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A FEW REMARKS ON QUADRATIC HAMILTON-POISSON SYSTEMS ON THE HEISENBERG LIE-POISSON SPACE

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Abstract. Positive semidefinite quadratic Hamilton-Poisson systems on the three-dimensional Heisenberg Lie-Poisson space are classified. Stability and integration of each normal form are briefly covered. The relation of these systems to optimal control is also briefly discussed.

1. Introduction

The dual of a Lie algebra admits a natural Poisson structure, namely the Lie-Poisson structure. Lie-Poisson structures are in a one-to-one correspondence with linear Poisson structures [**16**]. Many interesting dynamical systems are naturally expressed as quadratic Hamilton-Poisson systems on Lie-Poisson spaces (see, e.g., [**13, 14, 15, 17**]). In the last decade or so, quadratic Hamilton-Poisson systems on some low-dimensional Lie-Poisson spaces have been investigated by several authors (see, e.g., [**1, 2, 3, 5, 6, 7, 9, 10, 12, 19**]).

In this paper, we classify the positive semidefinite quadratic Hamilton-Poisson systems on the three-dimensional Heisenberg Lie-Poisson space $\mathfrak{h}_3^*$; we show that there are only five systems up to affine equivalence. (Two systems are affinely equivalent if their Hamiltonian vector fields are compatible with an affine isomorphism.) This classification is based on the classification of the homogeneous positive semidefinite quadratic Hamilton-Poisson systems on $\mathfrak{h}_3^*$ [**11**]. Remarkably, most inhomogeneous systems are affinely equivalent to homogeneous systems.

It turns out that those quadratic Hamilton-Poisson systems on $\mathfrak{h}_3^*$ which are associated to invariant optimal control problems on the Heisenberg group H_3 are all equivalent; we briefly expand this point in the last section. We also tabulate the integral curves, equilibria and their stability nature for each of the five systems.

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2. Preliminaries

The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ admits a natural Poisson structure, the (minus) *Lie-Poisson structure* (see, e.g., [**17**]), given by

$$\{F, G\}(p) = -p\left([\mathrm{d}F(p), \mathrm{d}G(p)]\right),$$

where $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. Here $[\cdot, \cdot]$ denotes the Lie bracket on $\mathfrak{g}$; $\mathrm{d}F$ and $\mathrm{d}G$ are linear functions on $\mathfrak{g}^*$ and are therefore identified with elements of $\mathfrak{g}$. A *linear Poisson automorphism* is a linear isomorphism $\psi\colon \mathfrak{g}^* \to \mathfrak{g}^*$ that preserves the Poisson bracket, i.e., $\left\{F, G\right\} \circ \psi = \left\{F \circ \psi, G \circ \psi\right\}$ for $F, G \in C^\infty(\mathfrak{g}^*)$. Linear Poisson automorphisms are exactly the dual maps of the Lie algebra automorphisms. The *Hamiltonian vector field* $\vec{H}$ associated to a (Hamiltonian) function $H \in C^\infty(\mathfrak{g}^*)$ is defined by $\vec{H}[F] = \left\{F, H\right\}$ for $F \in C^\infty(\mathfrak{g}^*)$. We follow the custom of identifying a Hamilton-Poisson system with its Hamiltonian. A function $C \in C^\infty(\mathfrak{g}^*)$ is a *Casimir* function if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$.

A *quadratic Hamilton-Poisson system* on $\mathfrak{g}^*$ is a Hamilton-Poisson system for which its Hamiltonian is the sum of a linear function and a quadratic form on $\mathfrak{g}^*$; we write $H_{A,\mathcal{Q}} = L_A + \mathcal{Q}$, where $A \in \mathfrak{g}$, $L_A(p) = p(A)$, and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$. We consider only those systems for which $\mathcal{Q}$ is positive semidefinite (this restriction is motivated by considerations from optimal control). If $A = 0$, then the system is called *homogeneous* and denoted by $H_\mathcal{Q}$; if $A \neq 0$, then the system is called *inhomogeneous*.

Let $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ be quadratic Hamilton-Poisson systems on $\mathfrak{g}^*$. $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are *affinely equivalent* (or A-equivalent for short) if their associated Hamiltonian vector fields are compatible with an affine isomorphism, i.e., there exists an affine isomorphism $\psi\colon \mathfrak{g}^* \to \mathfrak{g}^*$ such that the push-forward $\psi_* \vec{H}_{A,\mathcal{Q}}$ equals $\vec{H}_{B,\mathcal{R}}$. It is easy to show that the following systems are all A-equivalent to $H_{A,\mathcal{Q}}$:

($\mathfrak{E}1$) $H_{A,\mathcal{Q}} \circ \psi$, for any linear Poisson automorphism $\psi\colon \mathfrak{g}^* \to \mathfrak{g}^*$.
($\mathfrak{E}2$) $H_{A,\mathcal{Q}} + C$, for any Casimir function $C\colon \mathfrak{g}^* \to \mathbb{R}$.
($\mathfrak{E}3$) $H_{A,r\mathcal{Q}}$, for any $r \neq 0$.

Proposition 2.1 (cf. [**7**])**.** *Let $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ be inhomogeneous quadratic Hamilton systems on $\mathfrak{g}^*$. If $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{B,\mathcal{R}}$, then $H_\mathcal{Q}$ is A-equivalent to $H_\mathcal{R}$.*

3. The Heisenberg Lie-Poisson space

The Heisenberg Lie algebra $\mathfrak{h}_3$ is the only three-dimensional two-step nilpotent Lie algebra (up to isomorphism). We find it convenient to represent $\mathfrak{h}_3$ as a matrix Lie algebra

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} = x_1 E_1 + x_2 E_2 + x_3 E_3 \ : \ x_1, x_2, x_3 \in \mathbb{R} \right\}$$

with commutator relations

$$[E_2, E_3] = E_1, \qquad [E_3, E_1] = 0, \qquad [E_1, E_2] = 0.$$

Let (E_1^*, E_2^*, E_3^*) denote the dual of the standard basis. We identify any element $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$ of $\mathfrak{h}_3^*$ with the row-vector $p = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. With respect to the ordered basis (E_1^*, E_2^*, E_3^*), the group of linear Poisson automorphisms of $\mathfrak{h}_3^*$ is given by

$$\left\{ p \mapsto p \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \; : \; v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \; v_2 w_3 - v_3 w_2 \neq 0 \right\}.$$

The equations of motion $\dot{p} = \vec{H}(p)$ for any Hamiltonian $H \in C^\infty(\mathfrak{h}_3)$ are given by

$$\dot{p}_1 = 0, \qquad \dot{p}_2 = -p_1 \frac{\partial H}{\partial p_3}, \qquad \dot{p}_3 = p_1 \frac{\partial H}{\partial p_2}.$$

We note that $C \colon \mathfrak{h}_3^* \to \mathbb{R}$, $C(p) = p_1$ is a Casimir function.

4. Classification

We classify the positive semidefinite quadratic Hamilton-Poisson systems on $\mathfrak{h}_3^*$. The homogeneous systems were classified in [**11**]: Any homogeneous positive semidefinite quadratic Hamilton-Poisson system on $\mathfrak{h}_3^*$ is A-equivalent to exactly one of the systems

$$H_0(p) = 0, \quad H_1(p) = \frac{1}{2} p_2^2, \quad H_2(p) = \frac{1}{2}\left(p_2^2 + p_3^2\right).$$

In fact, it can be shown that the following somewhat stronger result holds.

Lemma 4.1 (cf. [**11**])**.** *Let $H_\mathcal{Q}$ be a positive semidefinite quadratic Hamilton-Poisson system on $\mathfrak{h}_3^*$. There exist a linear Poisson automorphism ψ and constants $r, k \in \mathbb{R}, r > 0$ such that $rH_\mathcal{Q} \circ \psi + kC^2 = H_i$ for exactly one index $i \in \{0, 1, 2\}$.*

Let $\mathsf{S}(H_i)$ denote the subgroup of linear Poisson automorphisms $\psi \colon \mathfrak{h}_3^* \to \mathfrak{h}_3^*$ satisfying $H_i \circ \psi = rH_i + kC^2$ for some $r > 0$ and $k \in \mathbb{R}$. It is not difficult to show that the system $L_B + H_i$ is A-equivalent to $L_B \circ \psi + H_i$ for any $\psi \in \mathsf{S}(H_i)$.

Lemma 4.2. *The subgroups $\mathsf{S}(H_i)$, $i \in \{0, 1, 2\}$, are given by*

$$\mathsf{S}(H_0) = \left\{ \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \; : \; \begin{array}{c} v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \\ v_2 w_3 - w_2 v_3 \neq 0 \end{array} \right\},$$

$$\mathsf{S}(H_1) = \left\{ \begin{bmatrix} v_2 w_3 & 0 & w_1 \\ 0 & v_2 & w_2 \\ 0 & 0 & w_3 \end{bmatrix} \; : \; v_2, w_1, w_2, w_3 \in \mathbb{R}, \; v_2 w_3 \neq 0 \right\},$$

$$\mathsf{S}(H_2) = \left\{ \begin{bmatrix} \sigma(v_2^2 + v_3^2) & 0 & 0 \\ 0 & v_2 & -\sigma v_3 \\ 0 & v_3 & \sigma v_2 \end{bmatrix} \; : \; \begin{array}{c} v_2, v_3 \in \mathbb{R}, \\ v_2^2 + v_3^2 \neq 0, \; \sigma = \pm 1 \end{array} \right\}.$$

Proof. The case $\mathsf{S}(H_0)$ is trivial. We determine $\mathsf{S}(H_1)$; the proof for $\mathsf{S}(H_2)$ is similar. Let

$$\psi\colon p \mapsto p \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}, \quad v_2w_3 - w_2v_3 \neq 0.$$

We have

$$(H_1 \circ \psi)(p) = \frac{1}{2} p \begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_1v_2 & v_2^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & v_3^2 \end{bmatrix} p^\top.$$

On the other hand,

$$rH_1(p) + kC^2(p) = p \begin{bmatrix} k & 0 & 0 \\ 0 & \frac{1}{2}r & 0 \\ 0 & 0 & 0 \end{bmatrix} p^\top.$$

Therefore, if $\psi \in \mathsf{S}(H_1)$, then $v_1 = v_3 = 0$, and so ψ is of the form given. If $v_1 = v_3 = 0$, then $(H_1 \circ \psi)(p) = v_2^2 H_1(p)$, and so $\psi \in \mathsf{S}(H_1)$. □

Theorem 4.3. *Any positive semidefinite quadratic Hamilton-Poisson system on* $\mathfrak{h}_3^*$ *is A-equivalent to exactly one of the following systems*

$$\begin{gathered} H_0(p) = 0, \qquad H_0'(p) = p_2, \qquad H_1(p) = \frac{1}{2}p_2^2 \\ H_1'(p) = p_3 + \frac{1}{2}p_2^2, \qquad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2). \end{gathered} \tag{4.1}$$

Proof. Let $H_{A,\mathcal{Q}}$ be a quadratic Hamilton Poisson system. By ($\mathfrak{E}$1), ($\mathfrak{E}$2), ($\mathfrak{E}$3) and Lemma 4.1, we have that $H_{A,\mathcal{Q}}$ is A-equivalent to a system $H = L_B + H_i$ for some $B \in \mathfrak{h}_3$ and $i \in \{0, 1, 2\}$. Furthermore, by Proposition 2.1, $L_B + H_i$ is not A-equivalent to $L_{\bar{B}} + H_j$ for any $\bar{B} \in \mathfrak{h}_3$ when $i \neq j$. Hence there are three cases to consider, namely, $H = L_B + H_i$, $i \in \{0, 1, 2\}$.

(Case $i = 0$) Let $H = L_B + H_0$ and let $B = \sum_{i=1}^3 b_i E_i$. If $b_2 = b_3 = 0$, then H is A-equivalent to the system $H_0(p) = 0$ (as $H = H_0 + K$ where $K(p) = -b_1p_1$ is a Casimir function). Suppose $b_2^2 + b_3^2 \neq 0$. Then

$$\psi\colon p \mapsto p \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{b_2}{b_2^2+b_3^2} & \frac{b_3}{b_2^2+b_3^2} \\ 0 & b_3 & -b_2 \end{bmatrix}$$

is an element of $\mathsf{S}(H_0)$ such that $L_B \circ \psi = L_{b_1E_1+E_2}$. Hence H is A-equivalent to $H_0' = p_2$. The systems H_0 and H_0' are not A-equivalent.

(Case $i = 1$) Let $H = L_B + H_1$ and let $B = \sum_{i=1}^3 b_i E_i$. If $b_2 = b_3 = 0$, then H is A-equivalent to the system $H_1(p) = \frac{1}{2}p_2^2$. Suppose $b_3 = 0, b_2 \neq 0$. Then

$$\psi\colon p \mapsto p \begin{bmatrix} \frac{1}{b_2} & 0 & 0 \\ 0 & \frac{1}{b_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an element of $\mathsf{S}(H_1)$ such that $L_B \circ \psi = L_{\frac{b_1}{b_2}E_1+E_2}$. Hence H is A-equivalent to $G(p) = p_2 + \frac{1}{2}p_2^2$. On the other hand, suppose $b_3 \neq 0$. Then

$$\psi\colon p \mapsto p \begin{bmatrix} -1 & 0 & 0 \\ 0 & -b_3 & b_2 \\ 0 & 0 & \frac{1}{b_3} \end{bmatrix}$$

is an element of $\mathsf{S}(H_1)$ such that $L_B \circ \psi = L_{-b_1E_1+E_3}$. Hence H is A-equivalent to $H_1'(p) = p_3 + \frac{1}{2}p_2^2$.

The systems H_1 and G are A-equivalent.
Indeed, $\psi\colon \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \mapsto \begin{bmatrix} p_1 & p_2 - 1 & p_3 \end{bmatrix}$ is an affine isomorphism such that $T_p\psi \cdot \vec{H}_1(p) = \vec{G} \circ \psi(p)$ (here $T_p\psi$ denotes the tangent map of ψ at p). However, the systems H_1 and H_1' are not A-equivalent. Indeed, suppose there exists an affine isomorphism $\psi\colon p \mapsto p\Psi + q, \Psi = [\Psi_{ij}]$ such that $\Psi \cdot \vec{H}_1' = \vec{H}_1 \circ \psi$. Then

$$\begin{cases} -\psi_{12}p_1 + \psi_{13}p_1p_2 &= 0 \\ -\psi_{22}p_1 + \psi_{23}p_1p_2 &= 0 \\ -\psi_{32}p_1 + \psi_{33}p_1p_2 &= \Big(\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1\Big) \\ & \quad \times\Big(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3 + q_2\Big) \end{cases}$$

for all $p \in \mathfrak{h}_3^*$. By inspection we have $\psi_{12}, \psi_{13}, \psi_{22}, \psi_{23} = 0$, hence $\det\Psi = 0$, a contradiction.

(Case $i = 2$) Let $H = L_B + H_2$ and let $B = \sum_{i=1}^{3} b_iE_i$. If $b_2 = b_3 = 0$, then H is A-equivalent to the system $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$. Suppose $b_2^2 + b_3^2 \neq 0$. Then

$$\psi_1 : p \mapsto p \begin{bmatrix} \frac{1}{b_2^2+b_3^2} & 0 & 0 \\ 0 & \frac{b_2}{b_2^2+b_3^2} & \frac{b_3}{b_2^2+b_3^2} \\ 0 & -\frac{b_3}{b_2^2+b_3^2} & \frac{b_2}{b_2^2+b_3^2} \end{bmatrix}$$

is an element of $\mathsf{S}(H_2)$ such that $L_B \circ \psi = L_{\frac{b_1}{b_2^2+b_3^2}E_1+E_2}$. Hence H is A-equivalent to $G(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$. The systems H_2 and G are A-equivalent. Indeed, $\psi\colon \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \mapsto \begin{bmatrix} p_1 & p_2 - 1 & p_3 \end{bmatrix}$ is an affine isomorphism such that $T_p\psi \cdot \vec{H}_2(p) = \vec{G} \circ \psi(p)$. □

5. Control, stability and integration

5.1. Optimal control

Given a (matrix) Lie group G, a *left-invariant control system* on G can be viewed as a family of left-invariant vector fields on G parametrized by controls. In classical notation, such a system is written as

$$\dot{g} = g\,\Xi(u), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell,$$

where the parametrization map $\Xi\colon \mathbb{R}^\ell \to \mathfrak{g}$ is an embedding and $\mathfrak{g}$ is the Lie algebra of G. An *optimal control problem* associated to such a system consists in minimizing a cost functional $\mathcal{J}(u(\cdot)) = \int_0^T L(u(t))\,\mathrm{d}t$ over all trajectories of the system subject to appropriate boundary conditions; here the Lagrangian $L\colon \mathbb{R}^\ell \to \mathbb{R}$ is smooth. Standard references for geometric control theory are [**4, 15**] (see also [**18**]).

When the parametrization map Ξ is affine and the Lagrangian L is a positive definite quadratic form, then one associates to the problem (via Pontryagin Maximum Principle) a positive semidefinite quadratic Hamilton-Poisson system on the Lie-Poisson space $\mathfrak{g}^*$. (For a normal extremal $(g(t), u(t))$, the control $u(t)$ is linearly related to an integral curve of this Hamiltonian.) Remarkably, for any such optimal control problem on the Heisenberg group H_3, it turns out that the associated Hamiltonian is A-equivalent to $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$ (cf. [**8**]).

5.2. Stability and integration

It is a simple matter to find the integral curves of the Hamilton-Poisson systems (4.1). It is also easy to determine the (Lyapunov) stability nature of the equilibria. (Instability does not follow from spectral stability but can be proven directly, whereas the energy-Casimir method is sufficient for showing stability). A summary is given in Table 1.

Table 1. Integral curves, equilibria and stability nature of equilibria (here equilibria are parametrized by $\eta, \mu \in \mathbb{R}$).

System	Integral curves	Equilibria	Stability
p_2	$(c_1, c_2, c_1 t + c_3)$	$(0, \eta, \mu)$	unstable
$\frac{1}{2}p_2^2$	$(c_1, c_2, c_1 c_2 t + c_3)$	$(0, \eta, \mu)$ $(\eta, 0, \mu)$	unstable unstable
$p_3 + \frac{1}{2}p_2^2$	$(c_1, -c_1 t + c_2, -\frac{1}{2}c_1^2 t^2 + c_1 c_2 t + c_3)$	$(0, \eta, \mu)$	unstable
$\frac{1}{2}(p_2^2 + p_3^2)$	$\big(c_1, c_2\cos(c_1 t) - c_3\sin(c_1 t),$ $c_3\cos(c_1 t) + c_2\sin(c_1 t)\big)$	$(0, \eta, \mu) \neq 0$ $(\mu, 0, 0)$	unstable stable

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