Acta Math. Univ. Comenianae
Vol. LXXXV, 1 (2016), pp. 1–7

FACTORISABLE MONOID OF GENERALIZED HYPERSUBSTITUTIONS OF TYPE $\tau = (n)$

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Abstract. A generalized hypersubstitution of type τ maps any operation symbols to the set of all terms. The extensions of generalized hypersubstitutions are mappings on the set of all terms. The set of all such generalized hypersubstitutions forms a monoid. In this paper, we determine the set of all unit-regular elements of this monoid of type $\tau = (n)$. We also conclude a submonoid of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ which is factorisable.

1. Introduction

The concept of generalized hypersubstitutions was introduced by S. Leeratanavalee and K. Denecke [**6**]. It is a convenient method to describe the considered tree transformations. A sequence of tree transformations can be described by products of generalized hypersubstitutions. In 2002, K. Denecke and S. Leeratanavalee used extensions of generalized hypersubstitutions to define tree transformations and describe algebraic properties of sets of tree transformations by algebraic properties of the set of all generalized hypersubstitutions [**5**]. In this paper, we study a monoid of generalized hypersubstitutions of type $\tau = (n)$. In 2015, A. Boonmee and S. Leeratanavalee [**2**] characterized all unit elements of this monoid of type $\tau = (n)$. The set of all regular elements of this monoid of type $\tau = (n)$ was studied by W. Puninagool and S. Leeratanavalee [**7**]. We used the concepts of unit elements and regular elements as tools to determine the set of all unit-regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$.

We recall first the concept of the monoid of all generalized hypersubstitutions of type $\tau = (n)$. Let $\{f_i : i \in I\}$ be an indexed set of operation symbols of type τ, where f_i is n_i-ary, $n_i \in \mathbb{N}$. Let $W_\tau(X)$ be the set of all terms of type τ built up by operation symbols from $\{f_i : i \in I\}$ and variables from $X := \{x_1, x_2, x_3, \dots\}$. A generalized hypersubstitution is a mapping σ which maps each n_i-ary operation symbol of type τ to a term of this type which does not necessarily preserve the arity. To define the extension $\widehat{\sigma}$ of σ, we define inducively the concept of generalized superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ as follows:

Received June 26, 2013; revised August 7, 2015.
2010 *Mathematics Subject Classification.* Primary 20B30, 20M05, 20M17.
Key words and phrases. generalized hypersubstitution; unit-regular element.

(i) If $t = x_j$, $1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.
(ii) If $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.
(iii) If $t = f_i(s_1, s_2, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Then we extend every generalized hypersubstitution σ to a mapping $\widehat{\sigma}\colon W_\tau(X) \to W_\tau(X)$ defined as follows:
(i) $\widehat{\sigma}[x] := x \in X$,
(ii) $\widehat{\sigma}[f_i(t_1, t_2, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ for any n_i-ary operation symbol f_i supposed that $\widehat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Let $\mathrm{Hyp}_G(\tau)$ be the set of all generalized hypersubstitutions of type τ. We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ for every $\sigma_1, \sigma_2 \in \mathrm{Hyp}_G(\tau)$, where $\circ$ denotes the usual composition of mappings. Let σ_{id} be a hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, \ldots, x_{n_i})$. In [**6**], S. Leeratanavalee and K. Denecke proved the following.

For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$, we have:
(i) $S^n(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) = \widehat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,
(ii) $(\widehat{\sigma}_1 \circ \sigma_2)\widehat{} = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$.

Then $\underline{\mathrm{Hyp}_G(\tau)} = (\mathrm{Hyp}_G(\tau), \circ_G, \sigma_{id})$ is a monoid and the set of all hypersubstitutions of type τ forms a submonoid of $\underline{\mathrm{Hyp}_G(\tau)}$.

2. Factorisable Monoid of Generalized Hypersubstitutions of Type $\tau = (n)$

From now on, we fix a type $\tau = (n)$. That means we have only one n-ary operation, say f. To factorize monoid of generalized hypersubstitutions of type $\tau = (n)$, we introduce some notations which will be used throughout this paper.

Let $t \in W_{(n)}(X)$, we denote

$\sigma_t :=$ the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,

$\mathrm{var}(t) :=$ the set of all variables occurring in the term t,

$\mathrm{vb}^t(x) :=$ number of occurrences of a variable x in t.

Then we introduce the following definitions.

Definition 2.1. Let $t \in W_{(n)}(X)$, a *subterm* of t, is defined inductively by the following.
(i) Every variable $x \in \mathrm{var}(t)$ is a subterm of t.
(ii) If $t = f(t_1, \ldots, t_n)$, then t itself, $t_1, \ldots, t_n$, and all subterms of $t_i, 1 \leq i \leq n$, are subterms of t.

We denote the set of all subterms of t by $\mathrm{sub}(t)$.

Definition 2.2. Let $t \in W_{(n)}(X) \setminus X$, where $t = f(t_1, \ldots, t_n)$ for some $t_1, \ldots, t_n \in W_{(n)}(X)$. For each $s \in \mathrm{sub}(t)$, $s \neq t$, a set $\mathrm{seq}^t(s)$ of sequences of s in t, is defined by

$$seq^t(s) = \{(i_1, \ldots, i_m) : m \in \mathbb{N} \text{ and } s = \pi_{i_m} \circ \cdots \circ \pi_{i_1}(t)\},$$

where $\pi_{i_l} : W_{(n)}(X) \setminus X \to W_{(n)}(X)$ by the formula $\pi_{i_l}(f(t_1, \dots, t_n)) = t_{i_l}$. Maps π_{i_l} are defined for $i_l = 1, 2, \dots, n$.

Lemma 2.3. *Let* $t, s \in W_{(n)}(X) \setminus X$, $x \in \mathrm{var}(t)$ *and* $\mathrm{var}(s) \cap X_n = \{x_{z_1}, \dots, x_{z_k}\}$. *If* $(i_1, \dots, i_m) \in \mathrm{seq}^t(x)$, *where* $i_1, \dots, i_m \in \{z_1, \dots, z_k\}$, *then* $x \in \mathrm{var}(\widehat{\sigma}_s[t]) = \mathrm{var}(\sigma_s \circ_G \sigma_t)$ *and there is* $(a_{i_1}, \dots, a_{i_m}) \in \mathrm{seq}^{\widehat{\sigma}_s[t]}(x)$, *where* a_{i_j} *is a sequence of natural numbers* $j_1, \dots, j_h$ *such that* $(j_1, \dots, j_h) \in \mathrm{seq}^s(x_{i_j})$ *for all* $j \in \{1, \dots, m\}$.

Proof. Let $t = f(t_1, \dots, t_n)$ for some $t_1, \dots, t_n \in W_{(n)}(X)$ and $(i_1, \dots, i_m) \in \mathrm{seq}^t(x)$, where $i_1, \dots, i_m \in \{z_1, \dots, z_k\}$. Let us proceed by mathematical induction on m. If $(i_1) \in \mathrm{seq}^t(x)$, where $i_1 \in \{z_1, \dots, z_k\}$, then $x = \pi_{i_1}(t) = t_{i_1}$, where $t_{i_1} \in \{t_1, \dots, t_n\}$. Hence $\widehat{\sigma}_s[t_{i_1}] = \widehat{\sigma}_s[x] = x$. Consider

$$\sigma_s \circ_G \sigma_t(f) = \widehat{\sigma}_s[t] = S^n(s, \widehat{\sigma}_s[t_1], \dots, \widehat{\sigma}_s[t_n]),$$

Since $x_{i_1} \in \mathrm{var}(s) \cap X_n$, $x = \widehat{\sigma}_s[t_{i_1}] \in \mathrm{var}(\widehat{\sigma}_s[t])$ and there is $(a_{i_1}) \in \mathrm{seq}^{\widehat{\sigma}_s[t]}(x)$, where a_{i_1} is a sequence of natural numbers $j_1, \dots, j_h$ such that $(j_1, \dots, j_h) \in \mathrm{seq}^s(x_{i_1})$. Let m be a natural number and assume that for each $u \in W_{(n)}(X) \setminus X$, $x \in \mathrm{var}(u)$ and $(l_1, \dots, l_p) \in \mathrm{seq}^u(x)$, where $l_1, \dots, l_p \in \{z_1, \dots, z_k\}$, then $x \in \mathrm{var}(\widehat{\sigma}_s[u]) = \mathrm{var}(\sigma_s \circ_G \sigma_u)$ and there is $(a_{l_1}, \dots, a_{l_p}) \in \mathrm{seq}^{\widehat{\sigma}_s[u]}(x)$, where a_{l_q} is a sequence of natural numbers $q_1, \dots, q_{h^*}$ such that $(q_1, \dots, q_{h^*}) \in \mathrm{seq}^s(x_{l_q})$ for all $q \in \{1, \dots, p\}$, is true for all natural numbers $p < m$. If $(i_1, \dots, i_m) \in \mathrm{seq}^t(x)$, where $i_1, \dots, i_m \in \{z_1, \dots, z_k\}$, then $x = \pi_{i_m} \circ \dots \circ \pi_{i_1}(t) = \pi_{i_m} \circ \dots \circ \pi_{i_2}(t_{i_1})$, i.e., $x \in \mathrm{var}(t_{i_1})$ and $(i_2, \dots, i_m) \in \mathrm{seq}^{t_{i_1}}(x)$. By our assumption, we get $x \in \mathrm{var}(\widehat{\sigma}_s[t_{i_1}])$ and there is $(a_{i_2}, \dots, a_{i_m}) \in \mathrm{seq}^{\widehat{\sigma}_s[t_{i_1}]}(x)$, where a_{i_j} is a sequence of natural numbers $j_1, \dots, j_h$ such that $(j_1, \dots, j_h) \in \mathrm{seq}^s(x_{i_j})$ for all $j \in \{2, \dots, m\}$. Since $x_{i_1} \in \mathrm{var}(s) \cap X_n$, $\widehat{\sigma}_s[t_{i_1}] \in \mathrm{sub}(S^n(s, \widehat{\sigma}_s[t_1], \dots, \widehat{\sigma}_s[t_n])) = \mathrm{sub}(\widehat{\sigma}_s[t])$ and $\mathrm{seq}^{\widehat{\sigma}_s[t]}(\widehat{\sigma}_s[t_{i_1}]) = \mathrm{seq}^s(x_{i_1})$. Hence $x \in \mathrm{var}(\widehat{\sigma}_s[t])$ and there is $(a_{i_1}, a_{i_2}, \dots, a_{i_m}) \in \mathrm{seq}^{\widehat{\sigma}_s[t]}(x)$ where a_{i_j} is a sequence of natural numbers $j_1, \dots, j_h$ such that $(j_1, \dots, j_h) \in \mathrm{seq}^s(x_{i_j})$ for all $j \in \{1, 2, \dots, m\}$. □

Definition 2.4 ([**4**])**.** For any monoid S, an element $u \in S$ is called *unit* if there exists $u^{-1} \in S$ such that $uu^{-1} = e = u^{-1}u$, where e is the identity element of S, and let $U(S)$ denotes the set of all unit elements of S.

Theorem 2.5 ([**2**])**.** *An element* $\sigma_t \in U(\mathrm{Hyp}_G(n))$ *if and only if* $t = f(x_{\pi(1)}, \dots, x_{\pi(n)})$, *where* $\pi \in S_n$ *and* S_n *is the set of all permutations on* $\{1, \dots, n\}$.

Definition 2.6 ([**4**])**.** An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. The semigroup S is called *regular* if all its elements are regular.

Lemma 2.7 ([**1**])**.** *Let* $\sigma_s, \sigma_t \in \mathrm{Hyp}_G(n)$, *where* $t = f(t_1, \dots, t_n)$ *such that* $t_{i_1} = x_{j_1}, \dots, t_{i_m} = x_{j_m}$ *for some* $i_1, \dots, i_m, j_1, \dots, j_m \in \{1, \dots, n\}$ *and* $\mathrm{var}(t) \cap X_n = \{x_{j_1}, \dots, x_{j_m}\}$. *Then* $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$ *if and only if* $s = f(s_1, \dots, s_n)$, *where* $s_{j_l} = x_{i_l}$ *for all* $l \in \{1, \dots, m\}$.

Let

$R_1 = \{\sigma_{x_i} : x_i \in X\}$,

$R_2 = \{\sigma_t \in \mathrm{Hyp}_G(n) : \mathrm{var}(t) \cap X_n = \emptyset\}$,

$R_3 = \{\sigma_t \in \mathrm{Hyp}_G(n) : t = f(t_1, \ldots, t_n)$, where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m, j_1, \ldots, j_m \in \{1, \ldots, n\}$ and $\mathrm{var}(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}\}$.

In 2010, W. Puninagool and S. Leeratanavalee [**7**] showed that $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $\mathrm{Hyp}_G(n)$.

Definition 2.8 ([**4**])**.** An element e of a semigroup S is called *idempotent* if $e^2 = ee = e$, and we denote the set of all idempotent elements in S by $E(S)$.

Let $E = \{\sigma_t \in \mathrm{Hyp}_G(n) : t = f(t_1, \ldots, t_n)$, where $t_{i_1} = x_{i_1}, \ldots, t_{i_m} = x_{i_m}$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $\mathrm{var}(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}\}$. Clearly, $E \subset R_3$.

In 2010, W. Puninagool and S. Leeratanavalee [**7**] showed that $E(\mathrm{Hyp}_G(n)) = R_1 \cup R_2 \cup E$ is the set of all idempotent elements in $\mathrm{Hyp}_G(n)$.

Definition 2.9 ([**4**])**.** An element a of a monoid S is called *unit-regular* if there exists $u \in U(S)$ such that $aua = a$. The monoid S is called *unit-regular* if all its elements are unit-regular.

Theorem 2.10. $\bigcup\limits_{i=1}^{3} R_i$ *is a set of all unit-regular elements in* $\mathrm{Hyp}_G(n)$.

Proof. Let $\sigma_t \in \bigcup_{i=1}^{3} R_i$. If $\sigma_t \in R_1 \cup R_2$, then $\sigma_t \in E(\mathrm{Hyp}_G(n))$, so $\sigma_t \circ_G \sigma_{id} \circ_G \sigma_t = \sigma_t \circ_G \sigma_t = \sigma_t$. If $\sigma_t \in R_3$, then $t = f(t_1, \ldots, t_n)$, where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m, j_1, \ldots, j_m \in \{1, \ldots, n\}$ and $\mathrm{var}(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}$. Choose $\sigma_u \in U(\mathrm{Hyp}_G(n))$, where $u = f(u_1, \ldots, u_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for some $\pi \in S_n$ such that $\pi(j_1) = i_1, \ldots, \pi(j_m) = i_m$. Then $u_{j_l} = x_{\pi(j_l)} = x_{i_l}$ for all $l \in \{1, \ldots, m\}$. By Lemma 2.7, $\sigma_t \circ_G \sigma_u \circ_G \sigma_t = \sigma_t$. Hence σ_t is a unit-regular element in $\mathrm{Hyp}_G(n)$. Since $\bigcup_{i=1}^{3} R_i$ is a set of all regular elements and all its elements are unit-regular, so $\bigcup_{i=1}^{3} R_i$ is a set of all unit-regular elements in $\mathrm{Hyp}_G(n)$. □

Lemma 2.11. *Let* $t = f(t_1, \ldots, t_n)$, *where* $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ *for some* $i_1, \ldots, i_m, j_1, \ldots, j_m \in \{1, \ldots, n\}$ *and* $\mathrm{var}(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}$. *If* $x_{j_l} \in \mathrm{var}(t_k)$ *for some* $l \in \{1, \ldots, m\}$ *and* $k \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$, *where* $(k_1, \ldots, k_p) \in \mathrm{seq}^{t_k}(x_{j_l})$ *for some* $k_1, \ldots, k_p \in \{1, \ldots, n\} \setminus \{i_l\}$, *then there exists* $\sigma_s \in \mathrm{Hyp}_G(n)$ *such that* $\sigma_s \circ_G \sigma_t$ *is not a unit-regular element in* $\mathrm{Hyp}_G(n)$.

Proof. Assume that the condition holds. Since $(k_1, \ldots, k_p) \in \mathrm{seq}^{t_k}(x_{j_l})$, then $(k, k_1, \ldots, k_p) \in \mathrm{seq}^{t}(x_{j_l})$. Let $h_1, \ldots, h_q$ be distinct from $k, k_1, \ldots, k_p$, then $q \leq n$. Choose $\sigma_s \in \mathrm{Hyp}_G(n)$, where $s = f(s_1, \ldots, s_n)$ such that $s_1 = x_{h_1}, \ldots, s_q = x_{h_q}$ and $s_{q+1}, \ldots, s_n \in W_{(n)}(X)$ and $\mathrm{var}(s_r) \cap X_n = \emptyset$ for all $r \in \{q+1, \ldots, n\}$. Then $s_i \neq x_{i_l}$ for all $i \in \{1, \ldots, n\}$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(t_1, \ldots, t_n)]\\ &= S^n(f(s_1, .., s_n), \widehat{\sigma}_s[t_1], \ldots, \widehat{\sigma}_s[t_n]) = f(u_1, \ldots, u_n),\end{aligned}$$

where $u_i = S^n(s_i, \widehat{\sigma}_s[t_1], \dots, \widehat{\sigma}_s[t_n])$ for all $i \in \{1, \dots, n\}$. Since $s_i \neq x_{i_l}$, $u_i \neq x_{j_l}$ for all $i \in \{1, \dots, n\}$. By Lemma 2.3 we get $x_{j_l} \in \text{var}(\sigma_s \circ_G \sigma_t)$ such that $x_{j_l} \in \text{var}(u_j)$, where $u_j \in W_{(n)}(X) \setminus X$ for some $j \in \{1, \dots, n\}$. Hence $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{3} R_i$, so $\sigma_s \circ_G \sigma_t$ is not a unit-regular element in $\text{Hyp}_G(n)$. □

Example 2.12. Let $\tau = (3)$ and $\sigma_t \in \bigcup_{i=1}^{3} R_i$, where $t = f(x_2, f(f(x_4, x_4, x_5), x_2, x_5), f(x_5, x_2, x_5))$. Choose $\sigma_s \in R_3$, where $s = f(x_2, x_3, x_4)$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, f(f(x_4, x_4, x_5), x_2, x_5), f(x_5, x_2, x_5))]\\ &= S^3(s, x_2, f(x_2, x_5, x_4), f(x_2, x_5, x_4))\\ &= f(f(x_2, x_5, x_4)), f(x_2, x_5, x_4), x_4).\end{aligned}$$

We see that $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{3} R_i$, so $\sigma_s \circ_G \sigma_t$ is not a unit-regular element in $\text{Hyp}_G(3)$. Therefore $\bigcup_{i=1}^{3} R_i$ is not closed under $\circ_G$.

Let $R_3^* = \{\sigma_t \in \text{Hyp}_G(n) : t = f(t_1, \dots, t_n)$, where $t_{i_1} = x_{j_1}, \dots, t_{i_m} = x_{j_m}$ for some $i_1, \dots, i_m, j_1, \dots, j_m \in \{1, \dots, n\}$ and $\text{var}(t) \cap X_n = \{x_{j_1}, \dots, x_{j_m}\}$ and if $x_{j_l} \in \text{var}(t_k)$ for some $l \in \{1, \dots, m\}$ and $k \in \{1, \dots, n\} \setminus \{i_1, \dots, i_m\}$, then there exists $(k_1, \dots, k_p) \in \text{seq}^{t_k}(x_{j_l})$ such that $k_q = i_l$ for some $q \in \{1, \dots, p\}\}$.

We denote $(UR)_{\text{Hyp}_G(n)} = R_1 \cup R_2 \cup R_3^*$.

Theorem 2.13. *$(UR)_{\text{Hyp}_G(n)}$ is a unit-regular submonoid of $\text{Hyp}_G(n)$.*

Proof. We have $(UR)_{\text{Hyp}_G(n)} \subset \text{Hyp}_G(n)$ and all its elements are unit-regular. So we will show that $(UR)_{\text{Hyp}_G(n)}$ is a submonoid of $\text{Hyp}_G(n)$, i.e., $\sigma_s \circ_G \sigma_t \in (UR)_{\text{Hyp}_G(n)}$ for all $\sigma_t, \sigma_s \in (UR)_{\text{Hyp}_G(n)}$.

If $\sigma_t \in R_1$, then $\sigma_s \circ_G \sigma_t \in R_1$ for all $\sigma_s \in (UR)_{\text{Hyp}_G(n)}$.

If $\sigma_t \in R_2$, then $\sigma_s \circ_G \sigma_t \in R_1$ for all $\sigma_s \in R_1$, and $\sigma_s \circ_G \sigma_t \in R_2$ for all $\sigma_s \in R_2 \cup R_3^*$.

If $\sigma_t \in R_3^*$, then $\sigma_s \circ_G \sigma_t \in R_1$ for all $\sigma_s \in R_1$ and $\sigma_s \circ_G \sigma_t \in R_2$ for all $\sigma_s \in R_2$. Denote $t = f(t_1, \dots, t_n)$, where $t_{i_1} = x_{j_1}, \dots, t_{i_m} = x_{j_m}$ for some $i_1, \dots, i_m, j_1, \dots, j_m \in \{1, \dots, n\}$ and $\text{var}(t) \cap X_n = \{x_{j_1}, \dots, x_{j_m}\}$. Let $\sigma_s \in R_3^*$, denote $s = f(s_1, \dots, s_n)$, where $s_{r_1} = x_{h_1}, \dots, s_{r_{m^*}} = x_{h_{m^*}}$ for some $r_1, \dots, r_{m^*}$, $h_1, \dots, h_{m^*} \in \{1, \dots, n\}$ and $\text{var}(s) \cap X_n = \{x_{h_1}, \dots, x_{h_{m^*}}\}$. Hence

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(t_1, \dots, t_n)]\\ &= S^n(f(s_1, \dots, s_n), \widehat{\sigma}_s[t_1], \dots, \widehat{\sigma}_s[t_n]) = f(u_1, \dots, u_n),\end{aligned}$$

where $u_i = S^n(s_i, \widehat{\sigma}_s[t_1], \dots, \widehat{\sigma}_s[t_n])$ for all $i \in \{1, \dots, n\}$.

<u>*Case 1.*</u> Let $i \in \{r_1, \dots, r_{m^*}\}$. Then $i = r_\alpha$ for some $\alpha \in \{1, \dots, m^*\}$. So

$$u_i = S^n(s_{r_\alpha}, \widehat{\sigma}_s[t_1], \dots, \widehat{\sigma}_s[t_n]) = S^n(x_{h_\alpha}, \widehat{\sigma}_s[t_1], \dots, \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_{h_\alpha}].$$

<u>*Case 1.1.*</u> Let $h_\alpha \in \{i_1, \dots, i_m\}$. Then $h_\alpha = i_\beta$ for some $\beta \in \{1, \dots, m\}$ and $t_{i_\beta} = x_{j_\beta}$, so $u_i = \widehat{\sigma}_s[t_{i_\beta}] = x_{j_\beta}$.

<u>*Case 1.2.*</u> Let $h_\alpha = k$, where $k \in \{1, \dots, n\} \setminus \{i_1, \dots, i_m\}$. Then $u_i = \widehat{\sigma}_s[t_k]$.

<u>*Case 1.2.1.*</u> Let $\text{var}(t_k) \cap X_n = \emptyset$. Then $\text{var}(u_i) \cap X_n = \emptyset$.

<u>*Case 1.2.2.*</u> Let $\text{var}(t_k) \cap X_n \neq \emptyset$. Then there exists $x_{j_\beta} \in \text{var}(t_k)$, where $t_{i_\beta} = x_{j_\beta}$ for some $\beta \in \{1, \dots, m\}$, and there exists $(k_1, \dots, k_p) \in \text{seq}^{t_k}(x_{j_\beta})$ such

that $k_q = i_\beta$ for some $q \in \{1, \ldots, p\}$. If $x_{k_1}, \ldots, x_{k_p} \in \text{var}(s)$, then $x_{i_\beta} = x_{k_q} \in \text{var}(s)$, where $k_q \neq k$, so $x_{k_q} = x_{i_\beta} = s_{r_\varepsilon}$ for some $\varepsilon \in \{1, \ldots, m^*\}$ and there exists $(r_\varepsilon) \in \text{seq}^s(x_{k_q})$ such that $r_\varepsilon \neq r_\alpha = i$. Hence $u_{r_\varepsilon} = S^n(s_{r_\varepsilon}, \widehat{\sigma}_s[t_1], \ldots, \widehat{\sigma}_s[t_n]) = S^n(x_{i_\beta}, \widehat{\sigma}_s[t_1], \ldots, \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_{i_\beta}] = x_{j_\beta}$. By Lemma 2.3, we get $x_{j_\beta} \in \text{var}(\widehat{\sigma}_s[t_k]) = \text{var}(u_i)$ and there exists $(a_{k_1}, \ldots, a_{k_P}) \in \text{seq}^{u_i}(x_{j_\beta})$, where $a_{k_q} = r_\varepsilon$, and a_{k_j} is a sequence $j_1, \ldots, j_h$ such that $(j_1, \ldots, j_h) \in \text{seq}^s(x_{k_j})$ for all $j \in \{1, \ldots, p\} \setminus \{q\}$. If $x_{k_\gamma} \notin \text{var}(s)$ for some $1 \leq \gamma \leq p$ then $x_{j_\beta} \notin \text{var}(u_i)$, so $\text{var}(u_i) \cap X_n = \emptyset$.

Case 2. Let $i = k^*$, where $k^* \in \{1, \ldots, n\} \setminus \{r_1, \ldots, r_{m^*}\}$. Then

$$u_i = S^n(s_i, \widehat{\sigma}_s[t_1], \ldots, \widehat{\sigma}_s[t_n]) = S^n(s_{k^*}, \widehat{\sigma}_s[t_1], \ldots, \widehat{\sigma}_s[t_n]).$$

Case 2.1. Let $\text{var}(s_{k^*}) \cap X_n = \emptyset$. Then $u_i = s_{k^*}$ and $\text{var}(u_i) \cap X_n = \emptyset$.

Case 2.2. Let $\text{var}(s_{k^*}) \cap X_n \neq \emptyset$. Then $x_{h_\alpha} \in \text{var}(s_{k^*})$ for some $\alpha \in \{1, \ldots, m^*\}$. So $s_{r_\alpha} = x_{h_\alpha}$ and there exists $(k_1^*, \ldots, k_p^*) \in \text{seq}^{s_{k^*}}(x_{h_\alpha})$ such that $k_q^* = r_\alpha$ for some $q \in \{1, \ldots, p\}$.

Case 2.2.1. Let $h_\alpha \in \{i_1, \ldots, i_m\}$. Then $h_\alpha = i_\beta$ for some $\beta \in \{1, \ldots, m\}$, so $x_{i_\beta} = x_{h_\alpha} \in \text{var}(s)$. By Lemma 2.3, we get $x_{j_\beta} \in \text{var}(\widehat{\sigma}_s[t])$ and $\text{seq}^s(x_{i_\beta}) \subseteq \text{seq}^{\widehat{\sigma}_s[t]}(x_{j_\beta})$. Since $(r_\alpha) \in \text{seq}^s(x_{i_\beta})$ and $(k^*, k_1^*, \ldots, k_p^*) \in \text{seq}^s(x_{i_\beta})$, then $(r_\alpha) \in \text{seq}^{\widehat{\sigma}_s[t]}(x_{j_\beta})$ and $(k^*, k_1^*, \ldots, k_p^*) \in \text{seq}^{\widehat{\sigma}_s[t]}(x_{j_\beta})$. Hence $u_{r_\alpha} = x_{j_\beta}$ $x_{j_\beta} \in \text{var}(u_{k^*}) = \text{var}(u_i)$ and there exists $(k_1^*, \ldots, k_p^*) \in \text{seq}^{u_i}(x_{j_\beta})$ such that $k_q^* = r_\alpha$.

Case 2.2.2. Let $h_\alpha = k$, where $k \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$, we can prove similar to Case 1.2.

Therefore, $\sigma_s \circ_G \sigma_t \in R_2 \cup R_3^* \subset (UR)_{\text{Hyp}_G(n)}$.

Hence $(UR)_{\text{Hyp}_G(n)}$ is closed under $\circ_G$ and we have $\sigma_{id} \in (UR)_{\text{Hyp}_G(n)}$, i.e., $(UR)_{\text{Hyp}_G(n)}$ is a submonoid of $\text{Hyp}_G(n)$. □

Theorem 2.14. *$(UR)_{\text{Hyp}_G(n)}$ is a maximal unit-regular semigroup of* $\text{Hyp}_G(n)$.

Proof. Let H be a proper unit-regular semigroup of $\text{Hyp}_G(n)$ such that $(UR)_{\text{Hyp}_G(n)} \subseteq H \subset \text{Hyp}_G(n)$. Let $\sigma_t \in H$, where $\sigma_t \in R_3 \setminus R_3^*$. By Lemma 2.11, we can choose $\sigma_s \in R_3^*$ such that $\sigma_s \circ_G \sigma_t$ is not unit-regular. So $\sigma_t \notin H$. Hence $H = (UR)_{\text{Hyp}_G(n)}$. □

Definition 2.15 ([**3**])**.** Let S be a semigroup and $E(S)$ be the set of all idempotents in S. We say S is *left* [*right*] *factorisable* if $S = GE(S)$ [$S = E(S)H$] for some subgroup $G[H]$ of S. S is *factorisable* if S is both left and right factorisable.

Theorem 2.16 ([**3**])**.** *A monoid S is factorisable if and only if it is unit-regular.*

Corollary 2.17. *$(UR)_{\text{Hyp}_G(n)}$ is factorisable.*

Acknowledgment. This research was supported by Chiang Mai University, Chaing Mai 50200, Thailand.

References

1. Boonmee A. snd Leeratanavalee S., *All Completely Regular Elements in* $\text{Hyp}_G(n)$, Discuss. Math. Gen. Algebra Appl. **33** (2013), 211–219.

2. Boonmee A. and Leeratanavalee S., *Factorisable Monoid of Generalized Hypersubstitutions of Type* $\tau = (2)$, Thai J. Math. **13(1)** (2015), 213–225.

3. D'Alarcao H., *Factorisable as a Finiteness Condition*, Semigroup Forum **20(1)** (1980), 281–282.
4. Howie J. M., *Fundamentals of Semigroup Theory*, Academic Press, London, 1995.
5. Denecke K. and Leeratanavalee S., *Tree Transformations Defined by Generalized Hypersubstitutions*, Sci. Math. Jpn. online, **6** (2002), 355–366.
6. Leeratanavalee S. and Denecke K., *Generalized Hypersubstitutions and Strongly Solid Varieties, General Algebra and Applications*, Proc. of the "59 th Workshop on General Algebra", 15-th Conference for Young Algebraists Potsdam 2000, Shaker Verlag (2000), 135–145.
7. Puninagool W. and Leeratanavalee S., *The Monoid of Generalized Hypersubstitutions of Type* $\tau = (n)$, Discuss. Math. Gen. Algebra Appl. **30** (2010), 173–191.

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