Acta Math. Univ. Comenianae
Vol. LXXXVI, 1 (2017), pp. 91–100

SEMI-SYMMETRIC TYPE OF α-SASAKIAN MANIFOLDS

K. K. BAISHYA AND P. R. CHOWDHURY

Abstract. Recently the present author introduced the notion of generalized quasi-conformal curvature tensor which bridges Conformal curvature tensor, Concircular curvature tensor, Projective curvature tensor and Conharmonic curvature tensor. This article attempts to characterize α-Sasakian manifolds with $\omega(X,Y)\cdot\mathcal{W}=0$. Based on this curvature conditions, we obtain and table the expression for the Ricci tensor for the respective semi-symmetric type of α-Sasakian manifolds.

1. Introduction

In 1968, Yano and Sawaki [**12**] introduced the notion of quasi-conformal curvature tensor in the context of Riemannian geometry. Recently, in tune with Yano and Sawaki [**12**], the present authors [**9**] introduced and studied a *generalized quasi-conformal curvature tensor in the context of* an $N(k,\mu)$-manifold. The beauty of such curvature tensor lies in the fact that it has the flavour of a Riemann curvature tensor R, a conformal curvature tensor C [**13**], a conharmonic curvature tensor $\hat{C}$ [**18**], a concircular curvature tensor E ([**11**, p. 84]), a projective curvature tensor P ([**11**, p. 84]) and an m-projective curvature tensor H [**8**] as particular cases. Hereafter in this paper, a *generalized quasi-conformal curvature tensor* $\mathcal{W}$ is called a *quasi-conformal like curvature tensor.* The components of a *quasi-conformal like curvature tensor* $\mathcal{W}$ in a Riemannian manifold (M^{2n+1},g) $n>1$, are given by

$$\begin{aligned}\mathcal{W}(X,Y)Z=&\frac{2n-1}{2n+1}\Big[((1+2na-b)-1+2n(a+b))c\Big]C(X,Y)Z\\&+\Big[1-b+2na\Big]E(X,Y)Z+2n(b-a)P(X,Y)Z\\&+\frac{2n-1}{2n+1}(c-1)(1+2n(a+b))\hat{C}(X,Y)Z,\end{aligned}\tag{1.1}$$

where a,b & c are real constants. The foregoing equation can also be written as

$$\begin{aligned}\mathcal{W}(X,Y)Z=&\,R(X,Y)Z+a\Big[S(Y,Z)X-S(X,Z)Y\Big]+b\Big[g(Y,Z)QX-g(X,Z)QY\Big]\\&-\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big[g(Y,Z)X-g(X,Z)Y\Big].\end{aligned}\tag{1.2}$$

Received October 12, 2015; revised December 24, 2015.
2000 *Mathematics Subject Classification.* Primary 53C15, 53C25.
Key words and phrases. generalized quasi-conformal curvature tensor; semi-symmetric type; α-Sasakian manifolds.

R, S, Q and r are a Christoffel Riemannian curvature tensor, a Ricci tensor, a Ricci operator and a scalar curvature, respectively. In particular, the *quasi conformal like curvature tensor* $\mathcal{W}$ induce to

(1) Riemann curvature tensor R if $a = b = c = 0$,
(2) conformal curvature tensor C if $a = b = -\frac{1}{2n-1}$, $c = 1$,
(3) conharmonic curvature tensor $\hat{C}$ if $a = b = -\frac{1}{2n-1}$, $c = 0$,
(4) concircular curvature tensor E if $a = b = 0$ and $c = 1$,
(5) projective curvature tensor P if $a = -\frac{1}{2n}$, $b = 0$, $c = 0$ and
(6) m-projective curvature tensor H, if $a = b = -\frac{1}{4n}$, $c = 0$.

An α-Sasakian manifold is said to be semi-symmetry type (respectively Ricci semi-symmetry type) if the *quasi-conformal like curvature tensor* $\mathcal{W}$ (respectively, Ricci tensor S) admits the condition [**19**]

$$(1.3)\quad \omega(X,Y)\cdot\mathcal{W} = 0, \quad \text{(respectively,} \quad \mathcal{W}(X,Y)\cdot S = 0) \quad \text{for any } X, Y \text{ on } M,$$

where the dot means that $\omega(X,Y)$ acts on $\mathcal{W}$ (respectively, on S) as derivation. Here, both ω and $\mathcal{W}$ stand for a *quasi-conformal like curvature tensor* with respect to the associated scalar triples $(\bar{a}, \bar{b}, \bar{c})$ and (a, b, c), respectively. In particular, a manifold satisfying the condition $R(X,Y)\cdot R = 0$ (obtained from (1.3) by setting $\bar{a} = \bar{b} = \bar{c} = 0 = a = b = c$) is said to be semi-symmetric in the sense of Cartan ([**5**, § 265]). A full classification of such space is given by Z. I. Szabő [**19**]. This type of the manifolds were studied by several authors such as Papantoniou [**1**], Perrone [**4**], and the references cited therein.

Our work is structured as follows. Section 2 is a very brief account of α-Sasakian manifolds. In Section 3, we investigate α-Sasakianmanifolds with $\omega(X,Y)\cdot\mathcal{W} = 0$. Based on this curvature condition, we obtain and table the expressions for the Ricci tensor for respective curvature restrictions. Finally, we bring out an expression for the Ricci tensor of an α-Sasakian manifolds with $\mathcal{W}(X,Y)\cdot S = 0$.

2. α-Sasakian manifolds

A contact manifold is a $(2n+1)$-dimensional C^∞-manifold M equipped with a global form η, called a contact form of M such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. In particular, $\eta \wedge (d\eta)^n \neq 0$ is a volume element of M so that a contact manifold is orientable. A contact manifold associated with the Riemannian metric g is called a contact metric manifold if it satisfies the following relation

$$(2.1)\qquad d\eta(X,Y) = g(X, \phi Y), \qquad \eta(X) = g(X, \xi), \qquad \phi^2 = -I + \eta \circ \xi,$$

where ϕ is a $(1,1)$-tensor field and ξ is a unique vector field such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. We denote the symbols ∇, R and Q by Levi-Civita connection, a curvature tensor and a Ricci operator of g, respectively. We define a (1,1) type tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$ and we know that h and $h\phi$ are trace free and $h\phi = -\phi h$. An almost contact manifold $M(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold if there exist two functions on M such that

$$(2.2)\qquad (\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

for any vector fields X, Y on M. If $\beta = 0$, then M is a α-Sasakian manifold. Sasakian manifolds are cases of α-Sasakian manifolds with $\alpha = 1$. If $\alpha = 0$, then M is called β-Kenmotsu manifold. Kenmotsu manifolds are cases of a β-Kenmotsu with $\beta = 1$. If both α and β vanish, then M is a cosymplectic manifold. In an α-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, the following relations hold [**2**]:

$$\nabla_X \xi = -\alpha\phi X, \tag{2.3}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.5}$$

for all vector fields X, Y on M^{2n+1} where α is a smooth function on M and we say that the α-Sasakian structure is of type $(\alpha, 0)$. From (2.4), it follows that

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y), \tag{2.6}$$

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.7}$$

$$\eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.8}$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.9}$$

$$S(X, \xi) = 2n\alpha^2\eta(X) \tag{2.10}$$

for any vector fields X, Y on M.

3. α-Sasakian manifolds with $\omega(X, Y) \cdot \mathcal{W} = 0$

Let us consider a $(2n + 1)$-dimensional α-Sasakian manifold M, satisfying the condition

$$(\omega(X, Y) \cdot \mathcal{W})(Z, U)V = 0 \tag{3.1}$$

i.e.,

$$\begin{aligned}\omega(X, Y)\mathcal{W}(Z, U)V = {} & \mathcal{W}(\omega(X, Y)Z, U)V + \mathcal{W}(Z, \omega(X, Y)U)V \\ & + \mathcal{W}(Z, U)\omega(X, Y)V\end{aligned} \tag{3.2}$$

which can be transformed into

$$\begin{aligned}& g(\omega(\xi, X(\mathcal{W}(Y, Z)U, \xi) - g(\mathcal{W}(\omega(\xi, X)Y, Z)U, \xi) \\ & -g(\mathcal{W}(Y, \omega(\xi, X)Z)U, \xi) - g(\mathcal{W}(Y, Z)\omega(\xi, X)U, \xi) = 0.\end{aligned} \tag{3.3}$$

Putting $X = Y = e_i$ in (3.3), where $\{e_1, e_2, e_3, \dots, e_{2n}, e_{2n+1} = \xi\}$ is an orthonormal basis of the tangent space at each point of the manifold M and taking the summation over i, $1 \le i \le 2n + 1$, we get

$$\begin{aligned}\sum_{i=1}^{2n+1} \Big[& g(\omega(\xi, e_i)\mathcal{W}(e_i, Z)U, \xi) - g(\mathcal{W}(\omega(\xi, e_i)e_i, Z)U, \xi) \\ & - g(\mathcal{W}(e_i, \omega(\xi, e)_i Z)U, \xi) - g(\mathcal{W}(e_i, Z)\omega(\xi, e_i)U, \xi)\Big] = 0.\end{aligned} \tag{3.4}$$

From the equation (1.2), we can easily bring out the following:

$$\begin{aligned}\eta(\mathcal{W}(\xi,U)Z) = \Big[&\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big) - 2n\alpha^2 a - 2n\alpha^2 b - \alpha^2\Big]\eta(Z)\eta(U)\\ &+ \Big[\alpha^2 + 2nb\alpha^2 - \frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]g(Z,U) + aS(Z,U),\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\sum_{i=1}^{2n+1}\bar{\mathcal{W}}(e_i,Z,U,e_i)\\ &= (1-b+2na)S(Z,U) + \Big(br - \frac{2ncr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big)g(Z,U),\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\sum_{i=1}^{2n+1}\eta)\mathcal{W}(e_i,Z)e_i)\\ &= \Big[-2n\alpha^2(1-a+2nb) - \Big\{ar - \frac{2ncr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big\}\Big]\eta(Z),\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\sum_{i=1}^{2n+1}S)\mathcal{W}(e_i,Z)U,e_i)\\ &= \Big(ar + \frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big)S(Z,U) - (a+b-1)S^2(Z,U)\\ &\quad + \Big(b\|Q\|^2 - \frac{cr^2}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big)g(Z,U),\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\sum_{i=1}^{2n+1}\eta(e_i)\eta(\mathcal{W}(Qe_iZ)U)\\ &= 2n\alpha^2\Big[\alpha^2 + 2n\alpha^2 b - \frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]g(Z,U) + 2n\alpha^2 aS(Z,U)\\ &\quad - 2n\alpha^2\Big[\alpha^2 + 2n\alpha^2(a+b) - \frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]\eta(Z)\eta(U),\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\sum_{i=1}^{2n+1}S(e_i,Z)\eta(\mathcal{W}(e_i,\xi)U)\\ &= 2n\alpha^2\Big[\alpha^2 + 2n\alpha^2(a+b) - \frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]\eta(Z)\eta(U)\\ &\quad - \Big\{\alpha^2 + 2n\alpha^2 b - \frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big\}S(Z,U) - aS^2(Z,U).\end{aligned} \tag{3.10}$$

Now,

$$\begin{aligned}&\sum_{i=1}^{2n+1}g(\omega(\xi,e_i)\mathcal{W}(e_i,Z)U,\xi)\\ &= \Big[\alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)\Big]\Big[\bar{\mathcal{W}}(e_i,Z,U,e_i) - \eta(\mathcal{W}(e_i,Z)U)\eta(e_i)\Big]\\ &\quad + \bar{a}\Big[S(\mathcal{W}(e_i,Z)U,e_i) - 2n\alpha^2\eta(\mathcal{W}(e_i,Z)U)\eta(e_i)\Big]\\ &= \Big[\alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)\Big]\bar{\mathcal{W}}(e_i,Z,U,e_i) + \bar{a}S(\mathcal{W}(e_i,Z)U,e_i)\\ &\quad + \Big[\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big) - 2n\alpha^2\bar{b} - \alpha^2 - 2n\alpha^2\bar{a}\Big]\eta(\mathcal{W}(\xi,U)Z).\end{aligned} \tag{3.11}$$

In view of (3.6), (3.8) and (3.11) becomes

$$
\begin{aligned}
\sum_{i=1}^{2n+1} g(\omega(\xi, e_i)\mathcal{W}(e_i, Z)U, \xi) &= \Big[\Big(\alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big)\Big)(1 + 2na - b) \\
&+ \bar{a}\Big(ar + \frac{cr}{2n+1}\Big(\frac{1}{2n} + a + b\Big)\Big)\Big]S(Z,U) + \bar{a}(1 - a - b)S^2(Z,U) \\
&+ \Big[\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big) - 2n\alpha^2(\bar{a} + \bar{b}) - \alpha^2\Big]\eta(\mathcal{W}(\xi, U)Z) \\
&+ \Big[\Big(\alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big)\Big)\Big(br - \frac{2ncr}{2n+1}\Big(\frac{1}{2n} + a + b\Big)\Big) \\
&+ \bar{a}\Big(b\|Q\|^2 - \frac{cr^2}{2n+1}\Big(\frac{1}{2n} + a + b\Big)\Big)\Big]g(Z,U).
\end{aligned}
\tag{3.12}
$$

In consequence of (3.5)–(3.10), we obtain the following:

$$
\begin{aligned}
\sum_{i=1}^{2n+1} g(\mathcal{W}(\omega(\xi, e_i)e_i, Z)U, \xi) &= \Big[(2n+1)\Big[\alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big)\Big] \\
&+ \bar{a}r\Big]\eta(\mathcal{W}(\xi, U)Z) + \Big[\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big) - 2n\alpha^2\bar{a} - \alpha^2\Big]g(\mathcal{W}(e_i, Z)U, \xi)\eta(e_i),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{2n+1} g(\mathcal{W}(Qe_i, Z)U, \xi)\eta(e_i) \\
&= \Big[2n\Big(\alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big)\Big) + \bar{a}r + 2n\alpha^2(\bar{b} - \bar{a})\Big]\eta(\mathcal{W}(\xi, U)Z) \\
&- 2n\alpha^2\bar{b}\Big[\alpha^2 + 2n\alpha^2 b - \frac{cr}{2n+1}\Big(\frac{1}{2n} + a + b\Big)\Big]g(Z,U) \\
&+ 2n\alpha^2\bar{b}\Big[\alpha^2 + 2n\alpha^2(a+b) - \frac{cr}{2n+1}\Big(\frac{1}{2n} + a + b\Big)\Big]\eta(Z)\eta(U) - 2n\alpha^2 a\bar{b}S(Z,U),
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&\sum_{i=1}^{2n+1} g(\mathcal{W}(e_i, \omega(\xi, e_i)Z)U, \xi) \\
&= \Big[\alpha^2 + 2n\alpha^2\bar{b} - \frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big)\Big]g(e_i, Z)\eta(\mathcal{W}(e_i, \xi)U) \\
&\quad \Big[\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big) - 2n\alpha^2\bar{a} - \alpha^2\Big]\eta(\mathcal{W}(e_i, e_i)U)\eta(Z) \\
&\quad + \bar{a}S(e_i, Z)\eta(\mathcal{W}(e_i, \xi)U) - \bar{b}\eta(\mathcal{W}(e_i, Qe_i)U)\eta(Z) \\
&= \Big[\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n} + \bar{a} + \bar{b}\Big) - 2n\alpha^2\bar{b} - \alpha^2\Big]\eta(\mathcal{W}(\xi, U)Z) \\
&\quad + 2n\alpha^2\bar{a}\Big[\alpha^2 + 2n\alpha^2(a+b) - \frac{cr}{2n+1}\Big(\frac{1}{2n} + a + b\Big)\Big]\eta(Z)\eta(U) \\
&\quad - \bar{a}\Big[\alpha^2 + 2n\alpha^2 b - \frac{cr}{2n+1}\Big(\frac{1}{2n} + a + b\Big)\Big]S(Z,U) - \bar{a}aS^2(Z,U),
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&\sum_{i=1}^{2n+1} g(\mathcal{W}(e_i,Z)\omega(\xi,e_i)U,\xi)\\
&=\Big[\alpha^2+2n\alpha^2\bar{b}-\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)\Big]g(e_i,U)\eta(\mathcal{W}(e_i,Z)\xi)\\
&\quad+\Big[\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)-2n\alpha^2\bar{a}-\alpha^2\Big]\eta(\mathcal{W})e_i,Z(e_i)\eta(U)\\
&\quad+\bar{a}S(e_i,U)\eta(\mathcal{W}(e_i,Z)\xi)\\
&=\Big[\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)-2n\alpha^2\bar{a}-\alpha^2\Big]\eta(\mathcal{W})e_i,Z)e_i)\eta)U)
\end{aligned}
\tag{3.15}
$$

which is equivalemt to

$$
\begin{aligned}
&\sum_{i=1}^{2n+1} g(\mathcal{W}(e_i,Z)\omega(\xi,e_i)U,\xi)\\
&=-\Big[\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)-2n\alpha^2\bar{a}-\alpha^2\Big]\Big[2n\alpha^2(1-a+2nb)\\
&\quad+\Big\{ar-\frac{2ncr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big\}\Big]\eta(U)\eta(Z).
\end{aligned}
\tag{3.16}
$$

By virtue of (3.12), (3.13), (3.14), and (3.16), the equation (3.4) yields

$$
\begin{aligned}
&\Big[\Big(\alpha^2+2n\alpha^2\bar{b}-\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)\Big)(1-b)+\alpha^2\bar{a}(1+2nb)\Big]S(Z,U)\\
&\quad+\Big[\Big(2n\alpha^2(1-a+2nb)+ar-\frac{2ncr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big)\\
&\quad\times\Big(\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)-2n\alpha^2\bar{a}-\alpha^2\Big)\\
&\quad+\Big\{\alpha^2+2n\alpha^2(a+b)-\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big\}\\
&\quad\times\Big[2n\Big[\alpha^2+2n\alpha^2\bar{b}-\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)\Big]+\bar{a}r-2n\alpha^2\bar{a}\Big]\Big]\eta(U)\eta(Z)\\
&\quad+\Big[\Big(\alpha^2+2n\alpha^2\bar{b}-\frac{\bar{c}r}{2n+1}\Big(\frac{1}{2n}+\bar{a}+\bar{b}\Big)\Big)\Big(br-2n\alpha^2-4n^2\alpha^2b\Big)\\
&\quad+\bar{a}\Big(b\|Q\|^2-\alpha^2r-2n\alpha^2rb\Big)\Big]g(Z,U)+\bar{a}(1-b)S^2(Z,U)=0.
\end{aligned}
\tag{3.17}
$$

Theorem 3.1. *Let (M^{2n+1},g), $n>1$, be an α-Sasakian manifold. Then for respective semi-symmetric type conditions, the Ricci tensor of the manifold M takes the respective forms as follows:*

Curvature condition	Expression for Ricci tensor
$R(X,Y)\cdot R=0$	$S=2n\alpha^2 g$
$R(X,Y)\cdot C=0$	$S=\big(\frac{r}{2n}-\alpha^2\big)g-\big(\frac{r}{2n}-(2n+1)\alpha^2\big)\eta\otimes\eta$
$R(X,Y)\cdot \hat{C}=0$	$S=\big(\frac{r}{2n}-\alpha^2\big)g-\big(\frac{r}{2n}-(2n+1)\alpha^2\big)\eta\otimes\eta$

Curvature condition	Expression for Ricci tensor
$R(X,Y)\cdot E=0$	$S=2n\alpha^2 g$
$R(X,Y)\cdot P=0$	$S=2n\alpha^2 g-(\frac{r}{2n}-2n-1)\eta\otimes\eta$
$R(X,Y)\cdot H=0$	$S=\big(\frac{r+4n^2\alpha^2}{4n+1}\big)g-\big(\frac{r-2n\alpha^2(2n+1)}{4n+1}\big)\eta\otimes\eta$
$E(X,Y)\cdot R=0$	$S=2n\alpha^2 g \quad \text{for } \alpha^2\neq-\frac{r}{2n(2n+1)}$
$E(X,Y)\cdot C=0$	$S=\big(\frac{r}{2n}-\alpha^2\big)g-\big(\frac{r}{2n}-(2n+1)\alpha^2\big)\eta\otimes\eta$ $\text{for } \ \alpha^2\neq-\frac{r}{2n(2n+1)}$
$E(X,Y)\cdot\hat{C}=0$	$S=\big(\frac{r}{2n}-\alpha^2\big)g-\big(\frac{r}{2n}-(2n+1)\alpha^2\big)\eta\otimes\eta$ $\text{for } \ \alpha^2\neq-\frac{r}{2n(2n+1)}$
$E(X,Y)\cdot E=0$	$S=2n\alpha^2 g \quad \text{for } \alpha^2\neq-\frac{r}{2n(2n+1)},$
$E(X,Y)\cdot P=0$	$S=2n\alpha^2 g-\big(\frac{r}{2n}-2\alpha^2\big)\eta\otimes\eta \quad \text{for } \alpha^2\neq-\frac{r}{2n(2n+1)}$
$E(X,Y)\cdot H=0$	$S=\big(\frac{r+4n^2\alpha^2}{4n+1}\big)g-\big(\frac{r-2n(2n+1)\alpha^2}{4n+1}\big)\eta\otimes\eta$ $\text{for } \alpha^2\neq-\frac{r}{2n(2n+1)}$
$\hat{C}(X,Y)\cdot R=0$	$\big(\frac{r}{2n}-2\alpha^2\big)S=-2n\alpha^4 g+\alpha^2\big(r-2n\alpha^2\big)\eta\otimes\eta+S^2$
$\hat{C}(X,Y)\cdot\hat{C}=0$	$(r-(2n-1)\alpha^2)S$ $=\big[\big(\frac{r}{2n}-2n\alpha^2\big)(r-2n\alpha^2)+\alpha^2 r-\\|Q\\|^2)\big]g$ $+2nS^2\big[\big(\frac{r}{2n}-\alpha^2\big)(2n\alpha^2(2n+1)-r)$ $-\alpha^2(2n+1)\big(r-2n\alpha^2\big)\big]\eta\otimes\eta$
$\hat{C}(X,Y)\cdot C=0$	$(r-(2n-1)\alpha^2)S$ $=\big[\big(\frac{r}{2n}-2n\alpha^2\big)(r-2n\alpha^2)+\alpha^2 r-\\|Q\\|^2]g$ $+2nS^2\big[\big(\frac{r}{2n}-\alpha^2\big)[2n\alpha^2(2n+1)-r]$ $+(\frac{r}{2n}-\alpha^2(2n+1))\big(r-2n\alpha^2\big)\big]\eta\otimes\eta$
$\hat{C}(X,Y)\cdot E=0$	$2S=-S^2+[r+2n]g+(2n-r)\big(\frac{r}{2n(2n+1)}-1\big)\eta\otimes\eta$
$\hat{C}(X,Y)\cdot P=0$	$\big(\frac{r}{2n}-2\alpha^2\big)S$ $=-2n\alpha^4 g+S^2+\big(\frac{r}{2n}-\alpha^2\big)\big((2n+1)\alpha^2-\frac{r}{2n}\big)\eta\otimes\eta$
$\hat{C}(X,Y)\cdot H=0$	$\frac{(4n+1)+2n}{4n}S=\big[\big(\frac{r}{4n}+n\big)+\frac{r}{2}+\frac{1}{4n}\\|Q\\|^2\big]g$ $-\big(1+\frac{1}{4n}\big)S^2+\big(\frac{r}{4n}-\frac{2n+1}{2}\big)\eta\otimes\eta$
$P(X,Y)\cdot R=0$	$\big(\frac{2n-1}{2n}\alpha^2\big)S=\alpha^2\big(-\frac{r}{2n}+2n\alpha^2\big)g+\frac{1}{2n}S^2$

Curvature condition	Expression for Ricci tensor
$P(X,Y)\cdot\hat{C}=0$	$\big(\frac{4n^2+1}{2n}\alpha^2\big)S = S^2 + \big((\alpha^2(r-2n\alpha^2)+\frac{1}{2n}(\alpha^2 r-\|Q\|^2)\big)g$ $+(2n+1)\big(-2n\alpha^4+\Big(\alpha^2-\frac{r}{2n}\Big)\alpha^2\big)\eta\otimes\eta$
$P(X,Y)\cdot C=0$	$\big(\frac{4n^2+1}{2n}\alpha^2\big)S = S^2 + \big(\alpha^2(r-2n\alpha^2)+\frac{1}{2n}(\alpha^2 r-\|Q\|^2)\big)g$ $+(2n+1)\big[\big(\alpha^2-\frac{r}{2n}\big)\alpha^2-\alpha^2\big(r+2n\alpha^2\big)\big]\eta\otimes\eta$
$P(X,Y)\cdot E=0$	$\big(\frac{2n-1}{2n}\alpha^2\big)S=\alpha^2\big(-\frac{r}{2n}+2n\alpha^2\big)g+\frac{1}{2n}S^2$ $+\frac{1}{2n}\big(2n\alpha^2-\frac{r}{2n+1}\big)\big(\frac{r}{2n}-(2n+1)\alpha^2\big)\eta\otimes\eta$
$P(X,Y)\cdot P=0$	$\big(\frac{2n-1}{2n}\alpha^2\big)S=\frac{1}{2n}S^2+\alpha^2\big(-\frac{r}{2n}+2n\alpha^2\big)g$
$P(X,Y)\cdot H=0$	$\alpha^2 S=\frac{1}{2n}\big(1+\frac{1}{4n}\big)S^2+\big(n\alpha^4-\frac{1}{8n^2}\|Q\|^2\big)$
$H(X,Y)\cdot R=0$	$\big(\frac{2n-1}{4n}\alpha^2\big)S=\alpha^2\big(-\frac{r}{4n}+n\alpha^2\big)g$ $+\alpha^2\big(\frac{r}{4n}-\frac{1}{2}\big)\eta\otimes\eta+\frac{1}{4n}S^2$
$H(X,Y)\cdot\hat{C}=0$	$\big(\frac{4n^2+1}{4n}\alpha^2\big)S=\frac{1}{2n}S^2+\big(\frac{\alpha^2}{2}(r-2n\alpha^2)+\frac{1}{4n}(\alpha^2 r-\|Q\|^2)\big)g$ $+\big[-r\frac{\alpha^2}{2}+\alpha^2(2n+1)\big(\frac{2n+1}{2}\alpha^2-\frac{r}{4n}\big)\big]\eta\otimes\eta$
$H(X,Y)\cdot E=0$	$\big(\frac{2n-1}{4n}\alpha^2\big)S=\alpha^2\big(-\frac{r}{4n}+n\alpha^2\big)g$ $+\big(\frac{(n+1)\alpha^2 r}{2n(2n+1)}-\frac{r^2}{8n^2(2n+1)}+\frac{\alpha^4}{2}\big)\eta\otimes\eta+\frac{1}{4n}S^2$
$H(X,Y)\cdot P=0$	$\big(\frac{2n-1}{4n}\alpha^2\big)S=\alpha^2\big(-\frac{r}{4n}+n\alpha^2\big)g$ $+\big(\frac{r}{2n}-(2n+1)\alpha^2\big)(-\frac{\alpha^2}{2})\eta\otimes\eta+\frac{1}{4n}S^2$
$H(X,Y)\cdot H=0$	$\big(\frac{1}{2}\alpha^2\big)S=\big(-\frac{\alpha^2 r}{8n}-\frac{1}{16n^2}\|Q\|^2+\frac{\alpha^2}{2}\big(\frac{r}{4n}+n\alpha^2\big)\big)g$ $+\frac{1}{4n}\big(1+\frac{1}{4n}\big)S^2+\big(\frac{r}{4n}-\frac{2n+1}{2}\alpha^2\big)(-\frac{\alpha^2}{2})\eta\otimes\eta$
$C(X,Y)\cdot R=0$	$\big(\frac{r}{2n}-2\alpha^2\big)S=-2n\alpha^4 g+\alpha^2\big(r-2n\alpha^2\big)\eta\otimes\eta+S^2$
$C(X,Y)\cdot\hat{C}=0$	$(r-(2n-1)\alpha^2)S=\big[\big(\frac{r}{2n}-2n\alpha^2\big)(r-2n\alpha^2)$ $+\alpha^2 r-\|Q\|^2\big]g\big[\big(\frac{r}{2n}-\alpha^2\big)(2n\alpha^2(2n+1)-r)$ $-\alpha^2(2n+1)(r-2n\alpha^2)\big]\eta\otimes\eta+2nS^2$
$C(X,Y)\cdot C=0$	$\{r-(2n-1)\alpha^2\}S=\big[\big(\frac{r}{2n}-2n\alpha^2\big)(r-2n\alpha^2)$ $+\alpha^2 r-\|Q\|^2\big]g\big[\big(\frac{r}{2n}-\alpha^2\big)(2n\alpha^2(2n+1)-r)$ $+\big(\frac{r}{2n}-\alpha^2(2n+1)\big)(r-2n\alpha^2)\big]\eta\otimes\eta+2nS^2$
$C(X,Y)\cdot P=0$	$\big(\frac{r}{2n}-2\alpha^2\big)S=-2n\alpha^4 g+S^2$ $+\big(\frac{r}{2n}-\alpha^2\big)\big((2n+1)\alpha^2-\frac{r}{2n}\big)\eta\otimes\eta$
$C(X,Y)\cdot E=0$	$\big(\frac{r}{2n}-2\alpha^2\big)S=-2n\alpha^4 g+\alpha^2\big(r-2n\alpha^2\big)\eta\otimes\eta+S^2$

4. α-Sasakian manifolds with $\mathcal{W}\cdot S=0$

Let $M^{2n+1}(\phi,\xi,\eta,g)$ $n>1$ be a quasi-conformal like α-Sasakian manifold, satisfying the condition

$$\mathcal{W}(\xi,Y)\cdot S=0, \tag{4.1}$$

i.e.,

$$\begin{aligned}\mathcal{W}(\xi,Y)S(Z,U)-S(\mathcal{W}(\xi,Y)Z,U)-S(Z,\mathcal{W}(\xi,Y)U)&=0,\\ S(\mathcal{W}(\xi,Y)Z,U)+S(Z,\mathcal{W}(\xi,Y)U)&=0.\end{aligned} \tag{4.2}$$

Setting $U=\xi$ in (4.2) and using (2.10), we get

$$2n\alpha^2\eta(\mathcal{W}(\xi,Y)Z)+S(Z,\mathcal{W}(\xi,Y)\xi)=0. \tag{4.3}$$

In view of (2.9), (2.10) and (1.2), we have:

$$\begin{aligned}\eta(\mathcal{W}(\xi,X)Y)=&\Big[\alpha^2(1+2nb)-\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]g(X,Y)\\ &-\Big[\alpha^2(1+2na+2nb)-\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]\eta(Y)\eta(X)\\ &+aS(X,Y).\end{aligned} \tag{4.4}$$

$$\begin{aligned}S(Y,\mathcal{W}(\xi,X)\xi)=&-\Big[\alpha^2(1+2na)-\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]S(X,Y)\\ &+2n\alpha^2\Big[\alpha^2(1+2na+2nb)-\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]\eta(Y)\eta(X)\\ &-bS^2(X,Y).\end{aligned} \tag{4.5}$$

By virtue of (4.4) and (4.5), (4.3) yields

$$\begin{aligned}S^2(Y,Z)=&\frac{2n\alpha^2}{b}\Big[\alpha^2(1+2nb)-\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]g(X,Y),\\ &-\frac{1}{b}\Big[\alpha^2-\frac{cr}{2n+1}\Big(\frac{1}{2n}+a+b\Big)\Big]S(X,Y)\end{aligned} \tag{4.6}$$

for all X,Y provided $b\neq 0$, where $S^2(X,Y)=S(QX,Y)$.

Theorem 4.1. *Let (M^{2n+1},g), $n>1$, be an α-Sasakian manifold with $\mathcal{W}(\xi,Y)\cdot S=0$. Then the Ricci tensor of the manifold M satisfies the relation (4.6) provided $b\neq 0$.*

Again, for $b=0$, the foregoing equation reduces to

$$S(X,Y)=2n\alpha^2g(X,Y)\quad\text{provided}\quad\alpha^2\neq\frac{cr}{2n+1}\Big(\frac{1}{2n}+a\Big). \tag{4.7}$$

Theorem 4.2. *Let (M^{2n+1},g), $n>1$, be an α-Sasakian manifold with $\mathcal{W}(\xi,Y)\cdot S=0$. Then the manifold is Einstein.*

References

1. Papantoniou B. J., *Contact Riemannian manifolds satisfying* $R(\xi, X) \cdot R = 0$ *and* $\xi \in (k, \mu)$*-nullity distribution*, Yokohama Math. J., **40** (1993), 149–161.

2. Blair D. E., *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics **509**, Springer-Verlag, Berlin, 1976.

3. Blair D. E., *Two remarks on contact metric structures*, Tohoku Math. J. **29** (1977), 319–324.

4. Perrone D., *Contact Riemannian manifolds satisfying* $R(\xi, X) \cdot R = 0$, Yokohama Math. J. **39(2)** (1992), 141–149.

5. Cartan E., *Sur une classe remarquable d'espaces de Riemannian*, Bull. Soc. Math. France **54** (1926), 214-264.

6. Boeckx E., Kowalski O. L. and Vanhecke L., *Riemannian Manifolds of Conullity Two*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

7. Calvaruso G. and Perrone D., *Semi-symmetric contact metric three-manifolds*, Yokohama Math. J. **49(2)** (2002), 149–161.

8. Pokhariyal G. P. and Mishra R. S., *Curvatur tensors′ and their relativistics significance II*, Yokohama Math J., **19** (1971), 97–103.

9. Baishya K. K. and Chowdhury P. R., *On generalized quasi-conformal* $N(k, \mu)$*-manifolds*, (to appear from Communications of the Korean Mathematical Society).

10. Nomizu K., *On the decomposition of generalized curvature tensor fields*, Differential Geometry in honor of K. Yano, Kinokuniya, Tokyo, 1972, 335–345.

11. Yano K. and Bochner S., *Curvature and Betti numbers*, Annals of Math. Studies 32, Princeton University Press, 1953.

12. Yano K. and Sawaki S., *Riemannian manifolds admitting a conformal transformation group,* J. Diff. Geom. **2** (1968), 161–184.

13. Eisenhart L. P., *Riemannian Geometry*, Princeton University Press, 1949.

14. Kowalski O. and Sekizawa M., *Three-dimensional Riemannian manifolds of conullity two, Riemannian Manifolds of Conullity Two,* Chapter 11, World Scientific, Singapore, 1996.

15. Haesen S. and Verstraelen L., *Natural intrinsic geometrical symmetries*, Symmetry, Integrability and Geometry: Methods and Applications SIGMA **5** (2009), 14 pp.

16. Sasaki S., *Lecture Notes on Almost Contact Manifolds*, *Part I*, Tohoku Univ, 1965.

17. Takahashi T., *Sasakian* ϕ*-symmetric spaces*, Tohoku Math. J., **29**(1977), 91–113.

18. Ishii Y., *On conharmonic transformations*, Tensor(N.S.), **7** (1957), 73–80.

19. Szabő Z. I., *Classification and construction of complete hypersurfaces satisfying* $R(X, Y) \cdot R = 0$, Acta. Sci. Math., **47** (1984), 321–348.

K. K. Baishya, Department of Mathematics, Kurseong College, Dowhill Road, Kurseong, Darjeeling 734 203, West Bengal, India, *e-mail*: `kanakkanti.kc@gmail.com`

P. R. Chowdhury, Shaktigarh Bidyapith(H.S), Siliguri, Darjeeling-734005, West Bengal, India, *e-mail*: `partha.raychowdhury81@gmail.com`