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L-DUAL OF RANDERS CHANGE OF MATSUMOTO METRIC

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Abstract. The study of L-duality of Lagrange and Finsler spaces was initiated by R. Miron in 1987. Some of the remarkable results obtained are the concrete L-duals of Randers and Kropina metrics. However, the importance of L-duality is by far limited to computing the dual of some Finsler fundamental functions. In this paper, we study L-dual of Randers change of Matsumoto metric.

1. Introduction

The (α, β)-metrics form an important class of Finsler metrics appearing iteratively in formulating Physics, Mechanics and Seismology, Biology, Control Theory, etc. This class of metrics was first introduced by Matsumoto [**11**]. An (α, β)-metric is a Finsler metric of the form $F := \alpha\phi\left(\frac{\beta}{\alpha}\right)$, where $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. The Randers and Matsumoto metrics are special and significant (α, β)-metrics which constitute a majority of actual research. The Matsumoto and Randers metrics are defined by $\phi(s) = \frac{1}{1-s}$ and $\phi(s) = 1 + s$, respectively. The Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$ is an important metric in Finsler geometry which is the Matsumotos slope-of-a-mountain metric. In the Matsumoto metric, the 1-form $\beta = b_i(x)y^i$ was originally induced by earth gravity. Hence, we could regard as the infinitesimals. This metric was introduced by Matsumoto as a realization of Finslers idea of "a slope measure of a mountain with respect to a time measure" (see [**10**]). In [**14**], Miron studied the L-duality of Lagrange and Finsler spaces. In this paper, the authors study L-dual of Randers change of Matsumoto metric and find the fundamental function of two major cases.

2. The Legendre Transformation

Let $F^n = (M, F)$ be an n-dimensional Finsler space. The fundamental function $F(x, y)$ is called an (α, β)-metric if F is a homogeneous function of α and β of degree one, where $\alpha^2 = a(y, y) = a_{ij}y^iy^j$, $y = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $\widetilde{TM} = TM \setminus \{0\}$.

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A Finsler space with fundamental function

$$F(x,y)=\alpha(x,y)+\beta(x,y), \tag{2.1}$$

is called a Randers space, whereas the space having the fundamental function

$$F(x,y)=\frac{\alpha^2(x,y)}{\beta(x,y)}, \tag{2.2}$$

is called a Kropina space.

A Finsler space with fundamental function

$$F(x,y)=\frac{\alpha^2(x,y)}{\alpha(x,y)-\beta(x,y)}, \tag{2.3}$$

is called a Matsumoto space.

The generalized metrics

$$F(x,y)=\frac{\alpha^{m+1}(x,y)}{\beta^m(x,y)}, \qquad (m\neq 0,-1), \tag{2.4}$$

and

$$F(x,y)=\frac{\alpha^{m+1}(x,y)}{(\alpha(x,y)-\beta(x,y))^m}, \qquad (m\neq 0,-1), \tag{2.5}$$

are called generalized Kropina and Matsumoto metrics, respectively, and the spaces equipped with the corresponding metrics are called generalized m-Kropina and generalized Matsumoto spaces, respectively.

Definition 2.1. A Cartan space C^n is a pair (M,H) which consists of a real n-dimensional C^∞-manifold M and a Hamiltonian function $H\colon T^x M\smallsetminus\{0\}\to\Re$, where $(T^m M,\pi^x,M)$ is the cotangent bundle of M such that $H(x,p)$ has the following properties:

1. It is two homogeneous with respect to p_i, $\quad(i,j,k=1,2,\dots,n)$.
2. The tensor field $g^{ij}(x,p)=\frac{1}{2}\frac{\partial^2 H}{\partial p_i\partial p_j}$ is nondegenerate.

Let $C^n=(M,K)$ be an n-dimensional Cartan space having the fundamental function $K(x,p)$. We also consider Cartan spaces having the metric function of the following forms (see [**5**]):

$$K(x,p)=\sqrt{a^{ij}(x)p_ip_j}+b^i(x)p_i, \tag{2.6}$$

or

$$K(x,p)=\frac{a^{ij}p_ip_j}{b^ip_i}, \tag{2.7}$$

or

$$K(x,p)=\frac{a^{ij}p_ip_j}{\sqrt{a^{ij}(x)p_ip_j}-b^i(x)p_i}, \tag{2.8}$$

with $a_{ij}a^{jk}=\delta_i^k$. We will again call these spaces Randers, Kropina and Matsumoto spaces, respectively, on the cotangent bundle T^*M.

Definition 2.2. A regular Lagrangian (Hamiltonian) on a domain $D \subset TM$ ($D^* \subset T^*M$) is a real smooth function $L\colon D \to \Re$ ($H\colon D^* \to \Re$) such that the matrix with entries

$$g_{ab}(x,y) = \dot{\partial}_a \dot{\partial}_b L(x,y) \qquad (g^{*ab}(x,y) = \dot{\partial}^a \dot{\partial}^b H(x,y))$$

is everywhere nondegenerate on $D(D^*)$, (see [**5**]).

A Lagrange (Hamilton) manifold is a pair $(M, L(H))$, where M is a smooth manifold and $L(H)$ is regular Lagrangian (Hamiltonian) on $D(D^*)$.

Example 1.

(a) Every Finsler space $F^n = (M, F(x,y))$ is a Lagrange manifold with $L = \frac{1}{2}F^2$.

(b) Every Cartan space $C^n = (M, \bar{F}(x,p))$ is a Hamilton manifold with $H = \frac{1}{2}\bar{F}^2$. (Here $\bar{F}$ is positively 1-homogeneous in p_i and the tensor $\bar{g}^{ab} = \frac{1}{2}\dot{\partial}_a \dot{\partial}_b \bar{F}^2$ is nondegenerate).

(c) (M, L) and (M, H) with

$$L(x,y) = \frac{1}{2} a_{ij}(x) y^i y^j + b_i(x) y^i + c(x)$$

and

$$H(x,y) = \frac{1}{2} \bar{a}^{ij}(x) p_i p_j + \bar{b}^i(x) p_i + \bar{c}(x)$$

are Lagrange and Hamilton manifolds, respectively. (Here $a_{ij}, \bar{a}^{ij}$ are the fundamental tensors of Riemannian manifold, b_i are components of covector field, $\bar{b}^i$ are the components of a vector fields, C and $\bar{C}$ are the smooth functions on M).

Let $L(x,y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x,p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$. If L is a differential map, we can consider the fiber derivative of L, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$ (see [**13, 14**])

$$\varphi(x,y) = (x^i, \dot{\partial}_a L(x,y)), \tag{2.9}$$

which is called the Legendre transformation. In this case, we can define the function $H\colon U^* \mapsto R$

$$H(x,y) = p_a y^a - L(x,y), \tag{2.10}$$

where $y = y^a$ is the solution of the equation

$$p_a = \dot{\partial}_a L(x,y). \tag{2.11}$$

In the same manner, the fiber derivative of H is locally given by

$$\varphi(x,p) = (x^i, \dot{\partial}^a H(x,p)), \tag{2.12}$$

where φ is a diffeomorphism between the same open sets $U \subset D$ and $U^* \subset D^*$. We can consider the function $L\colon U \mapsto R$

$$L(x,y) = p_a y^a - H(x,p), \tag{2.13}$$

where $p = (p_a)$ is the solution of the equation

$$y^a = \dot{\partial}^a H(x,p). \tag{2.14}$$

The Hamiltonian given by (2.10) is the Legendre transformation of the Lagrangian L and the Lagrangian given by (2.13) is called the Legendre transformation of the Hamiltonian H.

If (M,K) is a Cartan space, then (M,H) is a Hamiltonian manifold (see [**13, 14**]), where $H(x,p) = \frac{1}{2}K^2(x,p)$ is 2-homogenous on a domain of T^*M. So we get the following transformation of H on U

$$L(x,y) = p_a y^a - H(x,p) = H(x,p). \tag{2.15}$$

Theorem 2.1. *The scalar field $L(x,y)$ defined by (2.16) is a positively 2-homogeneous regular Lagrangian on U. Therefore, we get Finsler metric F of U, so that*

$$L = \frac{1}{2}F^2. \tag{2.16}$$

Thus for the Cartan space (M,K) we always can locally associate a Finsler space (M,F) which will be called the L-dual of a Cartan space $(M,C_{|U^*})$, and vice versa, we can associate locally a Cartan space to every Finsler space which will be called the L-dual of a Finsler space $(M,F_{|U})$.

3. The L-Dual of Randers change of Matsumoto Space

In this section, we consider the special (α,β)-metric $F = \frac{\alpha^2}{\alpha-\beta} + \beta$, we put $\alpha^2 = y_i y^i$, $\beta = b_i y^i$, $\beta^* = b^i p_i$, $p^i = a^{ij}p_j$, $\alpha^{*2} = p_i p^i = a^{ij}p_i p_j$. We have

$$p_i = \frac{1}{2}\dot{\partial}^i F^2 = F\left[\frac{2Fy^i}{\alpha^2} - \frac{F}{\alpha(\alpha-\beta)}(y^i - b_i\alpha)\right]. \tag{3.1}$$

Contracting (3.1) by p^i and b^i, respectively, we get

$$\alpha^{*2} = F\left[\frac{2F^3}{\alpha^2} - \frac{F}{\alpha(\alpha-\beta)}(F^2 - \alpha\beta^*) + \beta^*\right], \tag{3.2}$$

$$\beta^* = F\left[\frac{2F\beta}{\alpha^2} - \frac{F}{\alpha(\alpha-\beta)}(\beta - b^2\alpha) + b^2\right]. \tag{3.3}$$

In [**17**], for a Finsler (α,β)-metric F on a manifold M, there is a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$ with $\|\beta\|_x < b_0$ for all $x \in M$. ϕ satisfies $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, $(|s| \le b_0)$.

A Randers change of Matsumoto metric is a special (α,β)-metric with $\phi = [1 + s - \frac{1}{s}]$. Using Shen's notation [**19**] $s = \frac{\beta}{\alpha}$, (3.1) and (3.3) become

$$\alpha^{*2} = F\left[\frac{2F}{(1-s)^2} - \frac{1}{(1-s)^3} + \frac{\beta^*}{(1-s)^2} + \beta^*\right] \tag{3.4}$$

and

$$\beta^* = F\left[\frac{2s}{(1-s)} - \frac{1}{(1-s)^2}(s-b^2) + b^2\right]. \tag{3.5}$$

Putting $(1-s) = t$, so that $s = (1-t)$ in (3.4) and (3.5), we get

$$\alpha^{*2} = F\left[\frac{2F^2}{t^2} - \frac{F^2}{t^3} + F(1+\frac{1}{t^2})\beta^*\right] \tag{3.6}$$

and

$$\beta^* = F\left[\frac{2(1-t)}{t} - \frac{1}{t^2}(1-t-b^2) + b^2\right]. \tag{3.7}$$

Now, we have following two cases:

Case I. For $b^2 = 1$ from (3.7), we get

$$F = \left[\frac{\beta^* t}{(3-t)}\right]. \tag{3.8}$$

From (3.6) and (3.8), we get

$$(K-1)s^3 + 3Ks^2 + 4s - (4K+5) = 0, \tag{3.9}$$

where $K = \frac{\alpha^{*2}}{\beta^{*2}}$.

Solving (3.9) for s and using maple, we get

$$s = \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \frac{A}{3}, \tag{3.10}$$

where

$$p = \left[\frac{A^2 - 2A^3 + B}{3}\right], \qquad q = \left[\frac{3C - AB}{3}\right] \tag{3.11}$$

and

$$A = \frac{3K}{(k-1)}, \qquad B = \frac{4}{(K-1)}, \qquad C = \frac{4K+5}{1-K}. \tag{3.12}$$

From (3.8), we get

$$F = \frac{\beta^*\left[1 - \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \frac{A}{3}\right]}{2 + \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \frac{A}{3}}. \tag{3.13}$$

From (2.15) and (2.16), we get

$$H(x,p) = \frac{\frac{\beta^{*2}}{2}\left[1 - \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \frac{A}{3}\right]^2}{\left[2 + \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \frac{A}{3}\right]^2}. \tag{3.14}$$

Hence we have the following theorem.

Theorem 3.1. *Let (M,F) be a Randers change of Matsumoto space and $b=(a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then if $b^2=1$, the L-dual of (M,F) is the space having the fundamental function*

$$H(x,p)=\frac{\frac{\beta^{*2}}{2}\left[1-\left[\frac{-q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]^{\frac{1}{3}}-\left[\frac{-q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]^{\frac{1}{3}}+\frac{A}{3}\right]^2}{\left[2+\left[\frac{-q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]^{\frac{1}{3}}+\left[\frac{-q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]^{\frac{1}{3}}-\frac{A}{3}\right]^2}. \tag{3.15}$$

Case II. For $b^2\neq 1$, from (3.7), we get

$$F=\left[\frac{\beta^* t^2}{(b^2-2)t^2+3t+(b^2-1)}\right]. \tag{3.16}$$

From (3.6) and (3.16), we get

$$s^4+A_1s^3+A_2s^2+A_3s+A_4=0, \tag{3.17}$$

where

$$\begin{aligned}
A_1&=\frac{1}{A}(4Kb^4-10Kb^2-4b^2+4K+6),\\
A_2&=\frac{1}{A}(-8Kb^4+12Kb^2+8b^2-K-7),\\
A_3&=\frac{1}{A}(8Kb^4-4Kb^2-8b^2+1),\\
A_4&=\frac{1}{A}(-4Kb^4+4b^2+1),\\
A&=(-Kb^4+4Kb^2+b^2-4K-2).
\end{aligned}$$

Using maple, after long computations solving (3.17) for s, we get

$$s=\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1}-\frac{A_1}{4}\right], \tag{3.18}$$

where

$$\begin{aligned}
&H_1=B_1+2C, \qquad H_2=-B_2, \qquad H_3=B_1^2-B_3+2B_1C+C^2,\\
&B_1=\frac{-3A_1^2}{8}+A_2, \qquad B_2=\frac{A_1^3}{8}-\frac{A_1A_2}{8}+A_3,\\
&B_3=\frac{-3A_1^4}{256}-\frac{A_3A_1}{4}+\frac{A_1^2A_2}{16}+A_4\\
&C=\left(\frac{-P_2}{2}+\sqrt{\frac{P_2^2}{4}+\frac{P_1^3}{27}}\right)^{1/3}+\left(\frac{-P_2}{2}-\sqrt{\frac{P_2^2}{4}+\frac{P_1^3}{27}}\right)^{1/3}-\frac{D_1}{3},\\
&P_1=\left[\frac{D_1^2}{3}-\frac{2D_1^3}{3}+D_2\right], \qquad P_2=\left[D_3-\frac{D_1D_2}{3}\right],
\end{aligned}$$

and

$$D_1 = \frac{5}{2}B_1, \qquad D_2 = 2B_1^2 - B_3, \qquad D_3 = \frac{4B_1^3 - B_2^2 - 4B_1B_3}{8}.$$

From (3.16), we get

(3.19)

$$F = \frac{\beta^* \left[1 - \frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^2}{(b^2-2)\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^2 + 3\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right] + (b^2-1)}.$$

From (2.15) and (2.16), we get

(3.20)

$$H(x,p) = \frac{\frac{\beta^{*2}}{2}\left[1 - \frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^4}{\left[(b^2-2)\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^2 + 3\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right] + (b^2-1)\right]^2}.$$

Hence we have the following theorem.

Theorem 3.2. *Let (M,F) be a Randers change of Matsumoto space and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then if $b^2 \neq 1$, the L-dual of (M,F) is the space having the fundamental function*

(3.21)

$$H(x,p) = \frac{\frac{\beta^{*2}}{2}\left[1 - \frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^4}{\left[(b^2-2)\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^2 + 3\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right] + (b^2-1)\right]^2}.$$

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