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# CONTACT CR-WARPED PRODUCT SUBMANIFOLDS OF $(LCS)_n$-MANIFOLDS

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Abstract. The present paper deals with a study of doubly warped product contact CR-submanifolds of $(LCS)_n$-manifolds and warped product contact CR-submanifolds of $(LCS)_n$-manifolds. It is shown that there exists no doubly warped product contact CR-submanifolds of $(LCS)_n$-manifolds. However, we obtain some results for the existence or non-existence of warped product contact CR-submanifolds of $(LCS)_n$-manifolds and the existence is also ensured by an interesting example.

## 1. Introduction

In 2003, Shaikh [**15**] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [**10**] and also by Mihai and Rosca [**11**]. Then Shaikh and Baishya ([**16**], [**17**]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology.

The contact CR-submanifolds are a rich and very interesting subject. The study of the differential geometry of contact CR-submanifolds as a generalization of invariant and anti-invariant submanifolds of almost contact metric manifolds was initiated by Bejancu [**3**]. Thereafter, several authors studied contact CR-submanifolds of different classes of almost contact metric manifolds such as Atceken ([**1**], [**2**]), Chen ([**5**], [**6**]), Hasegawa and Mihai [**7**], Khan et. al. [**9**], Munteanu [**12**], Murathan et. al. [**13**] and many others.

The notion of warped product manifolds were introduced by Bishop and O'Neill [**4**] and later it has been studied by many mathematicians and physicists. These manifolds are generalization of Riemannian product manifolds. The existence or non-existence of warped product manifolds plays an important role in differential geometry as well as physics.

Motivated by the studies, the present paper deals with the study of contact CR-warped product submanifolds of $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. The notion of doubly warped products is introduced by Unal [**18**]. The doubly warped product contact CR-submanifolds were studied by Munteanu [**12**], Khan et. al [**9**] and many

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others. Section 3 is devoted to the study of doubly warped product contact CR-submanifolds of $(LCS)_n$-manifolds. It is shown that there exists no doubly warped product contact CR-submanifolds of $(LCS)_n$-manifolds. In [**8**], first two authors studied warped product semi-slant submanifolds of $(LCS)_n$-manifolds and in [**8**], all results are the nonexistence of warped product semi-slant submanifolds of $(LCS)_n$-manifolds, and all results of [**8**] are related to proper semi-slant submanifolds, that is, the slant distribution is neither invariant nor anti-invariant. Now in the last section, we study warped product contact CR-submanifolds $M = N_\perp \times_f N_T$ of $(LCS)_n$-Manifolds $\overline{M}$ such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$. We distinguish two cases:(i) $\xi$ tangent to $N_T$ and (ii) $\xi$ tangent to $N_\perp$. In case (i) it is proved that there do not exist warped product contact CR-submanifolds $M = N_\perp \times_f N_T$ such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is anti-invariant submanifold of $\overline{M}$. However, in the case (ii), it is proved that there exist warped product contact CR-submanifolds $M = N_\perp \times_f N_T$ of $(LCS)_n$-manifolds $\overline{M}$ such that $N_\perp$ is an anti-invariant submanifold of dimension $p$ tangent to $\xi$ and $N_T$ is an invariant submanifold of $\overline{M}$. Thus it is an interesting result for researchers. Moreover, it is important that finally we present an example of such type of a contact CR-warped product submanifold of $(LCS)_7$-manifolds.

## 2. Preliminaries

An $n$-dimensional Lorentzian manifold $\overline{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $g$, that is, $\overline{M}$ admits a smooth symmetric tensor field $g$ of type $(0, 2)$ such that for each point $p \in \overline{M}$, the tensor $g_p \colon T_p\overline{M} \times T_p\overline{M} \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at $p$ and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_p\overline{M}$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., $\leq 0$, $= 0$, $> 0$) [**14**].

**Definition 2.1.** [**19**] A vector field $P$ on $\overline{M}$ is said to be concircular if the $(1, 1)$ tensor field defined by $g(X, P) = A(X)$ for any $X \in \Gamma(T\overline{M})$, satisfies

$$(\overline{\nabla}_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\},$$

where $\alpha$ is a non-zero scalar, $\omega$ is a closed 1-form and $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$.

Let $\overline{M}$ be an $n$-dimensional Lorentzian manifold admitting a unit timelike concircular vector field $\xi$ called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since $\xi$ is a unit concircular vector field, it follows that there exists a non-zero 1-form $\eta$ such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\overline{\nabla}_X\eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \qquad \alpha \neq 0, \tag{2.3}$$

that is,

$$\overline{\nabla}_X\xi = \alpha\{X + \eta(X)\xi\}, \qquad \alpha \neq 0, \tag{2.4}$$

for all vector fields $X$, $Y$, where $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$ and $\alpha$ is a non-zero scalar function satisfying

$$\overline{\nabla}_X\alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

$\rho$ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\overline{\nabla}_X\xi, \tag{2.6}$$

then from (2.3) and (2.6), we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

from which it follows that $\phi$ is a symmetric $(1,1)$ tensor called the structure tensor of the manifold. Thus the Lorentzian manifold $\overline{M}$ together with the unit timelike concircular vector field $\xi$, its associated 1-form $\eta$ and an $(1,1)$ tensor field $\phi$ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [**15**]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [**10**]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold [**15**]:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.9}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.10}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.11}$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)[g(Y,Z)\xi - \eta(Z)Y], \tag{2.12}$$

$$(\overline{\nabla}_X\phi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.13}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.14}$$

$$R(X,Y)Z = \phi R(X,Y)Z + (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \tag{2.15}$$

for all $X,\ Y,\ Z \in \Gamma(T\overline{M})$.

Let $M$ be a submanifold of a $(LCS)_n$-manifold $\overline{M}$ with induced metric $g$. Also let $\nabla$ and $\nabla^\perp$ be the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.16}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{2.17}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $h$ and $A_V$ are second fundamental form and the shape operator (corresponding to the normal vector field $V$), respectively, for the immersion of $M$ into $\overline{M}$. The second fundamental form $h$ and the shape operator $A_V$ are related by

$$g(h(X,Y),V) = g(A_V X, Y) \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

For any $X \in \Gamma(TM)$, we can write

$$\phi X = EX + FX, \tag{2.19}$$

where $EX$ is the tangential component and $FX$ is the normal component of $\phi X$.

Also, for any $V \in \Gamma(T^{\perp}M)$, $\phi V$ can be written in the following way

$$\phi V = BV + CV, \tag{2.20}$$

where $BV$ and $CV$ are also the tangential and normal components of $\phi V$, respectively. From (2.19) and (2.20), we can derive that the tensor fields $E$, $F$, $B$ and $C$ are also symmetric because $\phi$ is symmetric. The covariant derivatives of the tensor fields of $E$ and $F$ are defined as

$$(\nabla_X E)Y = \nabla_X EY - E(\nabla_X Y) \tag{2.21}$$

and

$$(\overline{\nabla}_X F)Y = \nabla^{\perp}_X FY - F(\nabla_X Y) \tag{2.22}$$

for all $X,\ Y \in \Gamma(TM)$. The canonical structures $E$ and $F$ on a submanifold $M$ are said to be parallel if $\nabla E = 0$ and $\overline{\nabla} F = 0$, respectively.

A submanifold $M$ tangent to $\xi$ is called a contact CR-submanifold if it admits an invariant distribution $D$ whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, i.e., $TM = D \oplus D^{\perp} \oplus < \xi >$ with $\phi(D_p) \subseteq D_p$ and $\phi(D^{\perp}_p) \subset T^{\perp}_p M$ for every $p \in M$. It may be mentioned that the contact CR-submanifold is a special case of semi-slant submanifolds.

The notion of warped product manifolds was introduced by Bishop and O'Neill [**4**].

**Definition 2.2.** [**4**] Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds with Riemannian metric $g_1$ and $g_2$, respectively, and $f$ be a positive definite smooth function on $N_1$. The warped product of $N_1$ and $N_2$ is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{2.23}$$

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function $f$ is constant.

More explicitely, if the vector fields $X$ and $Y$ are tangent to $N_1 \times_f N_2$ at $(p, q)$, then

$$g(X,Y) = g_1(\pi_1 * X, \pi_1 * Y) + f^2(p) g_2(\pi_2 * X, \pi_2 * Y),$$

where $\pi_i$ $(i = 1, 2)$ are the canonical projections of $N_1 \times N_2$ onto $N_1$ and $N_2$, respectively, and * stands for the derivative map. It may be noted that the notion of a warped product is a particular case of a doubly warped product.

Let $M = N_1 \times_f N_2$ be a warped product manifold, which means that $N_1$ and $N_2$ are totally geodesic and totally umbilical submanifolds of $M$, respectively.

For warped product manifolds, we have [**14**] the following proposition.

**Proposition 2.1.** *Let $M = N_1 \times_f N_2$ be a warped product manifold. Then*
(I) $\nabla_X Y \in TN_1$ *is the lift of* $\nabla_X Y$ *on* $N_1$
(II) $\nabla_U X = \nabla_X U = (X \ln f)U$
(III) $\nabla_U V = \nabla'_U V - g(U,V)\nabla \ln f$
*for any $X$, $Y \in \Gamma(TN_1)$ and $U$, $V \in \Gamma(TN_2)$, where $\nabla$ and $\nabla'$ denote the Levi-Civita connections on $N_1$ and $N_2$, respectively.*

The notion of doubly warped products was introduced by Unal [**18**].

**Definition 2.3.** [**18**] Doubly warped products can be considered as a generalization of a warped product $(M, g)$ which is a warped product manifold of the form $M =_f B \times_b F$ with the metric $g = f^2 g_B + b^2 g_F$, where $b\colon B \to (0,\infty)$ and $f\colon F \to (0,\infty)$ are smooth maps and $g_B$, $g_F$ are the metrics on the Riemannian manifolds $B$ and $F$, respectively.

If either $b = 1$ or $f = 1$, but not both, then we obtain a (single) warped product. If both $b = 1$ and $f = 1$, then we have a product manifold. If neither $b$ nor $f$ is constant, then we have a non trivial doubly warped product.

For any $X \in \Gamma(TB)$ and $Z \in \Gamma(TF)$, on a doubly warped product manifold, the Levi-Civita connection is

$$\nabla_X Z = Z(\ln f)X + X(\ln b)Z. \tag{2.24}$$

Let $M =_{f_2} N_\perp \times_{f_1} N_T$ be doubly warped product contact CR-submanifolds of $(LCS)_n$-manifolds $\overline{M}$. Then such submanifolds are always tangent to the structure vector field $\xi$.

## 3. Doubly warped product contact CR-Submanifolds of $(LCS)_n$-Manifolds

In a similar way of [**9**], in this section, we study doubly warped product contact CR-submanifolds of $(LCS)_n$-manifolds and we prove the following theorem.

**Theorem 3.1.** *There exist no proper doubly warped product contact CR-submanifolds of $(LCS)_n$-manifolds.*

*Proof.* Let $M =_{f_2} N_\perp \times_{f_1} N_T$ be doubly warped product contact CR-submanifolds of $(LCS)_n$-manifolds $\overline{M}$. There arise two possible cases.

Case (i): If $\xi$ is tangent to $N_T$, then for $Z \in \Gamma(TN_\perp)$, we have

$$\nabla_Z \xi = Z(\ln f_1)\xi + \xi(\ln f_2)Z. \tag{3.1}$$

Now from (2.4), we have

$$\overline{\nabla}_Z \xi = \alpha\{Z + \eta(Z)\xi\} = \alpha Z. \tag{3.2}$$

Also from (2.16), we get

$$\overline{\nabla}_Z \xi = \nabla_Z \xi + h(Z,\xi). \tag{3.3}$$

From (3.2) and (3.3), we get

$$\nabla_Z\xi + h(Z,\xi) = \alpha Z,$$

which implies

$$\nabla_Z\xi = \alpha Z \qquad \text{and} \qquad h(Z,\xi) = 0. \tag{3.4}$$

In view of (3.4), it follows from (3.1) that

$$\alpha Z = Z(\ln f_1)\xi + \xi(\ln f_2)Z. \tag{3.5}$$

Since the two distributions are orthogonal, it follows from (3.5) that

$$Z(\ln f_1) = 0 \qquad \text{and} \qquad \xi(\ln f_2) = \alpha \tag{3.6}$$

for all $Z \in \Gamma(TN_\perp)$.

The first part of (3.6) shows that $f_1$ is constant on $TN_\perp$. So, doubly warped product contact CR-submanifolds of the form $M =_{f_2} N_\perp \times_{f_1} N_T$ of $(LCS)_n$-manifolds with $\xi$ tangent to $N_T$ does not exist.

Case(ii): If $\xi$ is tangent to $N_\perp$ and $X \in \Gamma(TN_T)$, then we get

$$\nabla_X\xi = \xi(\ln f_1)X + X(\ln f_2)\xi. \tag{3.7}$$

From (2.4) and (2.16), we get

$$\alpha\{X + \eta(X)\xi\} = \overline{\nabla}_X\xi = \nabla_X\xi + h(X,\xi), \tag{3.8}$$

i.e.,

$$\nabla_X\xi = \alpha X \qquad \text{and} \qquad h(X,\xi) = 0. \tag{3.9}$$

From (3.7) and (3.9), we get

$$\alpha X = \xi(\ln f_1)X + X(\ln f_2)\xi. \tag{3.10}$$

So by orthogonality of two distributions, (3.10) yields

$$\xi(\ln f_1) = \alpha \qquad \text{and} \qquad X(\ln f_2) = 0 \tag{3.11}$$

for all $X \in \Gamma(TN_T)$.

From the second part of (3.11) it follows that $f_2$ is constant on $TN_T$. So, in this case also doubly warped product contact CR-submanifolds do not exist. Hence the proof is complete. □

## 4. Warped product contact CR-Submanifolds of $(LCS)_n$-Manifolds

As there are not doubly warped product contact CR-submanifolds of $(LCS)_n$--manifolds, we will study warped product contact CR-submanifolds of $(LCS)_n$--manifolds in this section.

In 2001, Chen [**5**] introduced and studied the notion of warped product CR-submanifolds of Kaehler manifolds. Later Hasegawa and Mihai [**7**] studied contact CR-warped product submanifolds of Sasakian manifolds. Again Khan et. al [**9**] studied contact CR-warped product submanifolds of Kenmotsu manifolds.

Let $M = N_\perp \times_f N_T$ be contact warped product CR-submanifolds of $(LCS)_n$-manifold $\overline{M}$. Such submanifolds are always tangent to the structure vector field $\xi$. We distinguish two cases:

(i) $\xi$ tangent to $N_T$

(ii) $\xi$ tangent to $N_\perp$.

First we consider the case (i), where $\xi$ is tangent to $N_T$, and we prove the following theorem.

**Theorem 4.1.** *Let $\overline{M}$ be a $(LCS)_n$-manifold. Then there do not exist warped product contact CR-submanifolds $M = N_\perp \times_f N_T$ such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$.*

*Proof.* We now assume that $M = N_\perp \times_f N_T$ be contact warped product CR-submanifolds of a $(LCS)_n$-manifold $\overline{M}$ such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$. So by Proposition 2.1, we get

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X \tag{4.1}$$

for any vector fields $Z$ and $X$ tangent to $N_\perp$ and $N_T$, respectively.

Thus from (4.1), we get

$$\nabla_Z \xi = Z(\ln f)\xi \tag{4.2}$$

Also from (2.4) and (2.16), we get

$$\alpha Z = \overline{\nabla}_Z \xi = \nabla_Z \xi + h(Z, \xi).$$

which implies

$$\nabla_Z \xi = \alpha Z \qquad \text{and} \qquad h(Z, \xi) = 0. \tag{4.3}$$

From (4.2) and (4.3), we have $Z(\ln f) = 0$ for all $Z \in \Gamma(TN_\perp)$, i.e., $f$ is constant for all $Z \in \Gamma(TN_\perp)$. This means that $M$ is a usual Riemannian submanifold. This proves the theorem. □

*Remark* 4.2. In [**8**, Theorem 3], is valid for proper semi-slant submanifolds of $(LCS)_n$-manifolds. Thus Theorem 4.1 is completely different and not just a particular case of [**8**, Theorem 3].

Now we consider the case (ii), when $\xi$ is tangent to $N_\perp$. Assume that $\overline{M}$ is a $(LCS)_n$-manifold and consider the warped product contact CR-submanifold $M = N_\perp \times_f N_T$ such that $N_\perp$ is an anti-invariant submanifold of dimension $p$ tangent to $\xi$ and $N_T$ is an invariant submanifold of $\overline{M}$. Then for any $X \in \Gamma(TN_T)$, we have

$$g(\nabla_X \xi, X) = g(\overline{\nabla}_X \xi, X) = g(\alpha\{X + \eta(X)\xi\}, X),$$

i.e.,

$$\xi \ln f \|X\|^2 = \alpha \|X\|^2$$

or $\xi \ln f = \alpha$, i.e., $g(\nabla \ln f, \xi) = \alpha$, i.e.,

$$\nabla \ln f = -\alpha\xi, \tag{4.4}$$

where $\nabla \ln f$ denotes the gradient of $\ln f$ and defined by $g(\nabla \ln f, U) = U \ln f$ for all $U \in \Gamma(TM)$.

The relation (4.4) can also be written as

$$\sum_{i=1}^{p} \frac{\partial \ln f}{\partial x_i} = -\alpha\xi, \qquad i = 1, 2, \ldots, p, \tag{4.5}$$

which is the first order partial differential equation and has a unique solution, i.e., warped product exists. This leads to the following theorem.

**Theorem 4.3.** *Let $\overline{M}$ be a $(LCS)_n$-manifold. Then there exist warped product contact CR-submanifolds $M = N_\perp \times_f N_T$ such that $N_\perp$ is an anti-invariant submanifold of dimension p tangent to $\xi$, $N_T$ is invariant submanifold of $\overline{M}$ and the warping function f satisfying ((4.5)).*

**Example 4.4.** Let $\overline{M} = \mathbb{R}^7$ be the semi-Euclidean space endowed with the semi-Euclidean metric $g = \big[-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x3^2 + \mathrm{d}x_4^2 + \mathrm{d}x_5^2 + \mathrm{d}x_6^2\big]\,\mathrm{e}^{2t}$ with coordinate $(t, x_1, x_2, x_3, x_4, x_5, x_6)$. Define

$$\begin{aligned}
&\eta = \mathrm{e}^t\,\mathrm{d}t, && \xi = \mathrm{e}^t\,\frac{\partial}{\partial t}, && \phi\Big(\frac{\partial}{\partial t}\Big) = 0,\\
&\phi\Big(\frac{\partial}{\partial x_1}\Big) = -\frac{\partial}{\partial x_4}, && \phi\Big(\frac{\partial}{\partial x_2}\Big) = -\frac{\partial}{\partial x_5}, && \phi\Big(\frac{\partial}{\partial x_3}\Big) = -\frac{\partial}{\partial x_6},\\
&\phi\Big(\frac{\partial}{\partial x_4}\Big) = \frac{\partial}{\partial x_1}, && \phi\Big(\frac{\partial}{\partial x_5}\Big) = \frac{\partial}{\partial x_2}, && \phi\Big(\frac{\partial}{\partial x_6}\Big) = -\frac{\partial}{\partial x_3}.
\end{aligned}$$

Then it can be easily seen that the structure $(\phi, \xi, \eta, g)$ is a $(LCS)_7$-manifold on $\overline{M} = \mathbb{R}^7$.

Now we define a submanifold $M$ of $\overline{M}$ by $M = \{(x_1, 0, x_3, x_4, 0, x_6, t) \in \mathbb{R}^7\}$ endowed with the global vector fields

$$\begin{aligned}
&e_1 = \xi = \frac{\partial}{\partial t}, && e_2 = \frac{\partial}{\partial x_4}, && e_3 = \frac{\partial}{\partial x_2} + x_6\frac{\partial}{\partial t},\\
&e_4 = \frac{\partial}{\partial x_6}, && e_5 = \frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial t}.
\end{aligned}$$

Then the distributions $D_T = \operatorname{span}\{e_1, e_2, e_5\}$ and $D^\perp = \operatorname{span}\{e_3, e_4\}$ are invariant and anti-invariant distributions on $\overline{M}$, respectively. Let us denote their integral submanifolds by $N_T$ and $N_\perp$, respectively, then the submanifold $M = N_\perp \times_f N_T$ is a contact CR-warped product submanifold with warping function $f(t) = \mathrm{e}^t$.

**Conclusion.** Thus there exist warped product CR-submanifolds $M = N_\perp \times_f N_T$ of $(LCS)_n$-manifolds $\overline{M}$ such that $N_\perp$ is an anti-invariant submanifold tangent to $\xi$ and $N_T$ is an invariant submanifold of $\overline{M}$. Example 4.4 supports also the above result. So, there arises a natural question.

Do there exist warped product CR-submanifolds $M = N_T \times_f N_\perp$ of $(LCS)_n$-manifolds $\overline{M}$ such that $N_\perp$ is an anti-invariant submanifold tangent to $\xi$ and $N_T$ is an invariant submanifold of $\overline{M}$? This problem is still open.

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