SEMI-SYMMETRIC TYPE OF $\alpha$-SASAKIAN MANIFOLDS

K. K. BAISHYA and P. R. CHOWDHURY

Abstract. Recently the present author introduced the notion of generalized quasi-conformal curvature tensor which bridges Conformal curvature tensor, Concircular curvature tensor, Projective curvature tensor and Conharmonic curvature tensor. This article attempts to characterize $\alpha$-Sasakian manifolds with $\omega(X,Y) \cdot W = 0$. Based on this curvature conditions, we obtain and table the expression for the Ricci tensor for the respective semi-symmetric type of $\alpha$-Sasakian manifolds.

1. Introduction

In 1968, Yano and Sawaki [12] introduced the notion of quasi-conformal curvature tensor in the context of Riemannian geometry. Recently, in tune with Yano and Sawaki [12], the present authors [9] introduced and studied a generalized quasi-conformal curvature tensor in the context of an $N(k,\mu)$-manifold. The beauty of such curvature tensor lies in the fact that it has the flavour of a Riemann curvature tensor $R$, a conformal curvature tensor $C$ [13], a conharmonic curvature tensor $\hat{C}$ [18], a concircular curvature tensor $E$ ([11, p. 84]), a projective curvature tensor $P$ ([11, p. 84]) and an $m$-projective curvature tensor $H$ [8] as particular cases. Hereafter in this paper, a generalized quasi-conformal curvature tensor $W$ is called a quasi-conformal like curvature tensor. The components of a quasi-conformal like curvature tensor $W$ in a Riemannian manifold $(M^{2n+1},g)$, $n > 1$, are given by

$$W(X,Y)Z = \frac{2n-1}{2n+1} \left[ ((1 + 2na - b) - 1 + 2n(a + b))c \right] C(X,Y)Z$$

$$+ \left[ 1 - b + 2na \right] E(X,Y)Z + 2n(b-a)P(X,Y)Z$$

$$+ \frac{2n-1}{2n+1} (c-1)(1+2n(a+b))\hat{C}(X,Y)Z,$$

where $a, b \& c$ are real constants. The foregoing equation can also be written as

$$W(X,Y)Z = R(X,Y)Z + a \left[ S(Y,Z)X - S(X,Z)Y \right] + b \left[ g(Y,Z)QX - g(X,Z)QY \right]$$

$$- \frac{cr}{2n+1} \left( \frac{1}{2n} + a + b \right) \left[ g(Y,Z)X - g(X,Z)Y \right],$$

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$R, S, Q$ and $r$ are a Christoffel Riemannian curvature tensor, a Ricci tensor, a Ricci operator and a scalar curvature, respectively. In particular, the quasi conform like curvature tensor $W$ induce to

1. Riemann curvature tensor $R$ if $a = b = c = 0$,
2. conformal curvature tensor $C$ if $a = b = -\frac{1}{2n-7}, c = 1$,
3. conharmonic curvature tensor $\hat{C}$ if $a = b = -\frac{1}{2n-7}, c = 0$,
4. concircular curvature tensor $E$ if $a = b = 0$ and $c = 1$,
5. projective curvature tensor $P$ if $a = -\frac{1}{2n}, b = 0, c = 0$ and
6. $m$-projective curvature tensor $H$, if $a = b = -\frac{1}{2n}, c = 0$.

An $\alpha$-Sasakian manifold is said to be semi-symmetry type (respectively Ricci semi-symmetry type) if the quasi-conform like curvature tensor $W$ (respectively, Ricci tensor $S$) admits the condition [19]

\[ \omega(X, Y) \cdot W = 0, \quad (\text{respectively, } W(X, Y) \cdot S = 0) \]

for any $X, Y$ on $M$, where the dot means that $\omega(X, Y)$ acts on $W$ (respectively, on $S$) as derivation. Here, both $\omega$ and $W$ stand for a quasi-conform like curvature tensor with respect to the associated scalar triples $(\hat{a}, \hat{b}, \hat{c})$ and $(a, b, c)$, respectively. In particular, a manifold satisfying the condition $R(X, Y) \cdot R = 0$ (obtained from (1.3) by setting $\hat{a} = \hat{b} = \hat{c} = 0 = a = b = c$) is said to be semi-symmetric in the sense of Cartan ([5, § 265]). A full classification of such space is given by Z. I. Szabó [19]. This type of the manifolds were studied by several authors such as Papantoniou [1], Perrone [4], and the references cited therein.

Our work is structured as follows. Section 2 is a very brief account of $\alpha$-Sasakian manifolds. In Section 3, we investigate $\alpha$-Sasakian manifolds with $\omega(X, Y) \cdot W = 0$. Based on this curvature condition, we obtain and table the expressions for the Ricci tensor for respective curvature restrictions. Finally, we bring out an expression for the Ricci tensor of an $\alpha$-Sasakian manifolds with $W(X, Y) \cdot S = 0$.

2. $\alpha$-Sasakian manifolds

A contact manifold is a $(2n+1)$-dimensional $C^\infty$-manifold $M$ equipped with a global form $\eta$, called a contact form of $M$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$. In particular, $\eta \wedge (d\eta)^n \neq 0$ is a volume element of $M$ so that a contact manifold is orientable. A contact manifold associated with the Riemannian metric $g$ is called a contact metric manifold if it satisfies the following relation

\[
(2.1) \quad d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \circ \xi,
\]

where $\phi$ is a $(1,1)$-tensor field and $\xi$ is a unique vector field such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. We denote the symbols $\nabla$, $R$ and $Q$ by Levi-Civita connection, a curvature tensor and a Ricci operator of $g$, respectively. We define a $(1,1)$ type tensor field $h$ by $h = \frac{1}{2}\xi \phi$ and we know that $h$ and $h\phi$ are trace free and $h\phi = -\phi h$. An almost contact manifold $M(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold if there exist two functions on $M$ such that

\[
(2.2) \quad (\nabla_X \phi)(Y) = \alpha \{g(X, Y)\xi - \eta(Y)X\} + \beta \{g(\phi X, Y)\xi - \eta(Y)\phi X\},
\]
for any vector fields $X, Y$ on $M$. If $\beta = 0$, then $M$ is a $\alpha$-Sasakian manifold. Sasakian manifolds are cases of $\alpha$-Sasakian manifolds with $\alpha = 1$. If $\alpha = 0$, then $M$ is called $\beta$-Kenmotsu manifold. Kenmotsu manifolds are cases of a $\beta$-Kenmotsu with $\beta = 1$. If both $\alpha$ and $\beta$ vanish, then $M$ is a cosymplectic manifold. In an $\alpha$-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, the following relations hold [2]:

(2.3) \[ \nabla_X \xi = -\alpha \phi X, \]
(2.4) \[ (\nabla_X \phi)(Y) = \alpha \{g(X,Y)\xi - \eta(Y)X\}, \]
(2.5) \[ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \]

for all vector fields $X, Y$ on $M^{2n+1}$ where $\alpha$ is a smooth function on $M$ and we say that the $\alpha$-Sasakian structure is of type $(\alpha, 0)$. From (2.4), it follows that

(2.6) \[ (\nabla_X \eta)(Y) = -\alpha g(\phi X, Y), \]
(2.7) \[ R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \]
(2.8) \[ \eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \]
(2.9) \[ R(\xi, X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X], \]
(2.10) \[ S(\xi, \xi) = 2n\alpha^2 \eta(\xi) \]

for any vector fields $X, Y$ on $M$.

3. $\alpha$-Sasakian manifolds with $\omega(X, Y) \cdot W = 0$

Let us consider a $(2n + 1)$-dimensional $\alpha$-Sasakian manifold $M$, satisfying the condition

(3.1) \[ (\omega(X, Y) \cdot W)(Z, U)V = 0 \]

i.e.,

(3.2) \[ \omega(X, Y)W(Z, U)V = W(\omega(X, Y)Z, U)V + W(Z, \omega(X, Y)U)V + W(Z, U)\omega(X, Y)V \]

which can be transformed into

(3.3) \[ g(\omega(\xi, X)W(Y, Z)U, \xi) - g(W(\omega(\xi, X)Y, Z)U, \xi) - g(W(Y, \omega(\xi, X)Z)U, \xi) - g(W(\omega(\xi, X)Z, U, \xi) = 0. \]

Putting $X = Y = e_i$ in (3.3), where $\{e_1, e_2, e_3, \ldots, e_{2n}, e_{2n+1} = \xi\}$ is an orthonormal basis of the tangent space at each point of the manifold $M$ and taking the summation over $i$, $1 \leq i \leq 2n + 1$, we get

(3.4) \[ \sum_{i=1}^{2n+1} \left[ g(\omega(\xi, e_i)W(e_i, Z)U, \xi) - g(W(\omega(\xi, e_i)e_i, Z)U, \xi) - g(W(e_i, \omega(\xi, e_i)Z)U, \xi) - g(W(e_i, Z)\omega(\xi, e_i)U, \xi) \right] = 0. \]
From the equation (1.2), we can easily bring out the following:

\[
\eta(W(\xi, U)Z) = \left[ \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) - 2n\alpha^2 a - 2n\alpha^2 b - \alpha^2 \right] \eta(Z)\eta(U) \\
+ \left[ \alpha^2 + 2n\alpha^2 - \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right] g(Z, U) + aS(Z, U),
\]

\[
\sum_{i=1}^{2n+1} \hat{W}(e_i, Z, U, \xi)
\]

\[
= (1 - b + 2na)S(Z, U) + \left( br - \frac{2ncr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right) g(Z, U),
\]

\[
\sum_{i=1}^{2n+1} \eta W(e_i, Z, U, \xi)
\]

\[
= \left[ - 2n\alpha^2(1 - a + 2nb) - \left\{ ar - \frac{2ncr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right\} \right] \eta(Z),
\]

\[
\sum_{i=1}^{2n+1} S W(e_i, Z, U, \xi)
\]

\[
= \left( ar + \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right) S(Z, U) - (a + b - 1)S^2(Z, U)
\]

\[
+ \left( b|Q|^2 - \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right) g(Z, U),
\]

\[
\sum_{i=1}^{2n+1} \eta(e_i)\eta(W(Qe_i)|U)
\]

\[
= 2n\alpha^2 \left[ \alpha^2 + 2n\alpha^2 a - \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right] g(Z, U) + 2n\alpha^2 aS(Z, U)
\]

\[
- 2n\alpha^2 \left[ \alpha^2 + 2n\alpha^2 (a + b) - \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right] \eta(Z)\eta(U),
\]

\[
\sum_{i=1}^{2n+1} S(e_i, Z)\eta(W(e_i, \xi)|U)
\]

\[
= 2n\alpha^2 \left[ \alpha^2 + 2n\alpha^2 (a + b) - \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right] \eta(Z)\eta(U)
\]

\[
- \left\{ \alpha^2 + 2n\alpha^2 b - \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right\} S(Z, U) - aS^2(Z, U).
\]

Now,

\[
\sum_{i=1}^{2n+1} g(\omega(\xi, e_i)|W(e_i, Z)|U, \xi)
\]

\[
= \left[ \alpha^2 + 2n\alpha^2 b - \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right] \left\{ \hat{W}(e_i, Z, U, e_i) - \eta(W(e_i, Z)\eta(e_i)) \right\}
\]

\[
+ \hat{a} \left[ S(W(e_i, Z)U, e_i) - 2n\alpha^2 \eta(W(e_i, Z)U)\eta(e_i) \right]
\]

\[
= \left[ \alpha^2 + 2n\alpha^2 b - \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) \right] \hat{W}(e_i, Z, U, e_i) + \hat{a}S(W(e_i, Z)U, e_i)
\]

\[
+ \left[ \frac{cr}{2n + 1} \left( \frac{1}{2n} + a + b \right) - 2n\alpha^2 b - \alpha^2 - 2n\alpha^2 a \right] \eta(W(\xi, U)|Z).
\]
In view of (3.6), (3.8) and (3.11) becomes

\[ \sum_{i=1}^{2n+1} g(\omega(\xi,\varepsilon_i)\mathcal{W}(e_i, Z)(U, \xi)) = \left( \alpha^2 + 2n\alpha^2b - \frac{\bar{\varepsilon}r}{2n+1} \left( \frac{1}{2n} + \bar{a} + \bar{b} \right) \right)(1 + 2n\alpha - b) + \hat{a}(ar + \frac{cr}{2n+1} \left( \frac{1}{2n} + a + b \right)) \bigg] S(Z, U) + \bar{a}(1 - a - b)S^2(Z, U) \]

(3.12)

+ \left[ \frac{\bar{\varepsilon}r}{2n+1} \left( \frac{1}{2n} + \bar{a} + \bar{b} \right) - 2n\alpha^2(\bar{a} + \bar{b}) - \alpha^2 \right] \eta(\mathcal{W}(\xi, U)Z) + \left[ \left( \alpha^2 + 2n\alpha^2b - \frac{\bar{\varepsilon}r}{2n+1} \left( \frac{1}{2n} + \bar{a} + \bar{b} \right) \right) \left( \bar{b}r - \frac{2acr}{2n+1} \left( \frac{1}{2n} + a + b \right) \right) + \hat{a} \left[ \frac{\bar{b}|Q|^2}{2n+1} \left( \frac{1}{2n} + a + b \right) \right] \bigg] g(Z, U). \]

In consequence of (3.5)-(3.10), we obtain the following:

\[ \sum_{i=1}^{2n+1} g(\mathcal{W}(\omega(\xi, \varepsilon_i), Z)(U, \xi)) = \left[ (2n+1) \left( \alpha^2 + 2n\alpha^2b - \frac{\bar{\varepsilon}r}{2n+1} \left( \frac{1}{2n} + \bar{a} + \bar{b} \right) \right) \right] + \bar{a}r \eta(\mathcal{W}(\varepsilon_i, U)Z) + \left[ \frac{\bar{\varepsilon}r}{2n+1} \left( \frac{1}{2n} + \bar{a} + \bar{b} \right) - 2n\alpha^2(\bar{a} - \alpha^2) \right] g(\mathcal{W}(e_i, Z)(U, \xi)\eta(\varepsilon_i),)

(3.13)

\[ \sum_{i=1}^{2n+1} g(\mathcal{W}(\mathcal{Q}e_i, Z)(U, \xi)\eta(\varepsilon_i)) = \left[ 2n \left( \alpha^2 + 2n\alpha^2b - \frac{\bar{\varepsilon}r}{2n+1} \left( \frac{1}{2n} + \bar{a} + \bar{b} \right) \right) \right] + \bar{a}r + 2n\alpha^2(\bar{b} - \bar{a}) \eta(\mathcal{W}(\xi, U)Z) - 2n\alpha^2\bar{a}\bar{b}S(Z, U), \]

(3.13)
Expression for Ricci tensor

\[ S_{ij} = \alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + \bar{a} + \bar{b} \right) g(e_i, U) \eta(W(e_i, Z)\xi) \]

\[ + \left[ \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + \bar{a} + \bar{b} \right) - 2n\alpha^2\bar{a} - \alpha^2 \right] \eta(W)e_i, Z(e_i)\eta(U) \]

\[ + \bar{a}S(e_i, U)\eta(W(e_i, Z)\xi) \]

\[ = \left[ \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + \bar{a} + \bar{b} \right) - 2n\alpha^2\bar{a} - \alpha^2 \right] \eta(W)e_i, Z(e_i)\eta(U) \]

which is equivalent to

\[ \sum_{i=1}^{2n+1} g(W(e_i, Z)\omega(\xi, e_i)U, \xi) \]

\[ = \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + \bar{a} + \bar{b} \right) - 2n\alpha^2\bar{a} - \alpha^2 \left[ 2n\alpha^2(1 - a + 2nb) \right. \]

\[ + \left( \alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + \bar{a} + \bar{b} \right) \right) (1 - b) + \alpha^2\bar{a}(1 + 2nb) \]

By virtue of (3.12), (3.13), (3.14), and (3.16), the equation (3.4) yields

\[ \left[ \left( \alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + \bar{a} + \bar{b} \right) \right) (1 - b) + \alpha^2\bar{a}(1 + 2nb) \right] S(Z, U) \]

\[ + \left[ 2n\alpha^2(1 - a + 2nb) + \alpha^2 - \frac{2n\alpha^2\bar{a}}{2n+1} \left( \frac{1}{2n+1} + a + b \right) \right] \eta(U) \eta(Z) \]

\[ \times \left( \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + \bar{a} + \bar{b} \right) - 2n\alpha^2\bar{a} - \alpha^2 \right) \]

\[ \times \left[ \alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + a + b \right) \right] \]

\[ + \left[ \alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}}{2n+1} \left( \frac{1}{2n+1} + \bar{a} + \bar{b} \right) \right] \left( br - 2n\alpha^2 - 4n^2\alpha^2\bar{a} \right) \]

\[ + \bar{a} \left( \left( e^2 - \alpha^2 \right)^2 - 2\left( e^2 - \alpha^2 \right) \right) g(Z, U) + \bar{a}(1 - b)S^2(Z, U) = 0. \]

**Theorem 3.1.** Let \((M^{2n+1}, g), n > 1\), be an \(\alpha\)-Sasakian manifold. Then for respective semi-symmetric type conditions, the Ricci tensor of the manifold \(M\) takes the respective forms as follows:

<table>
<thead>
<tr>
<th>Curvature condition</th>
<th>Expression for Ricci tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(X, Y) \cdot R = 0 )</td>
<td>( S = 2n\alpha^2g )</td>
</tr>
<tr>
<td>( R(X, Y) \cdot C = 0 )</td>
<td>( S = \left( \frac{e}{2^m} - \alpha^2 \right)g - \left( \frac{e}{2^m} - (2n+1)\alpha^2 \right) \eta \otimes \eta )</td>
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<tr>
<td>( R(X, Y) \cdot \hat{C} = 0 )</td>
<td>( S = \left( \frac{e}{2^m} - \alpha^2 \right)g - \left( \frac{e}{2^m} - (2n+1)\alpha^2 \right) \eta \otimes \eta )</td>
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<td>---------------------</td>
<td>-----------------------------</td>
</tr>
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<td>$S = 2na^2 g$</td>
</tr>
<tr>
<td>$R(X, Y) \cdot P = 0$</td>
<td>$S = 2na^2 g - \left(\frac{r}{2n} - 2n - 1\right)\eta \otimes \eta$</td>
</tr>
<tr>
<td>$R(X, Y) \cdot H = 0$</td>
<td>$S = \left(\frac{r+4na^2}{4n+1}\right) g - \left(\frac{r-2na^2(2n+1)}{4n+1}\right)\eta \otimes \eta$</td>
</tr>
<tr>
<td>$E(X, Y) \cdot R = 0$</td>
<td>$S = 2na^2 g$ for $\alpha^2 \neq -\frac{r}{2n(2n+1)}$</td>
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<td>$E(X, Y) \cdot C = 0$</td>
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<td>$E(X, Y) \cdot E = 0$</td>
<td>$S = 2na^2 g$ for $\alpha^2 \neq -\frac{r}{2n(2n+1)}$</td>
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<td>$E(X, Y) \cdot P = 0$</td>
<td>$S = \left(\frac{r}{2n} - 2\alpha^2\right) g - \left(\frac{r}{2n} - 2n\alpha^2\right)\eta \otimes \eta$ for $\alpha^2 \neq -\frac{r}{2n(2n+1)}$</td>
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<td>$E(X, Y) \cdot H = 0$</td>
<td>$S = \left(\frac{r+4na^2}{4n+1}\right) g - \left(\frac{r}{4n+1}\right)\eta \otimes \eta$ for $\alpha^2 \neq -\frac{r}{2n(2n+1)}$</td>
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<tr>
<td>$\hat{C}(X, Y) \cdot R = 0$</td>
<td>$(\frac{r}{2n} - 2\alpha^2) S = -2na^4 g + \alpha^2 (r - 2na^2)\eta \otimes \eta + S^2$</td>
</tr>
<tr>
<td>$\hat{C}(X, Y) \cdot \hat{C} = 0$</td>
<td>$(\frac{r}{2n} - 2\alpha^2) S = \left(\frac{r}{2n} - 2na^2\right)(r - 2na^2) + \alpha^2 r -</td>
</tr>
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<td>$\hat{C}(X, Y) \cdot C = 0$</td>
<td>$(\frac{r}{2n} - 2\alpha^2) S = \left(\frac{r}{2n} - 2na^2\right)(r - 2na^2) + \alpha^2 r -</td>
</tr>
<tr>
<td>$\hat{C}(X, Y) \cdot E = 0$</td>
<td>$2S = -S^2 + [r + 2n] g + (2n - r)\left(\frac{r}{2n(2n+1)} - 1\right) \eta \otimes \eta$</td>
</tr>
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<td>$\hat{C}(X, Y) \cdot P = 0$</td>
<td>$(\frac{r}{2n} - 2\alpha^2) S = -2na^4 g + S^2 + (\frac{r}{2n} - \alpha^2)((2n+1)\alpha^2 - \frac{r}{2n}) \eta \otimes \eta$</td>
</tr>
<tr>
<td>$\hat{C}(X, Y) \cdot H = 0$</td>
<td>$(\frac{4n+1}{4n+2}) S = \left(\frac{r}{2n} + n\right) + r + \frac{1}{4n}</td>
</tr>
<tr>
<td>$P(X, Y) \cdot R = 0$</td>
<td>$(\frac{2n-1}{2n} \alpha^2) S = \alpha^2 (r - \frac{r}{2n} + 2na^2) g + \frac{1}{2n} S^2$</td>
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<tr>
<td>( P(X, Y) \cdot \mathcal{C} = 0 )</td>
<td>( (\tfrac{4n^2+1}{2n} \alpha^2) S = S^2 + (\alpha^2 (r-2n\alpha^2) + \tfrac{1}{2n} (\alpha^2 r - |Q|^2)) g + (2n+1)(-2n\alpha^2 + (\alpha^2 - \tfrac{r}{2n}) \alpha^2) \eta \otimes \eta )</td>
</tr>
<tr>
<td>( P(X, Y) \cdot C = 0 )</td>
<td>( (\tfrac{4n^2+1}{2n} \alpha^2) S = S^2 + (\alpha^2 (r-2n\alpha^2) + \tfrac{1}{2n} (\alpha^2 r - |Q|^2)) g + (2n+1)(\alpha^2 - \tfrac{r}{2n}) \alpha^2 - \alpha^2 (r+2n\alpha^2) \eta \otimes \eta )</td>
</tr>
<tr>
<td>( P(X, Y) \cdot E = 0 )</td>
<td>( (\tfrac{2n-1}{2n} \alpha^2) S = \alpha^2 \left( - \tfrac{r}{2n} + 2n\alpha^2 \right) g + \tfrac{1}{2n} S^2 + \left( \tfrac{1}{2n} (2n\alpha^2 - \tfrac{r}{2n}) \right) (\tfrac{1}{2n} - (2n+1)\alpha^2) \eta \otimes \eta )</td>
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<td>( P(X, Y) \cdot P = 0 )</td>
<td>( (\tfrac{2n-1}{2n} \alpha^2) S = \tfrac{1}{2n} S^2 + \alpha^2 \left( - \tfrac{r}{2n} + 2n\alpha^2 \right) g )</td>
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<td>( P(X, Y) \cdot H = 0 )</td>
<td>( \alpha^2 S = \tfrac{1}{2n} \left[ (1 + \tfrac{1}{2n}) S^2 + \left( na^4 - \tfrac{1}{4n} |Q|^2 \right) \right] )</td>
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<tr>
<td>( H(X, Y) \cdot R = 0 )</td>
<td>( (\tfrac{2n-1}{4n} \alpha^2) S = \alpha^2 (-\tfrac{r}{4n} + na^2) g + \left( \tfrac{r}{2n} - \tfrac{1}{4n} \right) \eta \otimes \eta + \tfrac{1}{4n} S^2 )</td>
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<tr>
<td>( H(X, Y) \cdot \mathcal{C} = 0 )</td>
<td>( (\tfrac{4n^2+1}{4n} \alpha^2) S = \tfrac{1}{2n} S^2 + \left( \tfrac{2n^2}{4n} (r-2n\alpha^2) + \tfrac{1}{2n} (\alpha^2 r - |Q|^2) \right) g + \left[ \left( r - \tfrac{r^2}{2n} \right) + \alpha^2 (2n+1) \left( \tfrac{2n^2}{4n} \alpha^2 - \tfrac{r}{2n} \right) \right] \eta \otimes \eta )</td>
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<td>( C(X, Y) \cdot R = 0 )</td>
<td>( \left( \tfrac{r}{2n} - 2 \alpha^2 \right) S = -2n\alpha^4 g + \alpha^2 \left( r - 2n\alpha^2 \right) \eta \otimes \eta + S^2 )</td>
</tr>
<tr>
<td>( C(X, Y) \cdot \mathcal{C} = 0 )</td>
<td>( (r - (2n - 1)\alpha^2) S = \left( \tfrac{r}{2n} - 2n\alpha^2 \right) \left( r - 2n\alpha^2 \right) + \alpha^2 \left( r - |Q|^2 \right) g \left[ \left( \tfrac{r}{2n} - \alpha^2 \right) (2n\alpha^2 (2n+1) - r) - \alpha^2 (2n+1) (r - 2n\alpha^2) \right] \eta \otimes \eta + 2n S^2 )</td>
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<td>( \left( r - (2n - 1) \alpha^2 \right) S = \left[ \left( \tfrac{r}{2n} - 2n\alpha^2 \right) (r - 2n\alpha^2) + \alpha^2 \left( r - |Q|^2 \right) g \left[ \left( \tfrac{r}{2n} - \alpha^2 \right) (2n\alpha^2 (2n+1) - r) + \left( \tfrac{r}{2n} - \alpha^2 (2n+1) \right) \right] \eta \otimes \eta + 2n S^2 )</td>
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</table>
4. α-Sasakian manifolds with $W \cdot S = 0$

Let $M^{2n+1}(\phi, \xi, \eta, g)$ $n > 1$ be a quasi-conformal like α-Sasakian manifold, satisfying the condition

\[(4.1) \quad W(\xi, Y) \cdot S = 0,\]

i.e.,

\[(4.2) \quad W(\xi, Y)S(Z, U) - S(W(\xi, Y)Z, U) - S(Z, W(\xi, Y)U) = 0,\]

\[S(W(\xi, Y)Z, U) + S(Z, W(\xi, Y)U) = 0.\]

Setting $U = \xi$ in (4.2) and using (2.10), we get

\[(4.3) \quad 2n\alpha^2 \eta(W(\xi, Y)Z) + S(Z, W(\xi, Y)\xi) = 0.\]

In view of (2.9), (2.10) and (1.2), we have:

\[(4.4) \quad \eta(W(\xi, X)Y) = \left[\alpha^2(1 + 2nb) - \frac{cr}{2n + 1} \left(\frac{1}{2n} + a + b\right)\right] g(X, Y)\]

\[+ \left[\alpha^2(1 + 2na + 2nb) - \frac{cr}{2n + 1} \left(\frac{1}{2n} + a + b\right)\right] \eta(Y)\eta(X)\]

\[+ aS(X, Y).\]

\[(4.5) \quad S(Y, W(\xi, X)\xi) = -\left[\alpha^2(1 + 2na) - \frac{cr}{2n + 1} \left(\frac{1}{2n} + a + b\right)\right] S(X, Y)\]

\[+ 2n\alpha^2 \left[\alpha^2(1 + 2na + 2nb) - \frac{cr}{2n + 1} \left(\frac{1}{2n} + a + b\right)\right] \eta(Y)\eta(X)\]

\[- bS^2(X, Y).\]

By virtue of (4.4) and (4.5), (4.3) yields

\[(4.6) \quad S^2(Y, Z) = \frac{2n\alpha^2}{b} \left[\alpha^2(1 + 2nb) - \frac{cr}{2n + 1} \left(\frac{1}{2n} + a + b\right)\right] g(X, Y),\]

\[+ \frac{1}{b} \left[\alpha^2 - \frac{cr}{2n + 1} \left(\frac{1}{2n} + a + b\right)\right] S(X, Y)\]

for all $X, Y$ provided $b \neq 0$, where $S^2(X, Y) = S(QX, Y)$.

**Theorem 4.1.** Let $(M^{2n+1}, g)$, $n > 1$, be an α-Sasakian manifold with $W(\xi, Y) \cdot S = 0$. Then the Ricci tensor of the manifold $M$ satisfies the relation (4.6) provided $b \neq 0$.

Again, for $b = 0$, the foregoing equation reduces to

\[(4.7) \quad S(X, Y) = 2n\alpha^2 g(X, Y) \quad \text{provided} \quad \alpha^2 \neq \frac{cr}{2n + 1} \left(\frac{1}{2n} + a\right).\]

**Theorem 4.2.** Let $(M^{2n+1}, g)$, $n > 1$, be an α-Sasakian manifold with $W(\xi, Y) \cdot S = 0$. Then the manifold is Einstein.
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