\pi(\sigma_k)} \geq \frac{\sigma_{k+2} - \sigma_{k+1}}{\sigma_{k+1} - \sigma_k} \tag{3.26}$$

for all $k = 1, 2, \ldots$, which implies that $\{\sigma_{k+1} - \sigma_k\}$ is decreasing. If (3.25) is assumed, then a parallel argument to the above shows that $\{s_{k+1} - s_k\}$ is increasing and $1/\pi(t)$ is concave, and finally leads to (3.26) with all the inequality signs reversed, ensuring that $\{\sigma_{k+1} - \sigma_k\}$ is increasing. This completes the proof. □

In order to say something about the sequence of extrema points of oscillatory solutions, we consider the equation

$$(Q(t)z')' + P(t)z = 0, \quad P(t) = \frac{1}{p(t)}, \quad Q(t) = \frac{1}{q(t)}. \tag{B}$$

As mentioned in Section 2, equations (A) and (B) are interrelated via $z = p(t)y'$ and $y = Q(t)z'$. This means that $y'(\tau) = 0$ implies $z(\tau) = 0$, and vice versa. Hence the sequence of extrema points of $y(t)$ coincides with that of zeros of the corresponding $z(t)$, and conversely. This observation suggests that Theorem 3.4 applied to (B) will provide information as to the monotonicity of the distances of consecutive extrema points of solutions of (A).

Theorem 3.5. *Suppose that* (A) *is oscillatory. Let* $\{\tau_k\}$ *denote the sequence of extrema points of its arbitrary solution.*

(i) *The sequence* $\{\tau_{k+1} - \tau_k\}$ *is decreasing or increasing according to* $p'(t) \leq 0$ *and* $q'(t) \geq 0$*, or* $p'(t) \geq 0$ *and* $q'(t) \leq 0$*.*

(ii) *Consider the case where* $p'(t) \leq 0$ *and* $q'(t) \leq 0$ *for* $t \geq 0$*. Suppose that* $\int_0^\infty q(t)\mathrm{d}t < \infty$*. Put* $\rho(t) = \int_t^\infty q(s)\mathrm{d}s$ *and assume that* $\rho(t)^2/q(t)$ *and* $\rho(t)^4/p(t)q(t)$ *are monotone for* $t \geq 0$*. Then, the sequence* $\{\tau_{k+1} - \tau_k\}$ *is decreasing or increasing according to whether*

$$(\rho(t)^2/q(t))' \leq 0 \qquad \text{and} \qquad (\rho(t)^4/p(t)q(t))' \geq 0 \qquad \text{for } t \geq 0,$$

or

$$(\rho(t)^2/q(t))' \geq 0 \qquad \text{and} \qquad (\rho(t)^4/p(t)q(t))' \leq 0 \qquad \text{for } t \geq 0.$$

To prove (i), apply (i) of Theorem 3.4 to equation (B) by noting that $Q'(t) = -q'(t)/q(t)^2$ and $P'(t) = -p'(t)/p(t)^2$.

To prove (ii), apply (ii) of Theorem 3.4 to equation (B) by using $Q(t), P(t)$ and $\rho(t)$ instead of $p(t), q(t)$ and $\pi(t)$, respectively.

This section concludes with some examples illustrating the main results stated above.

Example 3.1. Consider the linear differential equation

$$\big((t+1)^{\lambda}y'\big)' + \varphi(t)(t+1)^{-\lambda}y = 0 \tag{3.27}$$

on $[0,\infty)$, where $\lambda \in (0,1]$ is a constant and $\varphi\colon [0,\infty) \to (0,\infty)$ is a decreasing C^1-function such that $\varphi(\infty) > 0$. It is clear that (3.27) is oscillatory. Observe that

$$p'(t) \geq 0,\ q'(t) \leq 0, \qquad \text{and} \qquad (p(t)q(t))' = \varphi'(t) \leq 0.$$

Let $y(t)$ be a solution of (3.27) on $[0,\infty)$ satisfying the initial condition (3.5). Since $p(\infty) = \infty$ and $q(\infty) = 0$, (i) of Theorem 3.1 cannot be used to estimate $\mathcal{A}^*[y]$ and $\mathcal{A}_*[y]$. However, (iv) of Theorem 3.1 applies and gives

$$\mathcal{A}^*[y] \leq \sqrt{\frac{\varphi(0)\alpha^2+\beta^2}{\varphi(\infty)}}, \qquad \mathcal{A}_*[y] \geq \sqrt{\frac{\varphi(0)\alpha^2+\beta^2}{\varphi(0)}}. \tag{3.28}$$

This shows that all solutions of (3.27) are moderately bounded in accordance with the statement (iv) of Corollary 3.2. From (3.28), it follows that if in particular $\varphi(t) \equiv \omega^2 > 0$ a constant, then the upper and lower amplitudes coincide, that is,

$$\mathcal{A}^*[y] = \mathcal{A}_*[y] = \sqrt{\frac{\alpha^2\omega^2+\beta^2}{\omega^2}}. \tag{3.29}$$

This value may well be called the *amplitude* $\mathcal{A}[y]$ on $[0,\infty)$ of the solution $y(t)$ satisfying (3.5) of the equation

$$\big((t+1)^{\lambda}y'\big)' + \omega^2(t+1)^{-\lambda}y = 0. \tag{3.30}$$

As regards the slopes of $y(t)$ satisfying (3.5), using (i) of Theorem 3.2, one obtains an upper bound for $\mathcal{S}^*[y]$

$$\mathcal{S}^*[y] \leq \sqrt{\varphi(0)\alpha^2+\beta^2}. \tag{3.31}$$

However, there exists, no positive lower bounds for $\mathcal{S}_*[y]$ because (i) of Theorem 3.3 implies that $\lim_{t\to\infty} y'(t) = 0$ for all solutions of (A).

Finally, let $\{\sigma_k\}$ and $\{\tau_k\}$, respectively, be the sequences of zeros and extrema points of any given solution $y(t)$ of (3.27). The first statements of Theorems 3.4 and 3.5 guarantee that both $\{\sigma_{k+1}-\sigma_k\}$ and $\{\tau_{k+1}-\tau_k\}$ are increasing sequences.

Remark 3.1. Notice that the general solution of equation (3.30) is given explicitly by

$$y(t) = \begin{cases} A\cos(\omega\log(t+1)) + B\sin(\omega\log(t+1)) & \text{if } \lambda = 1 \\ A\cos\Big(\frac{\omega(t+1)^{1-\lambda}}{1-\lambda}\Big) + B\sin\Big(\frac{\omega(t+1)^{1-\lambda}}{1-\lambda}\Big) & \text{if } \lambda \in (0,1), \end{cases} \tag{3.32}$$

where A and B are arbitrary constants. It is a matter of elementary computation to verify directly that the solutions (3.32) satisfy (3.29), (3.31) (with $\varphi(0) = \omega^2$) and that the sequences of their zeros and extrema points enjoy the monotonicity properties as described above.

Example 3.2. Consider the equation

$$(\coth(t+a)\cdot y')' + \psi(t)\tanh(t+a)\cdot y = 0, \tag{3.33}$$

where $a > 0$ is a constant and $\psi\colon [0,\infty) \to (0,\infty)$ is an increasing C^1-function such that $\psi(\infty) < \infty$. This equation is clearly oscillatory. Since the functions $p(t) = \coth(t+a)$ and $q(t) = \psi(t)\tanh(t+a)$ satisfy

$$p'(t) \le 0, \quad q'(t) \ge 0, \qquad \text{and} \qquad (p(t)q(t))' = \psi'(t) \ge 0$$

and

$$p(0) = \coth a, \quad p(\infty) = 1, \quad q(0) = \psi(0)\tanh a, \quad q(\infty) = \psi(\infty),$$

all nontrivial solutions of equation (3.33) are moderately bounded by (ii) or (iii) of Corollary 3.2. To estimate $\mathcal{A}^*[y]$ and $\mathcal{A}_*[y]$ for a solution $y(t)$ satisfying (3.5), both (ii) and (iii) of Theorem 3.1 can be used. For example, from (iii), it follows that

$$\mathcal{A}^*[y] \le \sqrt{\alpha^2 + \frac{\coth^2 a}{\psi(0)}\beta^2}, \qquad \mathcal{A}_*[y] \ge \sqrt{\frac{\psi(0)}{\psi(\infty)}\alpha^2 + \frac{\coth^2 a}{\psi(\infty)}\beta^2},$$

which, when specialized to the case where $\psi(t) \equiv \omega^2 > 0$ a constant, gives the amplitude on $[0,\infty)$

$$\mathcal{A}[y] = \sqrt{\alpha^2 + \left(\frac{\coth a}{\omega}\right)^2 \beta^2}$$

for the solution $y(t)$ in question. It turns out that all oscillatory solutions of the equation

$$(\coth(t+a)\cdot y')' + \omega^2\tanh(t+a)\cdot y = 0 \tag{3.34}$$

have finite amplitudes on $[0,\infty)$.

It can also be shown that $\mathcal{S}^*[y] < \infty$ and $\mathcal{S}_*[y] > 0$ for all solutions of (3.33) by applying either (ii) or (iii) of Theorem 3.2. The results derived through (ii) for a solution $y(t)$ satisfying (3.5), read:

$$\mathcal{S}^*[y] \le \sqrt{\frac{\psi(\infty)}{\psi(0)}\bigl(\psi(0)\coth a\cdot\alpha^2 + \coth^3 a\cdot\beta^2\bigr)}, \tag{3.35}$$

$$\mathcal{S}_*[y] \ge \sqrt{\psi(0)\tanh^2 a\cdot\alpha^2 + \beta^2}. \tag{3.36}$$

It is easy to see that the estimates obtained via (iii) of Theorem 3.2 are not as sharp as (3.35), (3.36).

From the first statements of Theorems 3.4 and 3.5 applied to equation (3.33), it follows that the sequences $\{\sigma_k\}$ and $\{\tau_k\}$ of zeros and extrema points of any solution of it are arranged in such a way that both $\{\sigma_{k+1} - \sigma_k\}$ and $\{\tau_{k+1} - \tau_k\}$ are decreasing.

Remark 3.2. Note that equation (3.34) possesses general solutions expressed explicitly by

$$y(t) = A\cos(\omega\log\cosh(t+a)) + B\sin(\omega\log\cosh(t+a)),$$

where A and B are arbitrary constants.

4. Small or large oscillatory solutions

In this section, we study the existence of small or large oscillatory solutions of equation (A). It is known (cf. Corollary 3.2) that such solutions possibly exist only if the coefficients $p(t)$ and $q(t)$ satisfy one of the following conditions:

(i) $p'(t) \geq 0,\ q'(t) \leq 0,\ p(\infty) = \infty$ and/or $q(\infty) = 0$;

(ii) $p'(t) \leq 0,\ q'(t) \geq 0,\ p(\infty) = 0$ and/or $q(\infty) = \infty$;

(iii) $(p(t)q(t))' \geq 0,\ p(\infty)q(\infty) = \infty$;

(iv) $(p(t)q(t))' \leq 0,\ p(\infty)q(\infty) = 0$.

Our objective is to demonstrate that in each of the above cases there do exist small or large oscillatory solutions of (A) on the basis of a theory of Hartman [**4**, Remark] regarding the second order system of linear differential equations of the type

$$y'' + B(t)y' + A(t)y = 0, \tag{$*$}$$

where y is a d-dimensional vector, $A(t)$ is a continuously differentiable $d \times d$ matrix function on $[0, \infty)$ which is symmetric and positive definite for all t, and $B(t)$ is a continuous $d \times d$ matrix function on $[0, \infty)$.

We need the following two theorems due to Hartman in which, for any square matrix function $M(t)$, $M(t) \geq 0$ [or ≤ 0] means that $M(t)$ is non-negative definite [or non-positive definite] for any t, and $M(t)^T$ denotes the transpose of $M(t)$.

Theorem 4.1 (Hartman [**4**]). *Assume that*

$$A'(t) + B(t)A(t) + A(t)B(t)^T \geq 0 \quad [or\ \leq 0].$$

Then, if $y(t)$ is a solution of $(*)$,

$$\big(y(t) \cdot y(t) + A^{-1}(t)y'(t) \cdot y'(t)\big)' \leq 0 \quad [or\ \geq 0],$$

where the dot $\cdot$ denotes the scalar product. If in addition,

$$[\det A(t)] \exp\left[2 \int_0^t \operatorname{tr} B(s) \mathrm{d}s\right] \to \infty \quad [or\ 0] \quad as\ t \to \infty,$$

then $(*)$ possesses a solution $y_0(t)$ satisfying

$$y_0(t) \cdot y_0(t) + A^{-1}(t)y_0'(t) \cdot y_0'(t) \to 0 \quad [or\ \infty] \quad as\ t \to \infty.$$

Theorem 4.2 (Hartman [**4**]). *Assume that*

$$A'(t) \leq 0 \quad [or\ \geq 0]$$

and

$$B(t) + B(t)^T \geq 0 \quad [or\ \leq 0].$$

Then, if $y(t)$ is a solution of $(*)$,

$$\big(A(t)y(t) \cdot y(t) + y'(t) \cdot y'(t)\big)' \leq 0 \quad [or\ \geq 0].$$

If in addition,

$$[\det A(t)] \exp\left[-2 \int_0^t \operatorname{tr} B(s) \mathrm{d}s\right] \to 0 \quad [or\ \infty] \quad as\ t \to \infty,$$

then $(*)$ *possesses a solution* $y_1(t)$ *satisfying*

$$A(t)y_1(t)\cdot y_1(t) + y_1'(t)\cdot y_1'(t) \to 0 \quad [or\ \infty] \quad as\ t\to\infty.$$

Our first result pertaining to the cases (iii) and (iv) mentioned above is proved by means of the one-dimensional version of Theorem 4.1.

Theorem 4.3. *Assume that*

$$(p(t)q(t))' \ge 0 \quad [or\ \le 0].$$

Then, for any solution $y(t)$ *of* (A), *the functions* $\frac{p(t)}{q(t)}y'(t)^2+y(t)^2$ *and* $p(t)^2y'(t)^2+p(t)q(t)y(t)^2$ *are monotone, in fact,*

$$\left(\frac{p(t)}{q(t)}y'(t)^2 + y(t)^2\right)' \le 0 \qquad [or\ \ge 0],$$
$$\big(p(t)^2y'(t)^2 + p(t)q(t)y(t)^2\big)' \ge 0 \qquad [or\ \le 0].$$

Furthermore, if

$$p(\infty)q(\infty) = \infty \quad [or\ 0]\,,$$

then (A) *possesses linearly independent solutions* $y_0(t)$ *and* $y_1(t)$ *satisfying*

$$\lim_{t\to\infty}\left[\frac{p(t)}{q(t)}y_0'(t)^2 + y_0(t)^2\right] = 0 \qquad [or\ \infty],$$
$$\lim_{t\to\infty}\big[p(t)^2y_1'(t)^2 + p(t)q(t)y_1(t)^2\big] = \infty \qquad [or\ 0]. \tag{4.1}$$

Proof. The first statement readily follows from (2.3) and (2.4) of Lemma 2.1. See (2.7) and (2.8). Let $y_0(t)$ and $y_1(t)$ be linearly independent solutions of (A). Then we see from (2.9) of Lemma 2.2 that

$$0 < C^2 \le \left(\frac{p(t)}{q(t)}y_0'(t)^2 + y_0(t)^2\right)\big(p(t)^2y_1'(t)^2 + p(t)q(t)y_1(t)^2\big)$$

for some constant C^2. Hence, if there exists a solution $y_0(t)(\not\equiv 0)$ such that

$$\lim_{t\to\infty}\left[\frac{p(t)}{q(t)}y_0'(t)^2 + y_0(t)^2\right] = 0, \tag{4.1$_0$}$$

then any solution $y_1(t)$ which is linearly independent of $y_0(t)$ satisfies

$$\lim_{t\to\infty}\big[p(t)^2y_1'(t)^2 + p(t)q(t)y_1(t)^2\big] = \infty.$$

Similarly, if there exists a solution $y_1(t)(\not\equiv 0)$ satisfying

$$\lim_{t\to\infty}\big[p(t)^2y_1'(t)^2 + p(t)q(t)y_1(t)^2\big] = 0,$$

then any solution $y_0(t)$ which is linearly independent of $y_1(t)$ satisfies

$$\lim_{t\to\infty}\left[\frac{p(t)}{q(t)}y_0'(t)^2 + y_0(t)^2\right] = \infty.$$

First we consider the case where $(p(t)q(t))' \geq 0$ and $p(\infty)q(\infty) = \infty$. We begin by transforming equation (A) into

$$y'' + \frac{p'(t)}{p(t)}y' + \frac{q(t)}{p(t)}y = 0, \tag{4.2}$$

which can be regarded as a one-dimensional version of system $(*)$ with $A(t) = q(t)/p(t)$ and $B(t) = p'(t)/p(t)$. Note that

$$A'(t) + B(t)A(t) + A(t)B(t)^T = \frac{\big(p(t)q(t)\big)'}{p(t)^2} \geq 0,$$

$$[\det\ A(t)] \exp\left[2\int_0^t \mathrm{tr}\ B(s)\mathrm{d}s\right] = \frac{q(t)}{p(t)} \exp\left[2\int_0^t \frac{p'(s)}{p(s)}\mathrm{d}s\right]$$
$$= cp(t)q(t) \to \infty$$

as $t \to \infty$ for some constant $c > 0$, and

$$y(t) \cdot y(t) + A^{-1}(t)y'(t) \cdot y'(t) = \frac{p(t)}{q(t)}y'(t)^2 + y(t)^2.$$

Application of Theorem 4.1 to (4.2) ensures the existence of a solution $y_0(t)(\not\equiv 0)$ of (A) satisfying (4.1_0).

Next we treat the case where $(p(t)q(t))' \leq 0$ and $p(\infty)q(\infty) = 0$. Here Theorem 4.1 will be applied to equation (B) defined in Section 2. Since

$$\big(Q(t)P(t)\big)' = \left(\frac{1}{p(t)q(t)}\right)' \geq 0, \qquad Q(\infty)P(\infty) = \infty,$$

by the same arguments as in the case where $(p(t)q(t))' \geq 0$ and $p(\infty)q(\infty) = \infty$, we conclude that there exists a solution $z_1(t)(\not\equiv 0)$ of (B) such that

$$\lim_{t\to\infty}\left[\frac{Q(t)}{P(t)}z_1'(t)^2 + z_1(t)^2\right] = 0.$$

From this fact, using Lemma 2.3, we deduce that there exists a solution $y_1(t)(\not\equiv 0)$ of (A) satisfying

$$\lim_{t\to\infty}\left[p(t)^2y_1'(t)^2 + p(t)q(t)y_1(t)^2\right] = 0.$$

The proof is complete. □

We are now in a position to deal with the remaining cases (i) and (ii) with the help of Hartman's Theorem 4.2.

Theorem 4.4. *Assume that*

$$p'(t) \geq 0 \qquad [or\ \leq 0], \tag{4.3}$$

$$q'(t) \leq 0 \qquad [or\ \geq 0]. \tag{4.4}$$

Then, for any solution $y(t)$ of (A), *the functions $p(t)y'(t)^2 + q(t)y(t)^2$ and $\frac{p(t)^2}{q(t)}y'(t)^2 + p(t)y(t)^2$ are monotone, in fact,*

$$\big(p(t)y'(t)^2 + q(t)y(t)^2\big)' \leq 0 \qquad [or\ \geq 0],$$

$$\left(\frac{p(t)^2}{q(t)}y'(t)^2 + p(t)y(t)^2\right)' \geq 0 \qquad [or\ \leq 0].$$

Furthermore, if

(4.5) $$p(\infty) < \infty \qquad [or\ > 0],$$

(4.6) $$q(\infty) = 0 \qquad [or\ \infty],$$

then (A) *possesses linearly independent solutions* $y_0(t)$ *and* $y_1(t)$ *satisfying*

(4.7) $$\lim_{t\to\infty} \left[p(t)y_0'(t)^2 + q(t)y_0(t)^2\right] = 0 \qquad [or\ \infty],$$

(4.8) $$\lim_{t\to\infty} \left[\frac{p(t)^2}{q(t)}y_1'(t)^2 + p(t)y_1(t)^2\right] = \infty \qquad [or\ 0].$$

Proof. The first paragraph is a consequence of (2.1) and (2.2) of Lemma 2.1. See (2.5) and (2.6). Let $y_0(t)$ and $y_1(t)$ be linearly independent solutions of (A). Using (2.9) of Lemma 2.2, we obtain

$$0 < C^2 \leq \left(p(t)y_0'(t)^2 + q(t)y_0(t)^2\right)\left(\frac{p(t)^2}{q(t)}y_1'(t)^2 + p(t)y_1(t)^2\right).$$

Hence, if there exists a solution $y_0(t)(\not\equiv 0)$ such that

(4.7_0) $$\lim_{t\to\infty} \left[p(t)y_0'(t)^2 + q(t)y_0(t)^2\right] = 0,$$

then any solution $y_1(t)$ which is linearly independent of $y_0(t)$ satisfies

(4.8_∞) $$\lim_{t\to\infty} \left[\frac{p(t)^2}{q(t)}y_1'(t)^2 + p(t)y_1(t)^2\right] = \infty.$$

Analogously, if there exists a solution $y_1(t)(\not\equiv 0)$ satisfying

(4.8_0) $$\lim_{t\to\infty} \left[\frac{p(t)^2}{q(t)}y_1'(t)^2 + p(t)y_1(t)^2\right] = 0,$$

then any solution $y_0(t)$ which is linearly independent of $y_1(t)$ satisfies

(4.7_∞) $$\lim_{t\to\infty} \left[p(t)y_0'(t)^2 + q(t)y_0(t)^2\right] = \infty.$$

First we consider the case where $p'(t) \geq 0, q'(t) \leq 0$, $p(\infty) < \infty$ and $q(\infty) = 0$. To show the existence of a solution $y_0(t)(\not\equiv 0)$ satisfying (4.7_0), we apply Theorem 4.2 to equation (4.2) which is equivalent to (A). It is easy to check that

$$A'(t) = \frac{q'(t)p(t) - q(t)p'(t)}{p(t)^2} \leq 0,$$

$$B(t) + B(t)^T = 2\frac{p'(t)}{p(t)} \geq 0,$$

$$A(t)y(t)\cdot y(t) + y'(t)\cdot y'(t) = \frac{q(t)}{p(t)}y(t)^2 + y'(t)^2,$$

(4.9) $$[\det A(t)] \exp\left[-2\int_0^t \operatorname{tr} B(s)\mathrm{d}s\right] = c\frac{q(t)}{p(t)^3}$$

for some constant $c > 0$. Taking account of

$$0 \leq c\frac{q(t)}{p(t)^3} \leq c\frac{q(t)}{p(0)^3},$$

we find that (4.9) tends to 0 as $t \to \infty$. It follows from Theorem 4.2 that there exists a solution $y_0(t)(\not\equiv 0)$ of (A) satisfying

$$\lim_{t\to\infty}\left[\frac{q(t)}{p(t)}y_0(t)^2 + y_0'(t)^2\right] = 0.$$

Since

$$\begin{aligned} 0 &\leq p(t)y_0'(t)^2 + q(t)y_0(t)^2 \\ &= p(t)\left[y_0'(t)^2 + \frac{q(t)}{p(t)}y_0(t)^2\right] \leq p(\infty)\left[y_0'(t)^2 + \frac{q(t)}{p(t)}y_0(t)^2\right], \end{aligned}$$

we conclude that

$$\lim_{t\to\infty}\left[p(t)y_0'(t)^2 + q(t)y_0(t)^2\right] = 0,$$

and therefore there exists a solution $y_0(t)(\not\equiv 0)$ satisfying (4.7_0).

Next we deal with the case where $p'(t) \leq 0$, $q'(t) \geq 0$, $p(\infty) > 0$ and $q(\infty) = \infty$. Introduce the new independent variable s defined by

$$s = \int_0^t \frac{q(\xi)}{p(\xi)}\,\mathrm{d}\xi,$$

and let

$$\mathcal{P}(s) = \frac{1}{p(t)}, \qquad \mathcal{Q}(s) = \frac{1}{q(t)}.$$

Then we see that

$$\begin{aligned} \dot{\mathcal{P}}(s) &= \left(\frac{1}{p(t)}\right)' \Big/ \frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{p'(t)}{p(t)q(t)} \geq 0, \\ \dot{\mathcal{Q}}(s) &= \left(\frac{1}{q(t)}\right)' \Big/ \frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{p(t)q'(t)}{q(t)^3} \leq 0, \end{aligned}$$

and that

$$\mathcal{P}(\infty) = \frac{1}{p(\infty)} < \infty, \qquad \mathcal{Q}(\infty) = \frac{1}{q(\infty)} = 0.$$

We note that $\int_0^\infty \frac{q(\xi)}{p(\xi)}\,d\xi = \infty$ is satisfied in view of $q(\xi)/p(\xi) \geq q(0)/p(0) > 0$. Applying Theorem 4.2 to equation (C), we observe that there exists a solution $w_1(s)(\not\equiv 0)$ such that

$$\lim_{s\to\infty}\left[\mathcal{P}(s)\dot{w}_1(s)^2 + \mathcal{Q}(s)w_1(s)^2\right] = 0.$$

Using Lemma 2.4, we deduce that there exists a solution $y_1(t)(\not\equiv 0)$ of (A) satisfying

$$\lim_{t\to\infty}\left[\frac{p(t)^2}{q(t)}y_1'(t)^2 + p(t)y_1(t)^2\right] = 0,$$

which is (4.8_0). The proof is complete. □

The next theorem shows that the hypotheses (4.5), (4.6) can be replaced by the following “dual” limit conditions (4.10), (4.11).

Theorem 4.5. *Assume that* (4.3) *and* (4.4) *hold. If*

$$p(\infty) = \infty \qquad [or\ 0], \tag{4.10}$$

$$q(\infty) > 0 \qquad [or\ < \infty], \tag{4.11}$$

then (A) *possesses linearly independent solutions* $y_0(t)$ *and* $y_1(t)$ *satisfying* (4.7), (4.8).

Proof. First we deal with the case where $p'(t) \geq 0, q'(t) \leq 0$, $p(\infty) = \infty$ and $q(\infty) > 0$. Let $Q(t) = 1/q(t)$, $P(t) = 1/p(t)$. Since

$$Q'(t) \geq 0, \qquad P'(t) \leq 0, \qquad Q(\infty) < \infty, \qquad P(\infty) = 0,$$

Theorem 4.4 applies to equation (B) implying that the functions $Q(t)z'(t)^2 + P(t)z(t)^2$ and $\frac{Q(t)^2}{P(t)}z'(t)^2 + Q(t)z(t)^2$ are decreasing and increasing, respectively, for any solution $z(t)$ of (B), and that there exist linearly independent solutions $z_0(t)$ ($\not\equiv 0$) and $z_1(t)$ ($\not\equiv 0$) of equation (B) satisfying

$$\lim_{t\to\infty} \left[Q(t)z_0'(t)^2 + P(t)z_0(t)^2\right] = 0, \qquad \lim_{t\to\infty} \left[\frac{Q(t)^2}{P(t)} z_1'(t)^2 + Q(t)z_1(t)^2\right] = \infty,$$

respectively. We are now able to apply Lemma 2.3 concluding that equation (A) does possess linearly independent solutions $y_0(t)(\not\equiv 0)$ and $y_1(t)(\not\equiv 0)$ satisfying

$$\lim_{t\to\infty} \left[p(t)y_0'(t)^2 + q(t)y_0(t)^2\right] = 0, \qquad \lim_{t\to\infty} \left[\frac{p(t)^2}{q(t)} y_1'(t)^2 + p(t)y_1(t)^2\right] = \infty,$$

respectively. Next we consider the case where $p'(t) \leq 0$, $q'(t) \geq 0$, $p(\infty) = 0$ and $q(\infty) < \infty$. Since

$$Q'(t) \leq 0, \qquad P'(t) \geq 0, \qquad Q(\infty) > 0, \qquad P(\infty) = \infty,$$

from Theorem 4.4 applied to (B), it follows that there exist linearly independent solutions $z_0(t)(\not\equiv 0)$ and $z_1(t)(\not\equiv 0)$ of equation (B) satisfying

$$\lim_{t\to\infty} \left[\frac{Q(t)^2}{P(t)} z_0'(t)^2 + Q(t)z_0(t)^2\right] = 0, \qquad \lim_{t\to\infty} \left[Q(t)z_1'(t)^2 + P(t)z_1(t)^2\right] = \infty,$$

respectively. This fact combined with Lemma 2.3 then leads to the conclusion that (A) possesses linearly independent solutions $y_0(t)(\not\equiv 0)$ and $y_1(t)(\not\equiv 0)$ such that

$$\lim_{t\to\infty} \left[\frac{p(t)^2}{q(t)} y_0'(t)^2 + p(t)y_0(t)^2\right] = 0, \qquad \lim_{t\to\infty} \left[p(t)y_1'(t)^2 + q(t)y_1(t)^2\right] = \infty,$$

respectively. This completes the proof. □

The following theorem is a direct consequence of Theorem 4.2.

Theorem 4.6. *Assume that* (4.3) *and* (4.4) *hold. Then, for any solution* $y(t)$ *of* (A), *the function* $\frac{q(t)}{p(t)}y(t)^2 + y'(t)^2$ *is monotone, in fact,*

$$\left(\frac{q(t)}{p(t)} y(t)^2 + y'(t)^2\right)' \leq 0 \qquad [or\ \geq 0].$$

Furthermore, if

$$p(\infty) = \infty \qquad [\mathit{or}\ 0],$$
$$q(\infty) = 0 \qquad [\mathit{or}\ \infty],$$

then (A) *possesses a solution* $y(t)$ *satisfying*

$$\lim_{t\to\infty}\left[\frac{q(t)}{p(t)}y(t)^2 + y'(t)^2\right] = 0 \quad [\mathit{or}\ \infty].$$

Proof. The conclusion follows immediately from the one-dimensional version of Theorem 4.2 applied to equation (4.2). A crucial role is played by (4.9). □

It is natural to ask whether there really exist small or large oscillatory solutions of equation (A) among the solutions described in the above theorems. The answer is in the affirmative as long as Theorems 4.3, 4.4 and 4.5 are concerned.

Corollary 4.1. *Let equation* (A) *be oscillatory. If* $(p(t)q(t))' \geq 0$ [*or* ≤ 0] *and* $p(\infty)q(\infty) = \infty$ [*or* 0], *then there exists a small* [*or large*] *oscillatory solution of* (A).

Proof. Let $(p(t)q(t))' \geq 0$ and $p(\infty)q(\infty) = \infty$. It follows from Theorem 4.3 that there exists an oscillatory solution $y_0(t)$ of (A) such that

$$\lim_{t\to\infty}\left[\frac{p(t)}{q(t)}y_0'(t)^2 + y_0(t)^2\right] = 0.$$

This clearly implies that $\lim_{t\to\infty} y_0(t)^2 = 0$, i.e., $\lim_{t\to\infty} y_0(t) = 0$, which means that $y_0(t)$ is a small oscillatory solution of (A). On the other hand, if $(p(t)q(t))' \leq 0$ and $p(\infty)q(\infty) = 0$, then by Theorem 4.3 there exists an oscillatory solution $y_0(t)$ of (A) such that

$$\lim_{t\to\infty}\left[\frac{p(t)}{q(t)}y_0'(t)^2 + y_0(t)^2\right] = \infty,$$

which means that $\lim_{k\to\infty} |y_0(\tau_k)| = \infty$, where $\{\tau_k\}_{k=1}^{\infty}$ denotes the sequence of points at which $y_0(t)$ takes on extrema. This shows that $y_0(t)$ is a large oscillatory solution of (A). □

Corollary 4.2. *Let equation* (A) *be oscillatory. Assume that* $p'(t) \geq 0$ *and* $q'(t) \leq 0$. *If* $p(\infty) = \infty$ [*or* $< \infty$] *and* $q(\infty) > 0$ [*or* $= 0$], *then there exists a small* [*or large*] *oscillatory solution of* (A).

Proof. In case $p(\infty) = \infty$ and $q(\infty) > 0$, we see from Theorem 4.5 that there exists a solution $y_0(t)$ of (A) satisfying (4.7_0). Since

$$p(t)y_0'(t)^2 + q(t)y_0(t)^2 \geq q(\infty)y_0(t)^2 \geq 0,$$

we observe that $\lim_{t\to\infty} y_0(t)^2 = 0$, and hence, $\lim_{t\to\infty} y_0(t) = 0$. Therefore, there exists a small oscillatory solution $y_0(t)$ of (A). Next let $p(\infty) < \infty$ and $q(\infty) = 0$. Theorem 4.4 implies that there exists a solution $y_1(t)$ of (A) satisfying (4.8_∞). Since $p(t) \leq p(\infty)$, we find that

$$\lim_{t\to\infty}\left[\frac{p(t)^2}{q(t)}y_1'(t)^2 + p(\infty)y_1(t)^2\right] = \infty.$$

As in the proof of Corollary 4.1, we conclude that the solution $y_1(t)$ is a large oscillatory solution of (A). □

Corollary 4.3. *Let equation* (A) *be oscillatory. Assume that* $p'(t) \leq 0$ *and* $q'(t) \geq 0$. *If* $p(\infty) > 0$ [*or* $= 0$] *and* $q(\infty) = \infty$ [*or* $< \infty$], *then there exists a small* [*or large*] *oscillatory solution of* (A).

Proof. If $p(\infty) > 0$ and $q(\infty) = \infty$, then it follows from Theorem 4.4 that there exists a solution $y_1(t)$ of (A) satisfying (4.8_0). Since $p(t) \geq p(\infty)$, we obtain

$$\lim_{t\to\infty} p(\infty) y_1(t)^2 = 0,$$

which implies that $y_1(t)$ is a small oscillatory solution of (A). In the case where $p(\infty) = 0$ and $q(\infty) < \infty$, Theorem 4.5 means that there exists a solution $y_0(t)$ of (A) satisfying (4.7_∞). Since $q(t) \leq q(\infty)$, it can be shown that

$$\lim_{t\to\infty} \left[p(t) y_0'(t)^2 + q(\infty) y_0(t)^2\right] = \infty.$$

Arguing as in the proof of Corollary 4.1, we deduce that the solution $y_0(t)$ is a large oscillatory solution of (A). □

Remark 4.1. It remains to consider equation (A) in which $p(t)$ and $q(t)$ satisfy $p'(t)q'(t) \leq 0$ and $\{p(\infty) = \infty,\ q(\infty) = 0\}$ or $\{p(\infty) = 0,\ q(\infty) = \infty\}$. There seems to be no effective criteria for the presence of small or large oscillatory solutions for such equations, since the pair of functions $\{p(t), q(t)\}$ can be chosen so that (A) possesses any one of the three types of oscillatory solutions, moderately bounded, small or large. To observe that such a situation occurs consider the oscillatory equation

$$\left(\mathrm{e}^{-\alpha t}\, y'\right)' + k\,\mathrm{e}^{\beta t}\, y = 0, \tag{4.12}$$

where k, α and β are positive constants. Here $p(t) = \mathrm{e}^{-\alpha t}$ and $q(t) = k\,\mathrm{e}^{\beta t}$. Since $p(t)q(t) = k\,\mathrm{e}^{(\beta-\alpha)t}$, we are able to apply Corollary 4.1 to (4.12) and conclude that it has a small or large oscillatory solution according as $\beta > \alpha$ or $\beta < \alpha$. If $\beta = \alpha$, then (4.12) has linearly independent moderately bounded oscillatory solutions $\left(\cos(\sqrt{k}\,\mathrm{e}^{\alpha t}/\alpha), \sin(\sqrt{k}\,\mathrm{e}^{\alpha t}/\alpha)\right)$.

Example 4.1. Consider the equation

$$\left(\mathrm{e}^{\alpha t}\, y'\right)' + k\,\mathrm{e}^{\beta t}\, y = 0, \tag{4.13}$$

where α, β and k are positive constants. Assume that $\beta \geq 3\alpha$. Here $p(t) = \mathrm{e}^{\alpha t}$ and $q(t) = k\,\mathrm{e}^{\beta t}$. Observe that (4.13) is oscillatory via (1.2) with $\mu = 2$. Since $(p(t)q(t))' \geq 0$ and $p(\infty)q(\infty) = \infty$, it follows from Corollary 4.1 that there exists a small oscillatory solution of (4.13). If in particular $\beta = 3\alpha$ and $k = \alpha^2$, then this equation possesses linearly independent small oscillatory solutions $\mathrm{e}^{-\alpha t}\cos \mathrm{e}^{\alpha t}$, $\mathrm{e}^{-\alpha t}\sin \mathrm{e}^{\alpha t}$.

Analogously, it is shown that the equation

$$\left((t+a)^\alpha y'\right)' + k(t+a)^\beta y = 0, \tag{4.14}$$

a, k, α and β being positive constants such that $\alpha \geq 3/2$ and $\beta \geq 2\alpha - 3$, is oscillatory and possesses a small oscillatory solution by virtue of Corollary 4.1. It is easy to see that if in particular, $\beta = 3\alpha - 4$ and $k = (\alpha - 1)^4$, then the linearly independent functions

$$(t+a)^{1-\alpha}\cos\big((\alpha-1)(t+a)^{\alpha-1}\big), \quad (t+a)^{1-\alpha}\sin\big((\alpha-1)(t+a)^{\alpha-1}\big)$$

are small oscillatory solutions of (4.14).

Example 4.2. Consider the equation

$$\big((t+a)^{-\alpha}y'\big)' + k(t+a)^{-\beta}y = 0, \tag{4.15}$$

where a, k, α and β are positive constants. This equation is a special case of (A) with $p(t) = (t+a)^{-\alpha}$ and $q(t) = k(t+a)^{-\beta}$, and is shown to be oscillatory if $\beta \leq \alpha + 2$ and $k > (\alpha+1)^2/4$. Since $(p(t)q(t))' \leq 0$ and $p(\infty)q(\infty) = 0$, Corollary 4.1 ensures the existence of a large oscillatory solution of (4.15). If in particular, $\beta = \alpha + 2$ and $k = (\alpha+1)^2$, then (4.15) possesses linearly independent large oscillatory solutions

$$(t+a)^{\frac{\alpha+1}{2}}\cos\frac{\sqrt{3}}{2}\log\frac{(t+a)^{\alpha+1}}{\alpha+1}, \qquad (t+a)^{\frac{\alpha+1}{2}}\sin\frac{\sqrt{3}}{2}\log\frac{(t+a)^{\alpha+1}}{\alpha+1}.$$

Similarly, it can be shown that the equation

$$\big(\mathrm{e}^{-\alpha t}\,y'\big)' + k\,\mathrm{e}^{-\beta t}\,y = 0,$$

k, α and β being positive constants, is oscillatory if $\beta \leq 2\alpha$ and $k > \alpha^2/4$ and has a large oscillatory solution by way of Corollary 4.1. It is elementary to see that in the special case $\alpha = \beta$ this equation has two large oscillatory solutions

$$\mathrm{e}^{\frac{\alpha}{2}t}\cos\frac{\sqrt{4k-\alpha^2}}{2}t, \quad \mathrm{e}^{\frac{\alpha}{2}t}\sin\frac{\sqrt{4k-\alpha^2}}{2}t.$$

Example 4.3. Consider the equations

$$\big(\cosh(t+a)\cdot y'\big)' + k\coth(t+a)\cdot y = 0, \tag{4.16}$$

$$\Big(\frac{((t+a)^2-1)^{\frac{1}{2}}}{t+a}y'\Big)' + \frac{k(t+a)}{((t+a)^2-1)^{\frac{3}{2}}}y = 0, \tag{4.17}$$

where $a > 1$ and k are positive constants. It is easy to check that these equations are oscillatory and satisfy the hypotheses of Corollary 4.2. It follows that (4.16) has a small oscillatory solution while (4.17) has a large oscillatory solution. Note that if $k = 1/2$ (4.17), possesses two exact unbounded oscillatory solutions

$$((t+a)^2-1)^{\frac{1}{4}}\cos\frac{1}{4}\log((t+a)^2-1), \qquad ((t+a)^2-1)^{\frac{1}{4}}\sin\frac{1}{4}\log((t+a)^2-1).$$

Example 4.4. Corollary 4.3 is applicable to the equations

$$(\coth(t+a)\cdot y')' + k\cosh(t+a)\cdot y = 0, \tag{4.18}$$

$$\Big(\frac{t+a}{((t+a)^2-1)^{\frac{3}{2}}}y'\Big)' + k\frac{((t+a)^2-1)^{\frac{1}{2}}}{t+a}y = 0, \tag{4.19}$$

where $a > 1$ and k are positive constants. It is concluded that (4.18) has a small oscillatory solution while (4.19) has a large oscillatory solution.

It is well-known that Bessel's differential equation of order $\nu \in \mathbf{R}$, which can be expressed as

$$\big(ty'\big)' + \Big(t - \frac{\nu^2}{t}\Big)y = 0,$$

has two small oscillatory solutions $J_\nu(t)$ and $Y_\nu(t)$ termed, respectively, the Bessel functions of the first and second kind of order ν. Taking this fact into account together with some equations appearing in Examples 4.1–4.3, one would expect that under the specified hypotheses on $p(t)$ and $q(t)$ in Corollaries 4.1–4.3, equation (A) possesses two linearly independent solutions, small or large, with similar asymptotic behavior as $t \to \infty$.

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T. Kusano, Department of Mathematics, Faculty of Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan, *e-mail*: `ksntksjm4@gmail.com`

N. Yoshida, Department of Mathematics, Faculty of Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan, *e-mail*: `norio.yoshidajp@gmail.com`