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n-COHERENCE PROPERTY IN AMALGAMATED ALGEBRA ALONG AN IDEAL

K. A. ISMAILI and N. MAHDOU

Abstract. Let $f: A \to B$ be a ring homomorphism and let J be an ideal of B. The purpose of this article is to examine the transfer of the properties of n-coherence and strong n-coherence ($n \geq 2$) from a ring A to his amalgamated algebra $A \bowtie^f J$. Our results generate examples which enrich the current literature with new and original families of n-coherent rings that are not $(n-1)$-coherent rings.

1. Introduction

Throughout this paper, all rings are commutative with identity element and all modules are unitary.

Let A and B be two rings, let J be an ideal of B and let $f: A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$

$$A \bowtie^f J = \{(a, f(a) + j) : a \in A, j \in J\}$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [**9, 10**]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [**11, 12, 13**] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation [**9**, Examples 2.5 & 2.6] and other classical constructions such as the Nagata's idealization and the CPI extensions (in the sense of Boisen and Sheldon [**4**]) strictly related to it (see [**9**, Example 2.7 & Remark 2.8]).

Let R be a commutative ring. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules

$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0,$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules, respectively, are finitely generated and finitely presented R-modules.

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Let A be a ring, E be an A-module, and $R := A \propto E$ be the set of pairs (a, e) with componentwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E (also called the idealization of E over A). Considerable work, part of it summarized in Glaz [**18**] and Huckaba [**19**], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and noncommutative) ring theory. See, for instance, [**2, 3, 18, 19, 20, 21, 24**].

The ring R is n-coherent if each $(n-1)$-presented ideal of R is n-presented, and R is a strong n-coherent ring if each n-presented R-module is $(n+1)$-presented [**14, 15**] (This terminology is not the same as that of Costa (1994) [**7**], more precisely Costa's n-coherence is our strong n-coherence). In particular, 1-coherence coincides with coherence, and 0-coherence coincides with Noetherianity. Any strong n-coherent ring is n-coherent, and the converse holds for $n = 1$ or for coherent rings [**15**, Proposition 3.3].

In this paper, we study the amalgamated algebra $A \bowtie^f J$ to be an n-coherent ring ($n \geq 2$). Thereby, new examples are provided which particularly enriches the current literature with new classes of n-coherent rings ($n \geq 2$) that are not $(n-1)$-coherent rings.

2. Transfer of the properties of n-coherence and strong n-coherence ($n \geq 2$)

The main result (Theorem 2.2) examines the transfer of the properties of strong n-coherence and n-coherence ($n \geq 2$) to the amalgamated algebra along an ideal issued from local rings.

First, it is worthwhile noting that the function $f^n \colon A^n \to B^n$ defined by $f^n((\alpha_i)_{i=1}^{i=n}) = (f(\alpha_i))_{i=1}^{i=n}$ is a ring homomorphism, $(A \bowtie^f J)^n \cong A^n \bowtie^{f^n} J^n$ and $f^n(\alpha a) = f(\alpha) f^n(a)$ for all $\alpha \in A$ and $a \in A^n$.

Next, before we announce the main result of this section (Theorem 2.2), we make the following useful remark.

Proposition 2.1. *Let (A, M) be a local ring, $f \colon A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $J \subseteq \mathrm{Rad}(B)$.*

1. *$A \bowtie^f J$ is a local ring and $M \bowtie^f J$ is its maximal ideal.*
2. *$f(A) + J$ is a local ring and $f(M) + J$ is its maximal ideal.*

Proof. 1. Indeed, by [**10**, Proposition 2.6 (5)], $\mathrm{Max}(A \bowtie^f J) = \{m \bowtie^f J : m \in \mathrm{Max}(A)\} \cup \{\overline{Q}\}$ with $Q \in \mathrm{Max}(B)$ not containing $V(J)$ and $\overline{Q} := \{(a, f(a)+j) : a \in A, j \in J, f(a) + j \in Q\}$. Since $J \subseteq \mathrm{Rad}(B)$, then $J \subseteq Q$ for all $Q \in \mathrm{Max}(B)$. So, $\mathrm{Max}(A \bowtie^f J) = \{m \bowtie^f J : m \in \mathrm{Max}(A)\} = M \bowtie^f J$ since (A, M) is a local ring. So, $(A \bowtie^f J, M \bowtie^f J)$ is a local ring.

2. $f(A) + J$ is a local ring since $f(A) + J \cong \frac{A \bowtie^f J}{f^{-1}\{J\} \times \{0\}}$ and $A \bowtie^f J$ is a local ring by 1. Let M' be an ideal of $f(A) + J$ such that $f(M) + J \subsetneq M'$. Then there is $a \notin M$ and $j \in J$ such that $f(a) + j \in M'$. So, there is $(b, f(b) + g) \in A \bowtie^f J$ such that $(a, f(a) + j)(b, f(b) + g) = 1$. Hence, $(f(a) + j)(f(b) + g) = 1$, and then $f(a) + j \in U(f(A) + J)$, as desired. □

This is the main result of this paper.

Theorem 2.2. *Let (A, M) be a local ring, $f\colon A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $f(M)J = 0$.*

(1) *Assume that $J \subseteq \mathrm{Rad}(B)$.*

  (a) (i) *Assume that J is a finitely generated ideal of $f(A) + J$. If $A \bowtie^f J$ is (strong) 2-coherent ring, then so is A.*

   (ii) *Assume that $f(M) \subseteq J$ and M is a finitely generated ideal of A. If $A \bowtie^f J$ is a (a strong) 2-coherent ring, then so is $f(A) + J$.*

   (iii) *Assume that either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$, M and J are finitely generated ideals of A and $f(A)+J$, respectively, and $f(A)+J$ is a coherent ring. Then $A \bowtie^f J$ is (a strong) 2-coherent ring if and only if so is A.*

  (b) *Assume that $n > 2$.*

   (i) *Assume that either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$, M is $(n-3)$-finitely presented ideal of A, and J is $(n-2)$-finitely presented ideal of $f(A) + J$. If $A \bowtie^f J$ is (a strong) n-coherent ring, then so is A.*

   (ii) *Assume that $f(M) \subseteq J$, M is a $(n-2)$-finitely presented ideal of A, and J is a $(n-3)$-finitely presented ideal of $f(A) + J$. If $A \bowtie^f J$ is (strong) n-coherent ring, then so is $f(A) + J$.*

   (iii) *Assume that $f(M) \subseteq J$, M and J are $(n-2)$-finitely presented ideals of A and $f(A)+J$, respectively, and $f(A)+J$ is a strong $(n-1)$-coherent ring. Then $A \bowtie^f J$ is (a strong) n-coherent ring if and only if so is A.*

(2) *Assume that $J^2 = 0$, $n \geq 2$, and J is a finitely generated ideal of $(f(A) + J)$. Then $A \bowtie^f J$ is (a strong) n-coherent ring if and only if so is A.*

The proof of Theorem 2.2 draws on the following results.

Lemma 2.3. *Let (A, M) be a local ring, $f\colon A \to B$ be an injective ring homomorphism, and J be a proper ideal of B such that $f(M)J = 0$. Let $n, p \in \mathbb{N} \setminus \{0\}$, and let U be an A-module such that $U \subseteq M^p$.*

(1) *If J is a finitely generated ideal of $(f(A) + J)$ and U is a finitely presented A-module, then $f^p(U)$ is a finitely presented $(f(A) + J)$-module.*

(2) *Assume that either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$. If J is $(n-1)$-finitely presented ideal of $(f(A) + J)$ and U is n-finitely presented A-module, then $f^p(U)$ is n-finitely presented $(f(A) + J)$-module.*

Proof. **(1)** Assume that J is a finitely generated ideal of $(f(A) + J)$ and U is a finitely presented A-module. Let $\{(u_i)_{i=1}^{i=r}\}$ be a minimal generating set of U. Then $f^p(U) = \sum_{i=1}^{i=r}(f(A) + J)f^p(u_i)$ is a finitely generated $(f(A) + J)$-module. Consider the exact sequence of A-modules

$$0 \to \operatorname{Ker} v \to A^r \to U \to 0, \tag{1}$$

where $v((\alpha_i)_{i=1}^{i=r}) = \sum_{i=1}^{i=r} \alpha_i u_i$. On the other hand, consider the exact sequence of $(f(A)+J)$-modules

$$0 \to \operatorname{Ker} u \to (f(A)+J)^r \to f^p(U) \to 0, \tag{2}$$

where $u((f(\alpha_i)+j_i)_{i=1,\dots,r})) = \sum_{i=1}^{i=r}(f(\alpha_i)+j_i)f^p(u_i)$. So,

$$\begin{aligned}\operatorname{Ker} u &= \Big\{(f(\alpha_i)+j_i)_{i=1,\dots,r}) \in (f(A)+J)^r : \sum_{i=1}^{i=r} f(\alpha_i)f^p(u_i) = 0\Big\}\\ &\cong f^r(\operatorname{Ker} v) + J^r\end{aligned}$$

(since f is injective). By a sequence (1), $\operatorname{Ker} v$ is a finitely generated A-module. Then $f^r(\operatorname{Ker} v)$ a finitely generated $(f(A)+J)$-module (since $\operatorname{Ker} v \subseteq M^r$). So, $\operatorname{Ker} u$ is a finitely generated $(f(A)+J)$-module, and then $f^p(U)$ is a finitely presented $(f(A)+J)$-module.

(2) Proceed by induction on n. The property is true for $n=1$ by (1). Assume that the property is true for n, and assume that J is an n-finitely presented ideal of $(f(A)+J)$ and U is an $(n+1)$-finitely presented A-module. Let $\{(u_i)_{i=1}^{i=r}\}$ be a minimal generating set of U. Then $f^p(U) = \sum_{i=1}^{i=r}(f(A)+J)f^p(u_i)$. By the sequence (1), $\operatorname{Ker} v$ is an n-finitely presented A-module. So, $f^r(\operatorname{Ker} v)$ is an n-finitely presented $(f(A)+J)$-module by induction (since $\operatorname{Ker} v \subseteq M^r$).

Assume that $f(M) \cap J = \{0\}$. Then $\operatorname{Ker} u \cong f^r(\operatorname{Ker} v) \bigoplus J^r$ is an n-finitely presented $(f(A)+J)$-module.

Assume that $f(M) \subseteq J$. Then $\operatorname{Ker} u \cong J^r$ is an n-finitely presented $(f(A)+J)$--module.

So, in the both cases $\operatorname{Ker} u$ is an n-finitely presented $(f(A)+J)$-module, and then $f^p(U)$ is an $(n+1)$-finitely presented $(f(A)+J)$-module as desired. □

Lemma 2.4. *Let (A, M) be a local ring, $f\colon A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $f(M)J = 0$. Let $p \in \mathbb{N}^*$ and let U be an A-module and K be an $(f(A)+J)$-module such that $U \subseteq M^p$ and $K \subseteq J^p$.*

(1) *$U \bowtie^{f^p} K = \{(u, f^p(u)+k) : u \in U, k \in K\} (\subseteq M^p \bowtie^{f^p} J^p)$ is an $(A \bowtie^f J)$--module.*

(2) *$U \bowtie^{f^p} K$ is a finitely generated $(A \bowtie^f J)$-module if and only if U is a finitely generated A-module, and K is a finitely generated $(f(A)+J)$-module.*

(3) *Assume that $J \subseteq \operatorname{Rad}(B)$.*
   - (a) (i) *If $U \bowtie^{f^p} K$ is a finitely presented $(A \bowtie^f J)$-module, then U and K are finitely presented modules of A and $(f(A)+J)$, respectively, and M and J are finitely generated ideals of A and $f(A)+J$, respectively.*
     - (ii) *Assume that either $f(M) \cap J = \{0\}$ or $f(M) \subseteq J$. Then $U \bowtie^{f^p} K$ is a finitely presented $(A \bowtie^f J)$-module if and only if U and K are finitely presented modules of A and $(f(A)+J)$, respectively, and M and J are finitely generated ideals of A and $f(A)+J$, respectively.*
   - (b) *Let $n \geq 1$.*

(i) *Assume that* $U \bowtie^{f^p} K$ *is an* n*-finitely presented* $(A \bowtie^f J)$*-module, then* U *is an* n*-finitely presented* A*-module, and* M *is an* $(n-1)$*-finitely presented ideal of* A.

(ii) *Assume that either* $f(M) \cap J = \{0\}$ *or* $f(M) \subseteq J$. *If* U *and* K *are* n*-finitely presented modules of* A *and* $f(A)+J$*, respectively, and* M *and* J *are* $(n-1)$*-finitely presented ideals of* A *and* $f(A)+J$*, respectively, then* $U \bowtie^{f^p} K$ *is an* n*-finitely presented* $(A \bowtie^f J)$*-module.*

(iii) *Assume that either* $f^{-1}(J) = \{0\}$ *or* $f(M) \subseteq J$. *Then* $U \bowtie^{f^p} K$ *is an* n*-finitely presented* $(A \bowtie^f J)$*-module if and only if* U *and* K *are* n*-finitely presented modules of* A *and* $f(A)+J$*, respectively, and* M *and* J *are* $(n-1)$*-finitely presented ideals of* A *and* $f(A)+J$*, respectively.*

(4) *Assume that* $J^2 = 0$ *and* $n \in \mathbb{N}$. *Then* $U \bowtie^{f^p} J^p$ *is an* n*-finitely presented* $(A \bowtie^f J)$*-module if and only if* U *is an* n*-finitely presented* A*-module, and* J *is a finitely generated ideal of* $f(A)+J$.

Proof. (1) it is obvious.

(2) Assume that $U := \sum_{i=1}^{i=n} Au_i$ is a finitely generated A-module, where $u_i \in U$ for all $i \in \{1,\dots,n\}$ and $K := \sum_{i=1}^{i=m}(f(A)+J)k_i$ is a finitely generated $(f(A)+J)$--module, where $k_i \in K$ for all $i \in \{1,\dots,m\}$. We show easily that

$$U \bowtie^{f^p} K = \sum_{i=1}^{i=n}(A \bowtie^f J)(u_i, f^p(u_i)) + \sum_{i=1}^{i=m}(A \bowtie^f J)(0,k_i),$$

that is a finitely generated $(A \bowtie^f J)$-module. Conversely, assume that $U \bowtie^{f^p} K := \sum_{i=1}^{i=n}(A \bowtie^f J)(u_i, f^p(u_i)+k_i)$ is a finitely generated $(A \bowtie^f J)$-module, where $u_i \in U$ and $k_i \in K$ for all $1 \le i \le n$. It is clear that $U = \sum_{i=1}^{i=n} Au_i$ and $K = \sum_{i=1}^{i=n}(f(A)+J)k_i$.

(3) Assume that $J \subseteq \mathrm{Rad}(B)$.

(a)(i) Assume that $U \bowtie^{f^p} K$ is a finitely presented $(A \bowtie^f J)$-module. By (2), U is a finitely generated A-module and K is a finitely generated $(f(A)+J)$--module. Let $\{(u_i)_{i=1}^{i=r}\}$ be a minimal generating set of U, and let $\{(g_i)_{i=1}^{i=m}\}$ be a minimal generating set of K, where $u_i \in U$, $g_i \in K$. Consider the exact sequence of A-modules

$$0 \to \operatorname{Ker} v \to A^r \to U \to 0, \tag{3}$$

where $v((\alpha_i)_{i=1}^{i=r}) = \sum_{i=1}^{i=r} \alpha_i u_i$. On the other hand, consider the exact sequence of $(f(A)+J)$-modules

$$0 \to \operatorname{Ker} u \to (f(A)+J)^m \to K \to 0, \tag{4}$$

where $u((f(\alpha_i)+j_i)_{i=1,\dots,m}) = \sum_{i=1}^{i=m}(f(\alpha_i)+j_i)g_i$. So,

$$\begin{aligned}\operatorname{Ker} u &= \Big\{(f(\alpha_i)+j_i)_{i=1,\dots,m}) \in (f(A)+J)^m : \sum_{i=1}^{i=m}(f(\alpha_i)+j_i)g_i = 0\Big\}\\ &\cong f^m(M^m)+L_m\end{aligned}$$

(since $\{(g_i)_{i=1}^{i=m}\}$ is a minimal generating set of K and $(f(A)+J, f(M)+J)$ is a local ring by Proposition 2.1), where $L_m = \{(j_i)_{i=1}^{i=m} \in J^m : \sum_{i=1}^{i=m} j_i g_i = 0\}$. Consider the exact sequence of $(A \bowtie^f J)$-modules

$$0 \to \operatorname{Ker} w \to (A \bowtie^f J)^{r+m} \to U \bowtie^{f^p} K \to 0, \tag{5}$$

where

$$\begin{aligned}&w((\alpha_i, f(\alpha_i)+j_i)_{i=1,\dots,r}), (\beta_i, f(\beta_i)+k_i)_{i=1,\dots,m})\\ &= \sum_{i=1}^{i=r}(\alpha_i, f(\alpha_i)+j_i)(u_i, f^p(u_i)) + \sum_{i=1}^{i=m}(\beta_i, f(\beta_i)+k_i)(0, g_i)\\ &= \sum_{i=1}^{i=r}(\alpha_i u_i, f^p(\alpha_i u_i)) + \sum_{i=1}^{i=m}(0, (f(\beta_i)+k_i)g_i))\\ &= \Big(\sum_{i=1}^{i=r}\alpha_i u_i, \sum_{i=1}^{i=r} f^p(\alpha_i u_i) + \sum_{i=1}^{i=m}(f(\beta_i)+k_i)g_i\Big).\end{aligned}$$

It follows that $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} J^r \bigoplus M^m \bowtie^{f^m} L_m$.

By the sequence (5), $\operatorname{Ker} w$ is a finitely generated $(A \bowtie^f J)$-module. So, $\operatorname{Ker} v$ and L_m are finitely generated modules of A and $(f(A)+J)$, respectively, and M and J are finitely generated ideals of A and $(f(A)+J)$, respectively, by (4) (since $\operatorname{Ker} v \subseteq M^r$ and $L_m \subseteq J^m$). Therefore, U and K are finitely presented modules of A and $(f(A)+J)$, respectively, by sequences (3) and (4).

(ii) Assume that $U \bowtie^{f^p} K$ is a finitely presented $(A \bowtie^f J)$-module, then U and K are finitely presented modules of A and $(f(A)+J)$, respectively, and M and J are finitely generated ideals of A and $f(A)+J$, respectively, by (i). Conversely, assume that U and K are finitely presented modules of A and $(f(A)+J)$, respectively, and M and J are finitely generated ideals of A and $(f(A)+J)$, respectively.

Assume that $f(M) \cap J = \{0\}$. By the sequence (4), $\operatorname{Ker} u \cong f^m(M^m) \bigoplus L_m$ is a finitely generated $(f(A)+J)$-module. So L_m is a finitely generated $(f(A)+J)$-module.

Assume that $f(M) \subseteq J$. Then $L_m \cong \operatorname{Ker} u$ is a finitely generated $(f(A)+J)$-module.

Thus, in both cases $M^m \bowtie^{f^m} L_m$ is a finitely generated $(A \bowtie^f J)$-module by (2). So, $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} J^r \bigoplus M^m \bowtie^{f^m} L_m$ is a finitely generated $(A \bowtie^f J)$-module, and then $U \bowtie^{f^p} K$ is a finitely presented $(A \bowtie^f J)$-module by the sequence (5).

(b) Assume that $n \geq 1$.

(i) Proceed by induction on n. The property is true for $n = 1$ by (a)(i). Assume that the property is true for n, and assume that $U \bowtie^{f^p} K$ is an $(n+1)$-finitely presented $(A \bowtie^f J)$-module. By (2), U is a finitely generated A-module and K is a finitely generated $(f(A)+J)$-module. Let $\{(u_i)_{i=1}^{i=r}\}$ be a minimal generating set of U, and $\{(g_i)_{i=1}^{i=m}\}$ be a minimal generating set of K, where $u_i \in U, g_i \in K$. By the sequence (5), $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} J^r \bigoplus M^m \bowtie^{f^m} L_m$ is an n-finitely presented $(A \bowtie^f J)$-module. So, by induction $\operatorname{Ker} v$ is n-finitely presented A-module and M is an n-finitely presented ideal of A (since $\operatorname{Ker} v \subseteq M^r$). Therefore, U is an $(n+1)$-finitely presented A-module by the sequence (3).

(ii) The property is true for $n = 1$ by (a)(ii). Assume that the property is true for n, and assume that U and K are $(n+1)$-finitely presented modules of A and $(f(A)+J)$, respectively, and M and J are n-finitely presented ideals of A and $(f(A)+J)$, respectively. Then $\operatorname{Ker} v$ and $\operatorname{Ker} u$ are n-finitely presented modules of A and $(f(A)+J)$, respectively, by the sequence (3) and the sequence (4).

Assume that $f(M) \subseteq J$. Then $L_m \cong \operatorname{Ker} u$ is an n-finitely presented $(f(A)+J)$--module.

Assume that $f(M) \cap J = \{0\}$. Then L_m is an n-finitely presented $(f(A)+J)$--module since $\operatorname{Ker} u \cong f^m(M^m) \bigoplus L_m$ is an n-finitely presented $(f(A)+J)$--module.

Thus, in the both cases L_m is an n-finitely presented $(f(A)+J)$-module. Therefore, $\operatorname{Ker} v \bowtie^{f^r} J^r$ and $M^m \bowtie^{f^m} L_m$ are n-finitely presented $(A \bowtie^f J)$-modules by induction (since $\operatorname{Ker} v \subseteq M^r$ and $L_m \subseteq J^m$), and so $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} J^r \bigoplus M^m \bowtie^{f^m} L_m$ is an n-finitely presented $(A \bowtie^f J)$-module. Consequently, $U \bowtie^{f^p} K$ is an $(n+1)$-finitely presented $(A \bowtie^f J)$-module by the sequence (5).

(iii) Assume that U and K are n-finitely presented modules of A and $f(A)+J$, respectively, and M and J are $(n-1)$-finitely presented ideals of A and $f(A)+J$, respectively, then $U \bowtie^{f^p} K$ is an n-finitely presented $(A \bowtie^f J)$-module by (b)(ii). Conversely, the property is true for $n = 1$ by (a)(i). Assume that the property is true for n, and assume that $U \bowtie^{f^p} K$ is an $(n+1)$-finitely presented $(A \bowtie^f J)$--module. Let $\{(u_i)_{i=1}^{i=r}\}$ be a minimal generating set of U, and $\{(g_i)_{i=1}^{i=m}\}$ be a minimal generating set of K, where $u_i \in U, g_i \in K$. By (b)(i), U is an $(n+1)$-finitely presented A-module, and M is an n-finitely presented ideal of A. It remains to show that K is $(n+1)$-finitely presented $(f(A)+J)$-module and J is an n-finitely presented ideal of $f(A)+J$. By a sequence (5), $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} J^r \bigoplus M^m \bowtie^{f^m} L_m$ is an n-finitely presented $(A \bowtie^f J)$-module. So, by induction J is an n-finitely presented ideal of $f(A)+J$ and L_m is an n-finitely presented $(f(A)+J)$-module.

Assume that $f(M) \subseteq J$. Then $\operatorname{Ker} u \cong L_m$ is an n-finitely presented $(f(A)+J)$--module.

Assume that $f^{-1}(J) = \{0\}$. Then $\operatorname{Ker} u \cong f^m(M^m) \bigoplus L_m$ is an n-finitely presented $(f(A)+J)$-modules since $f^m(M^m)$ is also an n-finitely presented $(f(A)+J)$--module by Lemma 2.3.

So $\operatorname{Ker} u$ is an n-finitely presented $(f(A)+J)$-module in the both cases. Consequently, K is an $(n+1)$-finitely presented $(f(A)+J)$-module by a sequence (4).

(4) Proceed by induction on n. The property is true for $n = 0$ by (4). Assume that the property is true for n, and assume that $U \bowtie^{f^p} J^p = \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f^p(u_i)+k_i)$ is an $(n+1)$-finitely presented $(A \bowtie^f J)$-module, where $u_i \in U$

and $k_i \in J^p$ for all $i \in \{1, \dots, r\}$. Clearly, $U = \sum_{i=1}^{i=r} Au_i$. We may assume that $\{(u_i, f^p(u_i)+k_i)_{i=1}^{i=r}\}$ is a minimal generating set of $U \bowtie^{f^p} J^p$. Consider the exact sequence of A-modules

$$0 \to \operatorname{Ker} v \to A^r \to U \to 0, \tag{6}$$

where $v((\alpha_i)_{i=1}^{i=r}) = \sum_{i=1}^{i=r} \alpha_i u_i$. On the other, hand consider the exact sequence of $(A \bowtie^f J)$-modules

$$0 \to \operatorname{Ker} w \to (A \bowtie^f J)^r \to U \bowtie^{f^p} J^p \to 0, \tag{7}$$

where

$$\begin{aligned} w((\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=r}) &= \sum_{i=1}^{i=r} (\alpha_i, f(\alpha_i)+j_i)(u_i, f^p(u_i)+k_i) \\ &= \Big(\sum_{i=1}^{i=r} \alpha_i u_i, \sum_{i=1}^{i=r} f(\alpha_i)(f^p(u_i)+k_i)\Big). \end{aligned}$$

Then

$$\operatorname{Ker} w = \Big\{(\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=r} \in (A \bowtie^f J)^r : \sum_{i=1}^{i=r} \alpha_i u_i = 0, \sum_{i=1}^{i=r} f(\alpha_i)(f^p(u_i)+k_i) = 0\Big\}.$$

So,

$$\operatorname{Ker} u \cong \Big\{((\alpha_i)_{i=1}^{i=r}, f^r((\alpha_i)_{i=1}^{i=r}) + (j_i)_{i=1}^{i=r}) \in A^r \bowtie^{f^r} J^r : \sum_{i=1}^{i=r} \alpha_i u_i = 0, \sum_{i=1}^{i=r} f(\alpha_i) k_i = 0\Big\}.$$

Since $A \bowtie^f J$ is a local ring by Proposition 2.1 and $\{(u_i, f^p(u_i)+k_i)_{i=1}^{i=r}\}$ is a minimal generating set of $U \bowtie^{f^p} J^p$, then $\operatorname{Ker} w \subseteq M^r \bowtie^{f^r} J^r$. So,

$$\operatorname{Ker} w \cong \Big\{((\alpha_i)_{i=1}^{i=r}, f^r((\alpha_i)_{i=1}^{i=r}) + (j_i)_{i=1}^{i=r}) \in A^r \bowtie^{f^r} J^r : (\alpha_i)_{i=1}^{i=r} \in \operatorname{Ker} v\Big\}.$$

Therefore, $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} J^r$.

Since $U \bowtie^{f^p} J^p$ is an $(n+1)$-finitely presented $(A \bowtie^f J)$-module, then $\operatorname{Ker} w$ is an n-finitely presented $(A \bowtie^f J)$-module (by the sequence (7)). So, $\operatorname{Ker} v$ is an n-finitely presented A-module and J is a finitely generated ideal of $(f(A)+J)$ by induction (since $\operatorname{Ker} v \subseteq M^r$). Thus, U is an $(n+1)$-finitely presented A-module (by the sequence (6)). Conversely, assume that U is an $(n+1)$-finitely presented A-module and J is a finitely generated ideal of $(f(A)+J)$, then $U \bowtie^{f^p} J^p$ is a finitely generated $(A \bowtie^f J)$-module by (2), and then $U \bowtie^{f^p} J^p = \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f^p(u_i)+k_i)$, where $u_i \in U$ and $k_i \in J^p$ for all $i \in \{1, \dots, r\}$. We may assume that $\{(u_i, f^p(u_i)+k_i)_{i=1}^{i=r}\}$ is a minimal generating set of $U \bowtie^{f^p} J^p$. It is obvious that $U = \sum_{i=1}^{i=r} Au_i$. Since U is an $(n+1)$-finitely presented A-module, then $\operatorname{Ker} v$ is an n-finitely presented A-module (by the sequence (6)). So, $\operatorname{Ker} w (\cong \operatorname{Ker} v \bowtie^{f^r} J^r)$ is an n-finitely presented $(A \bowtie^f J)$-module by induction, and then $U \bowtie^{f^p} J^p$ is an $(n+1)$-finitely presented $(A \bowtie^f J)$-module (by the sequence (7)), as desired. □

Proof of Theorem 2.2. Recall that R is a strong n-coherent ring if and only if every $(n-1)$-finitely presented submodule of a finitely generated free R-module is n-presented.

(1) Assume that $J \subseteq \mathrm{Rad}(B)$.

(a)(i) Assume that J is a finitely generated ideal of $f(A)+J$, and $A \bowtie^f J$ is a strong 2-coherent ring. Let $U := \sum_{i=1}^{i=r} Au_i$ be a finitely presented A-module, where $\{(u_i)_{i=1}^{i=r}\}$ is a minimal generating set of U and $u_i \in M^p$ for all $i = 1, \ldots, r$, then $U \bowtie^{f^p} 0 = \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f^p(u_i))$ is a finitely generated $(A \bowtie^f J)$-module, where $(u_i, f^p(u_i)) \in M^p \bowtie^{f^p} J^p$. Consider the exact sequence of A-modules

$$0 \to \mathrm{Ker}\, v \to A^r \to U \to 0, \tag{8}$$

where $v((\alpha_i)_{i=1}^{i=r}) = \sum_{i=1}^{i=r} \alpha_i u_i$. On the other, hand consider the exact sequence of $(A \bowtie^f J)$-modules:

$$0 \to \mathrm{Ker}\, w \to (A \bowtie^f J)^r \to U \bowtie^{f^p} 0 \to 0, \tag{9}$$

where

$$\begin{aligned} w((\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=r}) &= \sum_{i=1}^{i=r} (\alpha_i, f(\alpha_i)+j_i)(u_i, f^p(u_i)) \\ &= \Big(\sum_{i=1}^{i=r} \alpha_i u_i, \sum_{i=1}^{i=r} (f(\alpha_i)+j_i) f^p(u_i)\Big). \end{aligned}$$

Then $\mathrm{Ker}\, w = \{(\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=r} \in (A \bowtie^f J)^r : \sum_{i=1}^{i=r} \alpha_i u_i = 0\}$. So, $\mathrm{Ker}\, w \cong \mathrm{Ker}\, v \bowtie^{f^r} J^r$. By the sequence (8), $\mathrm{Ker}\, v$ is a finitely generated A-module. So, $\mathrm{Ker}\, w$ is a finitely generated $(A \bowtie^f J)$-module by Lemma 2.4 (2) since $\mathrm{Ker}\, v \subseteq M^r$, and then $U \bowtie^{f^p} 0$ is a finitely presented $(A \bowtie^f J)$-module by a sequence (9). Hence, $U \bowtie^{f^p} 0$ is a 2-finitely presented $(A \bowtie^f J)$-module since $A \bowtie^f J$ is strong 2-coherent ring. Therefore U is a 2-finitely presented A-module by Lemma 2.4 (3)(b)(i) since $U \subseteq M^p$. Thus, A is strong 2-coherent ring.

The same reasoning as in the proof of (1)(a)(i) shows that if $A \bowtie^f J$ is a 2-coherent ring, then so is A.

(ii) Assume that M is a finitely generated ideal of A, $f(M) \subseteq J$, and $A \bowtie^f J$ is a strong 2-coherent ring. By Proposition 2.1, $(f(A)+J, J)$ is a local ring. Let $K := \sum_{i=1}^{i=r} (f(A)+J)k_i$ be a finitely presented $(f(A)+J)$-module, where $\{(k_i)_{i=1}^{i=r}\}$ be a minimal generating set of K and $k_i \in J^p$ for all $i = 1, \ldots, r$, then $0 \bowtie^{f^p} K = \sum_{i=1}^{i=r} A \bowtie^f J(0, k_i)$ is a finitely generated $(A \bowtie^f J)$-module, where $(0, k_i) \in M^p \bowtie^{f^p} J^p$. Consider the exact sequence of $(f(A)+J)$-modules

$$0 \to \mathrm{Ker}\, u \to (f(A)+J)^r \to K \to 0, \tag{10}$$

where $u((f(\alpha_i)+j_i)_{i=1}^{i=r}) = \sum_{i=1}^{i=r} (f(\alpha_i)+j_i)k_i$. So,

$$\begin{aligned} \mathrm{Ker}\, u &= \Big\{(f(\alpha_i)+j_i))_{i=1}^{i=r} \in (f(A)+J)^r : \sum_{i=1}^{i=r} (f(\alpha_i)+j_i))k_i = 0\Big\} \\ &\cong f^r(M^r) + L_r = L_r, \end{aligned}$$

where $L_r := \{(j_i))_{i=1}^{i=r} \in (J)^r : \sum_{i=1}^{i=r} j_i k_i = 0\}$. So, L_r is a finitely generated $(f(A)+J)$-module. On the other hand, consider the exact sequence of $(A \bowtie^f J)$--modules

$$0 \to \operatorname{Ker} w \to (A \bowtie^f J)^r \to 0 \bowtie^{f^p} K \to 0, \tag{11}$$

where

$$w((\alpha_i, f(\alpha_i) + j_i)_{i=1}^{i=r}) = \sum_{i=1}^{i=r} (\alpha_i, f(\alpha_i) + j_i)(0, k_i) = \Big(0, \sum_{i=1}^{i=r} (f(\alpha_i) + j_i)k_i\Big).$$

Then

$$\operatorname{Ker} w = \Big\{(\alpha_i, f(\alpha_i) + j_i)_{i=1}^{i=r} \in (A \bowtie^f J)^r : \sum_{i=1}^{i=r} (f(\alpha_i) + j_i)k_i = 0\Big\}.$$

So, $\operatorname{Ker} w \cong M^r \bowtie^{f^r} L_r$ is a finitely generated $(A \bowtie^f J)$-module by Lemma 2.4 (2), and then $0 \bowtie^{f^p} K$ is a finitely presented $(A \bowtie^f J)$-module by the sequence (11). Hence, $0 \bowtie^{f^p} K$ is a 2-finitely presented $(A \bowtie^f J)$-module since $A \bowtie^f J$ is strong a 2-coherent ring. Therefore, K is a 2-finitely presented $(f(A) + J)$-module by Lemma 2.4 (3)(b)(iii) since $K \subseteq J^p$. Thus, $f(A) + J$ is a strong 2-coherent ring.

The same reasoning as in the proof of (1)(a)(ii) shows that if $A \bowtie^f J$ is 2-coherent ring, then so is $f(A) + J$.

(iii) Assume that M and J are finitely generated ideals of A and $f(A) + J$, respectively, $f(A) + J$ is a coherent ring, and either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$. If $A \bowtie^f J$ is (a strong) 2-coherent ring, then so is A by (i). Conversely, assume that A is a strong 2-coherent ring and let $W := \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f^p(u_i) + k_i)$ is a finitely presented $(A \bowtie^f J)$-module, where $u_i \in M^p$, and $k_i \in J^p$. We may assume that $\{(u_i, f^p(u_i) + k_i)\}$ be a minimal generating set of W. Set $U := \sum_{i=1}^{i=r} Au_i$, and consider the exact sequence of A-modules

$$0 \to \operatorname{Ker} v \to A^r \to U \to 0, \tag{12}$$

where $v((\alpha_i)_{i=1}^{i=r}) = \sum_{i=1}^{i=r} \alpha_i u_i$. On the other hand consider the exact sequence of $(A \bowtie^f J)$-modules

$$0 \to \operatorname{Ker} w \to (A \bowtie^f J)^r \to W \to 0, \tag{13}$$

where

$$\begin{aligned} w((\alpha_i, f(\alpha_i) + j_i)_{i=1}^{i=r}) &= \sum_{i=1}^{i=r} (\alpha_i, f(\alpha_i) + j_i)(u_i, f^p(u_i) + k_i) \\ &= \Big(\sum_{i=1}^{i=r} \alpha_i u_i, \sum_{i=1}^{i=r} (f(\alpha_i) + j_i)(u_i, f^p(u_i) + k_i\Big). \end{aligned}$$

Then

$$\operatorname{Ker} w = \Big\{(\alpha_i, f(\alpha_i) + j_i)_{i=1}^{i=r} \in (A \bowtie^f J)^r : \sum_{i=1}^{i=r} \alpha_i u_i = 0, \sum_{i=1}^{i=r} (f(\alpha_i) + j_i)k_i = 0\Big\}.$$

Since $\operatorname{Ker} w \subseteq M^r \bowtie^{f^r} J^r$, then $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} L_r$, where $L_r = \{(j_i)_{i=1}^{i=r} \in J^r : \sum_{i=1}^{i=r} j_i k_i = 0\}$. By the sequence (13), $\operatorname{Ker} w$ is a finitely generated $(A \bowtie^f J)$--module. Then, $\operatorname{Ker} v$ and L_r are finitely generated modules of A and $(f(A) + J)$, respectively, by Lemma 2.4(2) (since $\operatorname{Ker} v \subseteq M^r$ and $L_r \subseteq J^r$). So L_r is a finitely presented $(f(A) + J)$-module since $(f(A) + J)$ is a coherent ring, and U is a 2-finitely presented A-module since A is strong 2-coherent ring. Thus, $\operatorname{Ker} v$ is a finitely presented A-module by a sequence (12). Consequently, $\operatorname{Ker} w$ is a finitely presented $(A \bowtie^f J)$-module by Lemma 2.4 (3)(a)(ii). Therefore, W is a 2-finitely presented $(A \bowtie^f J)$-module by the sequence (13).

The same reasoning as in the proof of (1)(a)(iii) shows that $A \bowtie^f J$ is a 2-coherent ring if and only if so is A.

(b) Assume that $n > 2$.

(i) Assume that M is an $(n-3)$ finitely presented A-module, J is an $(n-2)$-finitely presented ideal of $f(A)+J$, $A \bowtie^f J$ is a strong n-coherent ring, and assume that either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$. Let $U := \sum_{i=1}^{i=r} Au_i$ be an $(n-1)$-finitely presented A-module, where $\{(u_i)_{i=1}^{i=r}\}$ be a minimal generating set of U and $u_i \in M^p$ for all $i = 1, \ldots, r$, then $U \bowtie^{f^p} 0 = \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f^p(u_i))$ is a finitely generated $(A \bowtie^f J)$-module, where $(u_i, f^p(u_i)) \in M^p \bowtie^{f^p} J^p$. By the sequence (1), $\operatorname{Ker} v$ is an $(n-2)$-finitely presented A-module. So, $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} J^r$ is an $(n-2)$-finitely presented $(A \bowtie^f J)$-module by Lemma 2.4 (3)(b)(ii) since $\operatorname{Ker} v \subseteq M^r$, and then $U \bowtie^{f^p} 0$ is an $(n-1)$-finitely presented $(A \bowtie^f J)$-module by a sequence (2). Hence, $U \bowtie^{f^p} 0$ is an n-finitely presented $(A \bowtie^f J)$-module since $A \bowtie^f J$ is strong n-coherent ring. Therefore U is an n-finitely presented A-module by Lemma 2.4(3)(b)(i) since $U \subseteq M^p$. Thus, A is strong an n-coherent ring.

The same reasoning as in the proof of (1)(b)(i) shows that if $A \bowtie^f J$ is an n-coherent ring, then so is A.

(ii) Assume that M is an $(n-2)$-finitely presented ideal of A, J is an $(n-3)$--finitely presented ideal of $f(A)+J$, $f(M) \subseteq J$, and $A \bowtie^f J$ is a strong n-coherent ring. Let $K := \sum_{i=1}^{i=r}(f(A)+J)k_i$ be an $(n-1)$-finitely presented $(f(A)+J)$--module, where $\{(k_i)_{i=1}^{i=r}\}$ is a minimal generating set of K and $k_i \in J^p$ for all $i = 1, \ldots, r$, then $0 \bowtie^{f^p} K = \sum_{i=1}^{i=r} A \bowtie^f J(0, k_i)$ is a finitely generated $(A \bowtie^f J)$-module, where $(0, k_i) \in M^p \bowtie^{f^p} J^p$. By the sequence (10), $\operatorname{Ker} u \cong L_r$ is an $(n-2)$-finitely presented$(f(A)+J)$-module. So, $\operatorname{Ker} w \cong M^r \bowtie^{f^r} L_r$ is an $(n-2)$--finitely presented $(A \bowtie^f J)$-module by Lemma 2.4(3)(b)(ii), and then $0 \bowtie^{f^p} K$ is an $(n-1)$-finitely presented $(A \bowtie^f J)$-module by the sequence (11). Hence, $0 \bowtie^{f^p} K$ is an n-finitely presented $(A \bowtie^f J)$-module since $A \bowtie^f J$ is a strong n-coherent ring. Therefore K is n-finitely presented A-module by Lemma 2.4(3)(b)(iii) since $K \subseteq J^p$. Thus, $f(A)+J$ is strong an n-coherent ring.

The same reasoning as in the proof of (1)(b)(ii) shows that if $A \bowtie^f J$ is an n-coherent ring, then so is $f(A)+J$.

(iii) Assume that M and J are $(n-2)$-finitely presented ideals of A and $(f(A)+J)$, respectively, $f(M) \subseteq J$, and $f(A)+J$ is a strong $(n-1)$-coherent ring. If $A \bowtie^f J$ is (a strong) n-coherent ring, then so is A by (b)(i). Conversely, assume that A is a strong n-coherent ring. Let $W := \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f^p(u_i)+k_i)$ be an $(n-1)$-finitely presented $(A \bowtie^f J)$-module, where $u_i \in M^p$, and $k_i \in J^p$, and set $U := \sum_{i=1}^{i=r} Au_i$. We may assume that $\{(u_i, f^p(u_i)+k_i)\}$ be a minimal generating set of W. By the sequence (13), $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} L_r$ is an $(n-2)$-finitely presented $(A \bowtie^f J)$-module, where $L_r = \{(j_i)_{i=1}^{i=r} \in J^r : \sum_{i=1}^{i=r} j_i k_i = 0\}$. Then, $\operatorname{Ker} v$ and L_r are an $(n-2)$-finitely presented modules of A and $(f(A)+J)$, respectively, by Lemma 2.4(3)(b)(iii). (since $\operatorname{Ker} v \subseteq M^r$ and $L_r \subseteq J^r$). So L_r is an $(n-1)$-finitely presented $(f(A)+J)$-module since $(f(A)+J)$ is a strong $(n-1)$-coherent ring, and U is an n-finitely presented A-module since A is strong n-coherent ring. Thus, $\operatorname{Ker} v$ is an $(n-1)$-finitely presented A-module by the sequence (12). Consequently, $\operatorname{Ker} w$ is an $(n-1)$-finitely presented $(A \bowtie^f J)$-module by Lemma 2.4(3)(b)(ii). Therefore, W is an n-finitely presented $(A \bowtie^f J)$-module by the sequence (13).

The same reasoning as in the proof of (1)(b)(iii) shows that $A \bowtie^f J$ is an n-coherent ring if and only if so is A.

(2) Assume that $n \geq 2$, $J^2 = 0$, and J is a finitely generated ideal of $(f(A)+J)$. Assume that $A \bowtie^f J$ is strong n-coherent ring and let $U := \sum_{i=1}^{i=r} Au_i$ be a $(n-1)$--finitely presented A-module, where $u_i \in M^p$, then $W := U \bowtie^{f^p} J^p$ is an $(n-1)$--finitely presented $(A \bowtie^f J)$-module by Lemma 2.4(4). So, W is an n-finitely presented $(A \bowtie^f J)$-module since $A \bowtie^f J$ is a strong n-coherent ring. Therefore, U is an n-finitely presented A-module by Lemma 2.4(4). Thus, A is strong an n-coherent ring. Conversely, assume that A is a strong n-coherent ring and let $W := \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f^p(u_i)+k_i)$ be an $(n-1)$-finitely presented $(A \bowtie^f J)$-module, where $u_i \in M^p$, and $k_i \in J^p$ and set $U := \sum_{i=1}^{i=r} Au_i$. We may assume that $\{(u_i, f^p(u_i)+k_i)\}$ is a minimal generating set of W. By the sequence (13), $\operatorname{Ker} w \cong \operatorname{Ker} v \bowtie^{f^r} J^r$ (since $L_r = J^r$) is an $(n-2)$-finitely presented $(A \bowtie^f J)$--module. Then, $\operatorname{Ker} v$ is an $(n-2)$-finitely presented A-module by Lemma 2.4(4) (since $\operatorname{Ker} v \subseteq M^r$). So, by a sequence (12), U is an n-finitely presented A-module since A is a strong n-coherent ring. Thus, $\operatorname{Ker} v$ is an $(n-1)$-finitely presented A-module by the sequence (12). Consequently, $\operatorname{Ker} w$ is an $(n-1)$-finitely presented $(A \bowtie^f J)$-module by Lemma 2.4(4). Thus, W is an n-finitely presented $(A \bowtie^f J)$--module by the sequence (13).

The same reasoning as in the proof of (2) shows that $A \bowtie^f J$ is n-coherent ring if and only if so is A, and this completes the proof of Theorem 2.2. □

The following corollaries are an immediate consequence of Theorem 2.2.

Corollary 2.5. *Let (A, M) be a local ring, $f\colon A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $f(M)J = 0$.*

(1) *Assume that $J \subseteq \operatorname{Rad}(B)$.*
   - (a) *Assume that either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$, M and J are finitely generated ideals of A and $f(A)+J$, respectively, and $f(A)+J$ is a coherent ring. Then $A \bowtie^f J$ is a (strong) 2-coherent ring which is not coherent ring provided A is.*
   - (b) *Assume that $f(M) \subseteq J$, $n > 2$, M and J are $(n-2)$-finitely presented ideals of A and $f(A)+J$, respectively, and $f(A)+J$ is a strong $(n-1)$--coherent ring. Then $A \bowtie^f J$ is a (strong) n-coherent ring which is not a (strong) $(n-1)$-coherent ring provided A is.*

(2) *Assume that $J^2 = 0$, $n \geq 2$, and J is a finitely generated ideal of $(f(A)+J)$. Then $A \bowtie^f J$ is a (strong) n-coherent ring which is not a (strong) $(n-1)$--coherent ring provided A is.*

Proof. (1)(a) Follows immediately from Theorem 2.2(1)(a)(iii) and [**18**, Theorem 4.1.5].
(b) For $n > 3$, the proof follows immediately from Theorem 2.2(1)(b)(iii).
For $n = 3$, the proof follows immediately from Theorem 2.2(1)(b)(iii) and (1)(a)(i).

(2) For $n > 2$, the proof follows immediately from Theorem 2.2(2).

For $n = 2$, the proof follows immediately from Theorem 2.2 (2) and [**18**, Theorem 4.1.5]. □

Corollary 2.6. *Let (A, M) be a local ring, I be a finitely generated ideal of A such that $MI = 0$, and $n \geq 2$. Then $A \bowtie I$ is a (strong) n-coherent ring which is not a (strong) $(n-1)$-coherent ring provided A is.*

Theorem 2.2 enriches the literature with new examples of n-coherent rings that are not $(n-1)$-coherent rings ($n \geq 2$).

Example 2.7. Let $T := K + M$ be a Bézout domain, where K is a field and M is a nonzero maximal ideal of T, D is a subring of K, the quotient field of D is $k = qf(D) \subseteq K$, $R = D + M$, and $T_0 = k + M$. Let $m(\neq 0) \in M$, and consider the canonical ring homomorphism $f\colon T_0 \to T_0/M^2$ $(f(x) = \overline{x})$. Assume that either $[K : k] = \infty$ or $1 \neq [K : k] < \infty$ and M is not a principal ideal of T. Then, by Corollary 2.5 (2), $T_0 \bowtie^f (\overline{m})$ is a 2-coherent ring which is not a coherent ring since T_0 is by [**15**, Theorem 2.1].

Example 2.8. Let (R, m) be a local 2-coherent domain which is not a field, $K = qf(R)$, $A := R \propto K$ be the trivial ring extension of R by K and M its maximal ideal, and let E be an $\frac{A}{M}$ vector space with finite rank. Set $B := A \propto E$, $J := 0 \propto E$, and consider the ring homomorphism $f\colon A \to B$ $(f(x) = (x, 0))$. Then, by Corollary 2.5 (2), $A \bowtie^f J$ is a 2-coherent ring which is not a coherent ring since A is by [**21**, Theorem 3.1].

Example 2.9. Let (V, m) be a non discrete valuation domain, $A := V \propto \frac{V}{m}$ and $M := m \propto \frac{V}{m}$ its maximal ideal, $B := \frac{A}{M^2} \propto E$, where E is an $\frac{\frac{A}{M^2}}{\frac{M}{M^2}}$ vector space with finite dimension. Let $J := (\overline{m}) \propto E$, where $m(\neq 0) \in M$, and consider the ring homomorphism $f\colon A \to B$ $(f(x) = (\overline{x}, 0))$. Then, by Corollary 2.5(2), $A \bowtie^f J$ is a 3-coherent ring which is not a 2-coherent ring since A is by [**21**, Example 3.8].

Example 2.10. Let V be a non-Noetherian valuation ring with $rank(V) > 1$. Let $A = V[[T]]$ be the power series ring in one variable T and M its maximal ideal. Set $B := (A \propto \frac{A}{M}) \times \frac{A}{M^2}$, $J := (0 \propto \frac{A}{M}) \times \frac{M}{M^2}$, and consider the ring homomorphism $f\colon A \to B$ $(f(x) = ((x, 0), 0))$. Then, by Corollary 2.5(2) $A \bowtie^f J$ is a 2-coherent ring which is not a coherent ring since A is by [**7**, Example 4.4].

Example 2.11. Let K be a field, E be a K-vector space of infinite dimension, $A := K \propto E$ be the trivial extension ring of K by E, and $I := 0 \propto E^{'}$, where $E^{'}$ is a finitely generated K-subspace of E. Then, by Corollary 2.6 $A \bowtie I$ is a 2-coherent ring wich is not a coherent ring since A is by [**22**, Theorem 3.4] and [**21**, Theorem 2.6].

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K. A. Ismaili, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco, *e-mail*: alaouikarima2012@hotmail.fr

N. Mahdou, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco, *e-mail*: mahdou@hotmail.com