Acta Math. Univ. Comenianae
Vol. LXXXVI, 1 (2017), pp. 81–90

ON $N(k)$-CONTACT METRIC MANIFOLDS ADMITTING A TYPE OF A SEMI-SYMMETRIC NON-METRIC CONNECTION

A. BARMAN

Abstract. The object of the present paper is to study $N(k)$-contact metric manifolds admitting a type of a semi-symmetric non-metric connection.

1. Introduction

In 1988, Tanno ([**20**], [**21**]) introduced the notion of k-nullity distribution on a contact metric manifold. The k-nullity distribution of a Riemannian manifold (M, g) for a real number k is a distribution

$$N(k)\colon p \to N_p(k) = [Z \in \chi_p(M) : R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}] \tag{1.1}$$

for any $X, Y, Z \in \chi_p(M)$ and k being a constant, where R denotes the Riemannian curvature tensor and $\chi_p(M)$ denotes the tangent vector space of M^{2n+1} at any point $p \in M$.

If the characteristic vector field of a contact metric manifold belongs to the k-nullity distribution, then the relation

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] \tag{1.2}$$

holds. A contact metric manifold with $\xi \in N(k)$ is called a $N(k)$-contact metric manifold. Thus an $N(k)$-contact metric manifold is a contact metric manifold satisfying the relation (1.2). From (1.1) and (1.2), it follows that an $N(k)$-contact metric manifold is Sasakian if and only if $k = 1$.

In a recent paper [**17**], Majhi and De studied the classifications on $N(k)$-contact metric manifolds satisfying certain curvature conditions. The $N(k)$-contact metric manifolds have been also studied by several authors such as De and Gazi [**12**], Blair ([**6**], [**7**]), Blair, Koufogiorgos and Papantoniou [**9**], Ghosh, De and Taleshian [**14**] *Özgür* and Sular [**18**] and many others.

In 1924, Friedmann and Schouten [**13**] introduced the idea of a semi-symmetric connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor $\widetilde{T}$ of

Received September 5, 2015; revised July 14, 2016.
2000 *Mathematics Subject Classification.* Primary 53C15, 53C25.
Key words and phrases. $N(k)$-contact metric manifold; semi-symmetric non-metric connection; semisymmetric; Ricci semi-symmetric; Einstein manifold; η-Einstein manifold.

the connection $\widetilde{\nabla}$ satisfies $\widetilde{T}(X,Y) = u(Y)X - u(X)Y$, where u is a 1-form and ρ is a vector field defined by $u(X) = g(X,\rho)$ for all vector fields $X \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1932, Hayden [**15**] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M,g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if $\widetilde{\nabla}g = 0$.

After a long gap the study of a semi-symmetric connection $\overline{\nabla}$ satisfying

$$\overline{\nabla}g \neq 0 \tag{1.3}$$

was initiated by Prvanovič [**19**] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [**2**]. A semi-symmetric connection $\overline{\nabla}$ is said to be a semi-symmetric non-metric connection if it satisfies the condition (1.3).

In 1992, Agashe and Chafle [**1**] studied a semi-symmetric non-metric connection $\overline{\nabla}$, whose torsion tensor $\overline{T}$ satisfies $\overline{T}(X,Y) = u(Y)X - u(X)Y$ and $(\overline{\nabla}_X g)(Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y)$. They proved that the projective curvature tensor of the manifold with respect to these two connections are equal to each other.

The semi-symmetric non-metric connection has been further developed by several authors such as Barman ([**3**], [**4**]), Barman and De [**5**], De and Biswas [**10**], De and Kamilya [**11**], Liang [**16**] and many others.

A Riemannian manifold M is said to be a semisymmetric manifold if the relation

$$\bar{R}(X,Y) \cdot \bar{R} = 0$$

holds and a Riemannian manifold M is said to be a Ricci-semisymmetric manifold if the relation

$$\bar{R}(X,Y) \cdot \bar{S} = 0$$

holds, where $\bar{R}(X,Y)$ is the curvature operator, $\bar{R}$ and $\bar{S}$ denotes the curvature tensor and the Ricci tensor of the $N(k)$-contact metric manifold with respect to the semi-symmetric non-metric connection, respectively.

In this paper, we study a type of a semi-symmetric non-metric connection due to Agashe and Chafle [**1**] on $N(k)$-contact metric manifolds. The paper is organized as follows: After introduction in Section 2, we give a brief account of the $N(k)$-contact metric manifolds. In Section 3, we study the semi-symmetric non-metric connection on Riemannian manifolds. Section 4 is devoted to obtain the relation between the curvature tensor with respect to the semi-symmetric non-metric connection and the Levi-Civita connection. In the next section, we study $\bar{R} \cdot \bar{S} = 0$ in an $N(k)$-contact metric manifold with respect to the semi-symmetric non-metric connection. In Section 6, we investigate a semisymmetric condition in an $N(k)$-contact metric manifold with respect to the semi-symmetric non-metric connection and prove that the manifold is an η-Einstein manifold. Finally, we construct an example of a 3-dimensional $N(k)$-contact metric manifold admitting the semi-symmetric non-metric connection whose curvature tensor satisfies the skew-symmetric property in Section 4, and also supports the result obtained in Section 5.

2. N(k)-contact metric manifolds

A $(2n+1)$-dimensional manifold M is called an almost contact Riemannian manifold if either its structural group can be reduced to $U(n)\times 1$ or equivalently, there is an almost contact structure (ϕ,ξ,η) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \qquad g(X,\xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = 1, \qquad \phi(\xi) = 0, \qquad \eta(\phi(X)) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.4}$$

for any vector fields $X, Y \in \chi(M)$ [**7**].

On a contact metric manifold the relation

$$-\operatorname{div}\xi = \sum_{i=1}^{2n+1} g(\phi e_i, e_i) + \sum_{i=1}^{2n+1} g(\phi h e_i, e_i) = 0. \tag{2.5}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.6}$$

where $h = \frac{1}{2}\pounds_\xi \phi$, $\pounds$ denotes the Lie differentiation holds.

In an N(k)-contact metric manifold M^{2n+1} the following relations hold ([**8**], [**9**]):

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.7}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{2.8}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.9}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(X,\xi)Y = k[\eta(Y)X - g(X,Y)\xi], \tag{2.11}$$

$$\begin{aligned} S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad + 2[nk - (n-1)]\eta(X)\eta(Y), \qquad n \geq 1, \end{aligned} \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y), \tag{2.13}$$

$$S(Y,\xi) = 2kn\eta(X), \tag{2.14}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.15}$$

$$(\nabla_X h)(Y) = [(1-k)g(X,\phi Y) + g(X, h\phi Y)]\xi + \eta(Y)[h(\phi X + \phi h X)] \tag{2.16}$$

where R, S and r are the curvature tensor, the Ricci tensor and scalar curvature respectively with respect to the Levi-Civita connection.

3. Semi-symmetric non-metric connection

Let M be a $(2n+1)$-dimensional Riemannian manifold with the Levi-Civita connection ∇. If $\overline{\nabla}$ is the semi-symmetric non-metric connection of a Riemannian manifold M, a linear connection $\overline{\nabla}$ is given by [**1**]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \tag{3.1}$$

Using (3.1), the torsion tensor T of M with respect to the connection $\overline{\nabla}$ is given by

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = \eta(Y)X - \eta(X)Y. \tag{3.2}$$

Hence a relation satisfying (3.2) is called a semi-symmetric connection.

From (3.1), it yields

$$(\overline{\nabla}_U g)(X,Y) = -\eta(X)g(Y,U) - \eta(Y)g(X,U) \neq 0. \tag{3.3}$$

$\overline{\nabla}$ defined by (3.1), satisfying (3.2) and (3.3) is a type of a semi-symmetric non-metric connection.

Then $\bar{R}$ and R are related by [**1**]

$$\bar{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.4}$$

for all vector fields X, Y, Z on M, where α is a $(0,2)$ tensor field denoted by

$$\alpha(X,Z) = (\nabla_X \eta)(Z) - \eta(X)\eta(Z). \tag{3.5}$$

4. Curvature tensor of an $N(k)$-contact metric manifold with respect to the semi-symmetric non-metric connection

Using (2.8) in (3.5), we get

$$\alpha(X,Z) = g(X,\phi Z) + g(hX,\phi Z) - \eta(X)\eta(Z). \tag{4.1}$$

Combining (4.1) and (3.4), we have

$$\begin{aligned} \bar{R}(X,Y)Z = {} & R(X,Y)Z + g(X,\phi Z)Y + g(hX,\phi Z)Y \\ & - \eta(X)\eta(Z)Y - g(Y,\phi Z)X - g(hY,\phi Z)X + \eta(Y)\eta(Z)X. \end{aligned} \tag{4.2}$$

Putting $X = \xi$ in (4.2) and using (2.2), (2.4) and (2.9), we concern that

$$\begin{aligned} \bar{R}(\xi,Y)Z = {} & kg(Y,Z)\xi - (k+1)\eta(Z)Y - g(Y,\phi Z)\xi \\ & - g(hY,\phi Z)\xi + \eta(Y)\eta(Z)\xi. \end{aligned} \tag{4.3}$$

Now putting $Y = \xi$ in (4.3) and using (2.1), (2.2) and (2.4), imply that

$$\bar{R}(\xi,\xi)Z = 0. \tag{4.4}$$

Again putting $Z = \xi$ in (4.3) and using (2.1) and (2.2), it follows that

$$\bar{R}(\xi,Y)\xi = (k+1)[\eta(Y)\xi - Y]. \tag{4.5}$$

From (4.2), we derive

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z. \tag{4.6}$$

We call (4.6) the curvature tensor satisfying the skew-symmetric property with respect to the semi-symmetric non-metric connection $\overline{\nabla}$.

Taking the inner product in (4.2) with W and using (2.1), we obtain

(4.7)
$$\begin{aligned}\tilde{\bar{R}}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) &+ g(X,\phi Z)g(Y,W) \\ &+ g(hX,\phi Z)g(Y,W) - \eta(X)\eta(Z)g(Y,W) - g(Y,\phi Z)g(X,W) \\ &- g(hY,\phi Z)g(X,W) + \eta(Y)\eta(Z)g(X,W),\end{aligned}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Let $\{e_1,\dots,e_{2n},e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Putting $X = W = e_i$, where $1 \le i \le 2n+1$, in (4.7) and also using (2.1), we derive

$$\bar{S}(Y,Z) = S(Y,Z) - 2ng(Y,\phi Z) - 2ng(hY,\phi Z) + 2n\eta(Y)\eta(Z). \tag{4.8}$$

Putting $Z = \xi$ in (4.8) and using (2.2) and (2.14), we have

$$\bar{S}(Y,\xi) = 2n(k+1)\eta(Y). \tag{4.9}$$

Again putting $Y = \xi$ in (4.8) and using (2.2), (2.4) and (2.14), we conclude that

$$\bar{S}(\xi,Z) = 2n(k+1)\eta(Z). \tag{4.10}$$

Let $\{e_1,\dots,e_{2n},e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Putting $Y = Z = e_i$, where $1 \le i \le 2n+1$, in (4.8) and also using (2.4) and (2.5), we obtain

$$\bar{r} = r + 2n. \tag{4.11}$$

Combining (2.8) and (3.1), it follows that

$$(\overline{\nabla}_U\eta)(X) = g(U,\phi X) + g(hU,\phi X) - \eta(X)\eta(U). \tag{4.12}$$

Summing up we can state the following proposition

Proposition 4.1. For an $N(k)$-contact metric manifold M with respect to the semi-symmetric non-metric connection $\overline{\nabla}$,

(i) The curvature tensor $\bar{R}$ is given by (4.2).
(ii) The Ricci tensor $\bar{S}$ is given by (4.8).
(iii) $\bar{R}(\xi,Y)Z = kg(Y,Z)\xi - (k+1)\eta(Z)Y - g(Y,\phi Z)\xi - g(hY,\phi Z)\xi + \eta(Y)\eta(Z)\xi$.
(iv) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$.
(v) The scalar curvature tensor $\bar{r}$ is given by (4.11).
(vi) The Ricci tensor $\bar{S}$ is not symmetric.
(vii) $\bar{S}(Y,\xi) = 2n(k+1)\eta(Y) = S(\xi,Y)$.
(viii) $(\overline{\nabla}_U\eta)(X) = g(U,\phi X) + g(hU,\phi X) - \eta(X)\eta(U)$.

5. $N(k)$-CONTACT METRIC MANIFOLDS SATISFYING $\bar{R}\cdot\bar{S} = 0$

Definition 5.1. An $N(k)$-contact metric manifold is said to be an Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form $S(X,Y) = ag(X,Y)$, where a is a constant.

In this section, we suppose that the $N(k)$-contact metric manifold under consideration is Ricci-semisymmetric with respect to the semi-symmetric non-metric connection, that is,

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V)=0.$$

Then we have

$$\bar{S}(\bar{R}(X,Y)U,V)+\bar{S}(U,\bar{R}(X,Y)V)=0. \tag{5.1}$$

Putting $X=\xi$ in (5.1), it follows that

$$\bar{S}(\bar{R}(\xi,Y)U,V)+\bar{S}(U,\bar{R}(\xi,Y)V)=0. \tag{5.2}$$

Using (4.3), (4.9) and (4.10) in (5.2), we obtain

$$\begin{aligned}2nk(k+1)\eta(V)g(Y,U)-(k+1)\eta(U)\bar{S}(Y,V)-2n(k+1)\eta(V)g(Y,\phi U)\\ -2n(k+1)\eta(V)g(hY,\phi U)+2nk(k+1)\eta(U)g(Y,V)-(k+1)\eta(V)\bar{S}(U,Y)\\ -2n(k+1)\eta(U)g(Y,\phi V)-2n(k+1)\eta(U)g(hY,\phi V)\\ +4n(k+1)\eta(Y)\eta(U)\eta(V)=0.\end{aligned} \tag{5.3}$$

Again putting $U=\xi$ in (5.3) and using (2.1) and (2.2), we get

$$\begin{aligned}2n(k+1)\eta(V)\eta(Y)-(k+1)\bar{S}(Y,V)+2nk(k+1)g(Y,V)\\ -2n(k+1)g(Y,\phi V)-2n(k+1))g(hY,\phi V)=0.\end{aligned} \tag{5.4}$$

In view of (4.8) and (5.4), we conclude that

$$S(Y,V)=2nkg(Y,V). \tag{5.5}$$

Therefore, $S(Y,Z)=ag(Y,Z)$, where $a=2nk$. From which it follows that the manifold is an Einstein manifold.

Now, we are in a position to state the following theorem.

Theorem 5.1. *If an $N(k)$-contact metric manifold is Ricci-semisymmetric with respect to the semi-symmetric non-metric connection, then the manifold is an Einstein manifold.*

6. $N(k)$-contact metric manifolds satisfying $\bar{R}\cdot\bar{R}=0$

Definition 6.1. An $N(k)$-contact metric manifold M is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(Z,W)=ag(Z,W)+b\eta(Z)\eta(W),$$

where a and b are smooth functions on the manifold.

In this section, we suppose that the manifold under consideration is semi-symmetric with respect to the semi-symmetric non-metric connection M^{2n+1}, that is,

$$(\bar{R}(U,V)\cdot\bar{R})(X,Y)Z=0.$$

Then we have

$$\begin{aligned}\bar{R}(U,V)\bar{R}(X,Y)Z-\bar{R}(\bar{R}(U,V)X,Y)Z\\ -\bar{R}(X,\bar{R}(U,V)Y)Z-\bar{R}(X,Y)\bar{R}(U,V)Z=0.\end{aligned} \tag{6.1}$$

Putting $U = \xi$ in (6.1), it follows that

$$\begin{aligned} &\bar{R}(\xi,V)\bar{R}(X,Y)Z - \bar{R}(\bar{R}(\xi,V)X,Y)Z \\ &-\bar{R}(X,\bar{R}(\xi,V)Y)Z - \bar{R}(X,Y)\bar{R}(\xi,V)Z = 0. \end{aligned} \tag{6.2}$$

Combining (4.3) and (6.2), we obtain

$$\begin{aligned} \bar{R}(\xi,V)\bar{R}(X,Y)Z &- kg(X,V)\bar{R}(\xi,Y)Z + (k+1)\eta(X)\bar{R}(V,Y)Z \\ &+g(V,\phi X)\bar{R}(\xi,Y)Z + g(hV,\phi X)\bar{R}(\xi,Y)Z - \eta(X)\eta(V)\bar{R}(\xi,Y)Z \\ &-kg(Y,V)\bar{R}(X,\xi)Z + (k+1)\eta(Y)\bar{R}(X,V)Z + g(V,\phi Y)\bar{R}(X,\xi)Z \\ &+g(hV,\phi Y)\bar{R}(X,\xi)Z - \eta(Y)\eta(V)\bar{R}(X,\xi)Z - kg(V,Z)\bar{R}(X,Y)\xi \\ &+(k+1)\eta(Z)\bar{R}(X,Y)V + g(V,\phi Z)\bar{R}(X,Y)\xi + g(hV,\phi Z)\bar{R}(X,Y)\xi \\ &-\eta(V)\eta(Z)\bar{R}(X,Y)\xi. \end{aligned} \tag{6.3}$$

Again putting $X = \xi$ in (6.3), we get

$$\begin{aligned} \bar{R}(\xi,V)\bar{R}(\xi,Y)Z &- kg(\xi,V)\bar{R}(\xi,Y)Z + (k+1)\eta(\xi)\bar{R}(V,Y)Z \\ &+g(V,\phi\xi)\bar{R}(\xi,Y)Z + g(hV,\phi\xi)\bar{R}(\xi,Y)Z - \eta(\xi)\eta(V)\bar{R}(\xi,Y)Z \\ &-kg(Y,V)\bar{R}(\xi,\xi)Z + (k+1)\eta(Y)\bar{R}(\xi,V)Z + g(V,\phi Y)\bar{R}(\xi,\xi)Z \\ &+g(hV,\phi Y)\bar{R}(\xi,\xi)Z - \eta(Y)\eta(V)\bar{R}(\xi,\xi)Z - kg(V,Z)\bar{R}(\xi,Y)\xi \\ &+(k+1)\eta(Z)\bar{R}(\xi,Y)V + g(V,\phi Z)\bar{R}(\xi,Y)\xi + g(hV,\phi Z)\bar{R}(\xi,Y)\xi \\ &-\eta(V)\eta(Z)\bar{R}(\xi,Y)\xi. \end{aligned} \tag{6.4}$$

Using (2.1), (2.2), (4.3), (4.4), and (4.5) in (6.4), we have

$$\begin{aligned} &k(k+1)g(Y,Z)[\eta(V)\xi - V] - (k+1)g(Y,\phi Z)[\eta(V)\xi - V] \\ &-(k+1)g(hY,\phi Z)[\eta(V)\xi - V] + (k+1)\eta(Y)\eta(Z)[\eta(V)\xi - V] \\ &+(k+1)\bar{R}(V,Y)Z - k(k+1)\eta(Z)g(V,Y)\xi + (k+1)^2\eta(Y)\eta(Z)V \\ &+(k+1)\eta(Z)g(V,\phi Y)\xi + (k+1)\eta(Z)g(hV,\phi Y)\xi - (k+1)\eta(Y)\eta(Z)\eta(V)\xi \\ &-k(k+1)\eta(V)g(Z,Y)\xi + (k+1)^2\eta(V)\eta(Z)Y + (k+1)\eta(V)g(Y,\phi Z)\xi \\ &+(k+1)\eta(V)g(hY,\phi Z)\xi - (k+1)\eta(Y)\eta(Z)\eta(V)\xi + k(k+1)\eta(Y)g(V,Z)\xi \\ &-(k+1)^2\eta(Y)\eta(Z)V - (k+1)\eta(Y)g(V,\phi Z)\xi - (k+1)\eta(Y)g(hV,\phi Z)\xi \\ &+(k+1)\eta(Y)\eta(Z)\eta(V)\xi + k(k+1)\eta(Z)g(V,Y)\xi - (k+1)^2\eta(V)\eta(Z)Y \\ &-(k+1)\eta(Z)g(Y,\phi V)\xi - (k+1)\eta(Z)g(hY,\phi V)\xi + (k+1)\eta(Y)\eta(Z)\eta(V)\xi \\ &+(k+1)[g(V,\phi Z) + g(hV,\phi Z) - kg(V,Z) - \eta(V)\eta(Z)][\eta(Y)\xi - Y] = 0. \end{aligned} \tag{6.5}$$

Now contracting V in (6.5) and using (2.1), (2.2) and (4.8), we conclude that

$$S(Y,Z) = 2nkg(Y,Z) + (-k-1)\eta(Y)\eta(Z). \tag{6.6}$$

Therefore, $S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z)$, where $a = 2nk$ and $b = (-k-1)$, which implies that the manifold is an η-Einstein manifold.

In view of above discussions, we state the following proposition.

Proposition 6.1. *If an $N(k)$-contact metric manifold is semisymmetric with respect to the semi-symmetric non-metric connection, then the manifold is an η-Einstein manifold.*

Definition 6.2. An $N(k)$-contact metric manifold is said to have an η-parallel Ricci tensor if the Ricci tensor with respect to the Levi-Civita connection satisfies $(\nabla_U S)(\phi Y, \phi Z) = 0$.

Putting $Y = \phi Y$ and $Z = \phi Z$ in (6.6), implies that

$$S(\phi Y, \phi Z) = 2nkg(\phi Y, \phi Z).$$

From the above equation, it yields,

$$(\nabla_U S)(\phi Y, \phi Z) = -2nk[\eta(\phi Y)g(U, \phi Z) + \eta(\phi Z)g(U, \phi Y)] = 0.$$

Therefore, considering all the cases, we can state the following theorem.

Theorem 6.2. *If an $N(k)$-contact metric manifold is semisymmetric with respect to the semi-symmetric non-metric connection, then the manifold with respect to the Levi-Civita connection satisfies an η-parallel Ricci tensor condition.*

7. Example

In this section, we construct an example of an $N(k)$-contact metric manifold. We consider 3-dimensional manifold $M = (x, y, z) \in R^3$, where (x, y, z) are the standard coordinate in R^3. Let e_1, e_2, e_3 be three vector fields in R^3 which satisfy $[e_1, e_2] = (1 + \lambda)e_3$, $[e_2, e_3] = 2e_1$, $[e_3, e_1] = (1 - \lambda)e_2$, where λ is a real number.

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j; \ i, j = 1, 2, 3. \end{cases}$$

Let η be the 1-form defined by $\eta(U) = g(U, e_1)$ for any $U \in \chi(M)$.
Let ϕ be the (1, 1)-tensor field defined by $\phi e_1 = 0$, $\phi e_2 = e_3$, $\phi e_3 = -e_2$. Using the linearity of ϕ and g, we have

$$\eta(e_1) = 1,$$
$$\phi^2(U) = -U + \eta(U)e_1$$

and

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Moreover, $he_1 = 0$, $he_2 = \lambda e_2$, $he_3 = -\lambda e_3$. The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{7.1}$$

Using Koszul's formula, we get the following:

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 = -(1 + \lambda)e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = (1 + \lambda)e_1, \end{array}$$

$$\nabla_{e_3} e_1 = (1-\lambda)e_2, \qquad \nabla_{e_3} e_2 = -(1-\lambda)e_1, \qquad \nabla_{e_3} e_3 = 0.$$

In view of the above relations, we have $\nabla_X \xi = -\phi X - \phi hX$ for $e_1 = \xi$. Therefore, the manifold is a contact metric manifold with the contact structure (ϕ, η, ξ, g).

Using (3.1) in the above equations, we obtain

$$\begin{aligned}
&\overline{\nabla}_{e_1} e_1 = e_1, && \overline{\nabla}_{e_1} e_2 = 0, && \overline{\nabla}_{e_1} e_3 = 0,\\
&\overline{\nabla}_{e_2} e_1 = -(1+\lambda)e_3 + e_2, && \overline{\nabla}_{e_2} e_2 = 0, && \overline{\nabla}_{e_2} e_3 = (1+\lambda)e_1,\\
&\overline{\nabla}_{e_3} e_1 = (1-\lambda)e_2 + e_3, && \overline{\nabla}_{e_3} e_2 = -(1-\lambda)e_1, && \overline{\nabla}_{e_3} e_3 = 0.
\end{aligned}$$

Now, we can easily obtain the non-zero components of the curvature tensors with respect to the Levi-Civita connection as follows:

$$\begin{aligned}
&R(e_1,e_2)e_2 = (1-\lambda^2)e_1, && R(e_1,e_3)e_3 = (1-\lambda^2)e_1,\\
&R(e_3,e_2)e_2 = -(1-\lambda^2)e_3, && R(e_2,e_3)e_3 = -(1-\lambda^2)e_2,\\
&R(e_2,e_1)e_1 = (1-\lambda^2)e_2, && R(e_3,e_1)e_1 = (1-\lambda^2)e_3.
\end{aligned}$$

In view of the expressions of the curvature tensors, we conclude that the manifold is an $N(1-\lambda^2)$-contact metric manifold.

Thus, we get the non-zero components of the curvature tensors with respect to the semi-symmetric non-metric connection as follows:

$$\begin{aligned}
&\bar{R}(e_1,e_2)e_2 = (1-\lambda^2)e_1, && \bar{R}(e_1,e_3)e_3 = (1-\lambda^2)e_1,\\
&\bar{R}(e_1,e_2)e_3 = -(1+\lambda)e_1, && \bar{R}(e_3,e_2)e_2 = -(1-\lambda^2)e_3 + e_2,\\
&\bar{R}(e_2,e_3)e_3 = -(1-\lambda^2)e_2 - (1+\lambda)e_3, && \bar{R}(e_1,e_2)e_1 = -(2-\lambda^2)e_2,\\
&\bar{R}(e_3,e_1)e_1 = (2-\lambda^2)e_3, && \bar{R}(e_1,e_3)e_2 = -(1-\lambda)e_1.
\end{aligned}$$

With the help of the above results we find the Ricci tensors as follows:

$$S(e_1,e_1) = 2(1-\lambda^2), \qquad S(e_2,e_2) = S(e_3,e_3) = 0$$

and

$$\bar{S}(e_1,e_1) = 2(2-\lambda^2), \qquad \bar{S}(e_2,e_2) = \bar{S}(e_3,e_3) = 0.$$

Also, it follows that the scalar curvature tensors with respect to the Levi-Civita connection and the semi-symmetric non-metric connection are $r = 2(1-\lambda^2)$ and $\bar{r} = 2(2-\lambda^2)$, respectively.

Let X, Y, U and V be any four vector fields given by $X = a_1e_1 + a_2e_2 + a_3e_3$, $Y = b_1e_1 + b_2e_2 + b_3e_3$, $U = c_1e_1 + c_2e_2 + c_3e_3$ and $V = d_1e_1 + d_2e_2 + d_3e_3$, where a_i, b_i, c_i, d_i, for all $i = 1,2,3$ are all non-zero real numbers.
Using the above equations, we obtain

$$\begin{aligned}
\bar{R}(X,Y)Z = {} & [(1-\lambda^2)a_1b_2c_2 + (1-\lambda^2)a_1b_3c_3 - (1+\lambda)a_1b_2c_3 - (1-\lambda)a_1b_3c_2]e_1\\
& + [a_3b_2c_2 - (1-\lambda^2)a_2b_3c_3 - (2-\lambda^2)a_1b_2c_1]e_2\\
& + [-(1-\lambda^2)a_3b_2c_2 - (1+\lambda)a_2b_3c_3 + (2-\lambda^2)a_3b_1c_1]e_3\\
= {} & -\bar{R}(Y,X)Z.
\end{aligned}$$

Hence, the curvature tensor of an $N(k)$-contact metric manifold with respect to the semi-symmetric non-metric connection satisfies the skew-symmetric property.

From (5.1), we obtain

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V) = 2(2-\lambda^2)[(1-\lambda^2)a_1b_2c_2d_1 + (1-\lambda^2)a_1b_3c_3d_1 - (1+\lambda)a_1b_2c_3d_1 - (1-\lambda)a_1b_3c_2d_1].$$

Therefore, the $N(k)$-contact metric manifold has the property $\bar{R}\cdot\bar{S} = 0$ with respect to the semi-symmetric non-metric connection if $\lambda^2 = 2$.

Acknowledgement. The author is thankful to the referee for his/her valuable comments towards the improvement of my paper.

References

1. Agashe N. S. and Chafle M. R., *A semi-symmetric non-metric connection on a Riemannian Manifold*, Indian J. Pure Appl. Math., **23(6)** (1992), 399–409.
2. Andonie, O. C., *On semi-symmetric non-metric connection on a Riemannian manifold*, Ann. Fac. Sci. De Kinshasa, Zaire Sect. Math. Phys., **2** (1976).
3. Barman A., *Semi-symmetric non-metric connection in a P-Sasakian manifold*, Novi Sad J. Math., **43** (2013), 117–124.
4. Barman A., *A type of semi-symmetric non-metric connection on non-degenerate hypersurfaces of semi-Riemannian manifolds*, Facta Univer. (NIS), **29** (2014), 13–23.
5. Barman A. and De U. C., *Semi-symmetric non-metric connections on Kenmotsu manifolds*, Romanian J. Math. and Comp. Sci., **5** (2014), 13–24.
6. Blair D. E., *Inversion theory and conformal mapping*, Student Mathematical Library 9, American Mathematical Society, 2000.
7. Blair D. E., *Contact manifolds in Riemannian geometry*, Lecture Note in Mathematics, 509. Springer-Verlag, Berlin, New-York, 1976.
8. Blair D. E., *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics 203, Boston, MA: Birkhauser Boston, Inc. 2002.
9. Blair D. E., Koufogiorgos T. and Papantoniou B. J., *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., **91** (1995), 1–3, 189–214.
10. De U. C. and Biswas S. C., *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Ganita, **48** (1997), 91–94.
11. De U. C. and Kamilya D., *Hypersurfaces of a Riemannian manifold with semi-symmetric non-metric connection*, J. Indian Inst. Sci., **75** (1995), 707–710.
12. De U. C. and Gazi A. K., On ϕ-recurrent $N(k)$-contact metric manifolds, Math. J. Okayama Univ., **50** (2008), 101–112.
13. Friedmann A. and Schouten J. A., *Über die Geometric der halbsymmetrischen Übertragung*, Math., Zeitschr., **21** (1924), 211–223.
14. Ghosh S., De U. C. and Taleshian A., *Conharmonic curvature tensor on $N(k)$-contact metric manifolds*, ISRN Geometry, 2011, Art. ID 423798, 11 pages.
15. Hayden H. A., *Subspaces of space with torsion*, Proc. London Math. Soc., **34** (1932), 27–50.
16. Liang Y., *On semi-symmetric recurrent-metric connection*, Tensor, N. S.**55** (1994), 107–112.
17. Majhi P. and De U. C., *Classifications on $N(k)$-contact metric manifolds satisfying certain curvature conditions*, Acta Math. Univ. Comenianae, **84** (2015), 167–178.
18. Özgür C. and Sular S., *On N(k)-contact metric manifolds satisfying certain conditions*. Sut. J. Math., **44** (2008), 89–99.
19. Prvanovic M., *On pseudo metric semi-symmetric connections*, Pub. De L' Institut Math., Nouvelle serie, **18** (1975), 157–164.
20. Tanno S., *Ricci curvatures of contact Riemannian manifolds*, Tohoku Math. J., **40** (1988), 441–448.
21. Tanno S., *Variational problems on contact metric manifolds*, Trans. Amer. Math. Soc., **314** (1989), 349–379.

A. Barman, Department of Mathematics, Kabi-Nazrul Mahavidyalaya, P.O.: Sonamura-799181, P.S.-Sonamura, Dist-Sepahijala, Tripura, India, *e-mail*: `ajitbarmanaw@yahoo.in`