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EXPONENTIAL POMPEIU'S TYPE INEQUALITIES WITH APPLICATIONS TO OSTROWSKI'S INEQUALITY

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Abstract. In this paper, some exponential Pompeiu's type inequalities for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.

1. Introduction

In 1946, Pompeiu [**6**] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [**8**, p. 83]).

Lemma 1 (Pompeiu, 1946 [**6**]). *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1.1}$$

The following inequality is useful to derive some Ostrowski type inequalities.

Corollary 1 (Pompeiu's Inequality). *With the assumptions of Lemma 1 and if $\|f - \ell f'\|_\infty = \sup_{t\in(a,b)} |f(t) - tf'(t)| < \infty$ where $\ell(t) = t$, $t \in [a,b]$, then*

$$|tf(x) - xf(t)| \le \|f - \ell f'\|_\infty |x - t| \tag{1.2}$$

for any $t, x \in [a,b]$.

The inequality (1.2) was obtained by the author in [**3**].

In 1938, A. Ostrowski [**4**] proved the following result by the estimating the integral mean.

Theorem 1 (Ostrowski, 1938 [**4**]). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $|f'(t)| \le M < \infty$ for all $t \in (a,b)$. Then for any $x \in [a,b]$, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a). \tag{1.3}$$

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The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller quantity.*

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 2 (Dragomir, 2005 [**3**])**.** *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $[a,b]$ *not containing* 0. *Then for any* $x \in [a,b]$, *we have the inequality*

$$\begin{aligned}(1.4)\qquad &\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \\ &\le \frac{b-a}{|x|}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left\|f-\ell f'\right\|_\infty ,\end{aligned}$$

where $\ell(t) = t,\ \ t \in [a,b]$.

The constant $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

In [**7**], using a mean value theorem, E. C. Popa obtained a generalization of (1.4) as follows:

Theorem 3 (Popa, 2007 [**7**])**.** *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *Assume that* $\alpha \notin [a,b]$. *Then for any* $x \in [a,b]$, *we have the inequality*

$$\begin{aligned}(1.5)\qquad &\left|\left(\frac{a+b}{2}-\alpha\right)f(x)+\frac{\alpha-x}{b-a}\int_a^b f(t)\mathrm{d}t\right| \\ &\le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f-\ell_\alpha f'\right\|_\infty ,\end{aligned}$$

where $\ell_\alpha(t) = t-\alpha,\ \ t \in [a,b]$.

In [**5**], J. Pečarić and S. Ungar proved a general estimate with the p-norm, $1 \le p \le \infty$ which for $p = \infty$, gives Dragomir's result.

Theorem 4 (Pečarić & Ungar, 2006 [**5**])**.** *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0 < a < b$. *Then for* $1 \le p,q \le \infty$ *with* $\frac{1}{p}+\frac{1}{q} = 1$, *we have the inequality*

$$(1.6)\qquad \left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \le PU(x,p)\left\|f-\ell f'\right\|_p$$

for $x \in [a,b]$, *where*

$$PU(x,p) := (b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q}\right.$$
$$\left.+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q}\right].$$

In the cases $(p,q)=(1,\infty)$, $(\infty,1)$ *and* $(2,2)$, *the quantity* $PU(x,p)$ *has to be taken as the limit as* $p\to 1,\infty$ *and* 2, *respectively.*

For other inequalities in terms of the p-norm of the quantity $f-\ell_\alpha f'$, where $\ell_\alpha(t)=t-\alpha$, $t\in[a,b]$ and $\alpha\notin[a,b]$, see [**1**] and [**2**].

In this paper, some exponential Pompeiu's type inequalities for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.

2. Exponential Inequalities

We can provide some similar results for complex-valued functions with the exponential instead of ℓ.

Lemma 2. *Let* $f\colon[a,b]\to\mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *and* $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}(\alpha)\neq 0$. *Then for any* $t,x\in[a,b]$, *we have*

$$\left|\frac{f(x)}{\exp(\alpha x)}-\frac{f(t)}{\exp(\alpha t)}\right|$$
$$\leq\begin{cases}|\operatorname{Re}(\alpha)|\,\|f'-\alpha f\|_\infty \times\left|\frac{1}{\exp(t\operatorname{Re}(\alpha))}-\frac{1}{\exp(x\operatorname{Re}(\alpha))}\right| & \text{if } f'-\alpha f\in L_\infty[a,b],\\[2ex] q^{1/q}\,|\operatorname{Re}(\alpha)|^{1/q}\,\|f'-\alpha f\|_p \times\left|\frac{1}{\exp(tq\operatorname{Re}(\alpha))}-\frac{1}{\exp(xq\operatorname{Re}(\alpha))}\right|^{1/q} & \text{if } f'-\alpha f\in L_p[a,b]\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \|f'-\alpha f\|_1\,\frac{1}{\min\{\exp(t\operatorname{Re}(\alpha)),\exp(x\operatorname{Re}(\alpha))\}},\end{cases}\tag{2.1}$$

or equivalently,

$$|\exp(\alpha t)f(x)-f(t)\exp(\alpha x)|$$
$$\leq\begin{cases}|\operatorname{Re}(\alpha)|\,\|f'-\alpha f\|_\infty \times|\exp(x\operatorname{Re}(\alpha))-\exp(t\operatorname{Re}(\alpha))| & \text{if } f'-\alpha f\in L_\infty[a,b],\\[2ex] q^{1/q}\,|\operatorname{Re}(\alpha)|^{1/q}\,\|f'-\alpha f\|_p \times|\exp(xq\operatorname{Re}(\alpha))-\exp(tq\operatorname{Re}(\alpha))|^{1/q} & \text{if } f'-\alpha f\in L_p[a,b]\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \|f'-\alpha f\|_1 \max\{\exp(t\operatorname{Re}(\alpha)),\exp(x\operatorname{Re}(\alpha))\}.\end{cases}\tag{2.2}$$

Proof. If f is absolutely continuous, then $f/\exp(\alpha\cdot)$ is absolutely continuous on the interval $[a,b]$ and

$$\int_t^x \left(\frac{f(s)}{\exp(\alpha s)}\right)' \mathrm{d}s = \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}$$

for any $t,x \in [a,b]$ with $x \neq t$.

Since

$$\begin{aligned}\int_t^x \left(\frac{f(s)}{\exp(\alpha s)}\right)' \mathrm{d}s &= \int_t^x \frac{f'(s)\exp(\alpha s) - \alpha f(s)\exp(\alpha s)}{\exp(2\alpha s)}\mathrm{d}s \\ &= \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)}\mathrm{d}s,\end{aligned}$$

then we get the following identity

$$\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} = \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)}\mathrm{d}s \tag{2.3}$$

for any $t,x \in [a,b]$ with $x \neq t$.

Taking the modulus in (2.3) we have

$$\begin{aligned}\left|\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}\right| &= \left|\int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)}\mathrm{d}s\right| \\ &\le \left|\int_t^x \frac{|f'(s) - \alpha f(s)|}{|\exp(\alpha s)|}\mathrm{d}s\right| := I\end{aligned} \tag{2.4}$$

and utilizing Hölder's integral inequality, we deduce

$$\begin{aligned} I &\le \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s) - \alpha f(s)| \left|\int_t^x \frac{1}{|\exp(\alpha s)|}\mathrm{d}s\right|, \\ \left|\int_t^x |f'(s) - \alpha f(s)|^p \,\mathrm{d}s\right|^{1/p} \left|\int_t^x \frac{1}{|\exp(\alpha s)|^q}\mathrm{d}s\right|^{1/q}, \\ \left|\int_t^x |f'(s) - \alpha f(s)|\,\mathrm{d}s\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(\alpha s)|}\right\}, \end{cases} \\ &\le \begin{cases} \|f' - \alpha f\|_\infty \left|\int_t^x \frac{1}{|\exp(\alpha s)|}\mathrm{d}s\right|, \\ \|f' - \alpha f\|_p \left|\int_t^x \frac{1}{|\exp(\alpha s)|^q}\mathrm{d}s\right|^{1/q}, \\ \|f' - \alpha f\|_1 \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(\alpha s)|}\right\}. \end{cases} \end{aligned} \tag{2.5}$$

Now, since $\alpha = \mathrm{Re}(\alpha) + \mathrm{i}\,\mathrm{Im}(\alpha)$ and $s \in [a,b]$, then

$$\begin{aligned}|\exp(\alpha s)| &= \exp(s\,\mathrm{Re}(\alpha) + \mathrm{i}\,s\,\mathrm{Im}(\alpha)) = |\exp(s\,\mathrm{Re}(\alpha))\exp(\mathrm{i}\,s\,\mathrm{Im}(\alpha))| \\ &= |\exp(s\,\mathrm{Re}(\alpha))|\,|\exp(\mathrm{i}\,s\,\mathrm{Im}(\alpha))| = \exp(s\,\mathrm{Re}(\alpha)).\end{aligned}$$

We have

$$\begin{aligned}\int_t^x \frac{1}{|\exp(\alpha s)|}\mathrm{d}s &= \int_t^x \frac{1}{\exp(s\operatorname{Re}(\alpha))}\mathrm{d}s = \int_t^x \exp(-s\operatorname{Re}(\alpha))\,\mathrm{d}s \\ &= -\operatorname{Re}(\alpha)\exp(-s\operatorname{Re}(\alpha))|_t^x \\ &= -\operatorname{Re}(\alpha)\exp(-x\operatorname{Re}(\alpha)) + \operatorname{Re}(\alpha)\exp(-t\operatorname{Re}(\alpha)) \\ &= \operatorname{Re}(\alpha)\left[\exp(-t\operatorname{Re}(\alpha)) - \exp(-x\operatorname{Re}(\alpha))\right] \\ &= \operatorname{Re}(\alpha)\left[\frac{1}{\exp(t\operatorname{Re}(\alpha))} - \frac{1}{\exp(x\operatorname{Re}(\alpha))}\right].\end{aligned}$$

By (2.4) and (2.5), we get

$$\left|\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}\right| \le \|f' - \alpha f\|_\infty |\operatorname{Re}(\alpha)| \left|\frac{1}{\exp(t\operatorname{Re}(\alpha))} - \frac{1}{\exp(x\operatorname{Re}(\alpha))}\right|$$

and the first part of (2.1) is proved.

We have

$$\begin{aligned}\int_t^x \frac{1}{|\exp(\alpha s)|^q}\mathrm{d}s &= \int_t^x \frac{1}{\exp(sq\operatorname{Re}(\alpha))}\mathrm{d}s \\ &= q\operatorname{Re}(\alpha)\left[\frac{1}{\exp(tq\operatorname{Re}(\alpha))} - \frac{1}{\exp(xq\operatorname{Re}(\alpha))}\right].\end{aligned}$$

By (2.4) and (2.5), we get the second part of (2.1).

We have

$$\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|\exp(\alpha s)|}\right\} = \frac{1}{\min\{\exp(t\operatorname{Re}(\alpha)), \exp(x\operatorname{Re}(\alpha))\}}.$$

By (2.4) and ((2.5), we get the last part of (2.1).

The inequality (2.2) follows from (2.1) by multiplying by $|\exp(\alpha x)\exp(\alpha t)|$ and performing the required calculation. □

The following particular case is of interest.

Corollary 2. *Let $f\colon [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$. Then for any $t, x \in [a,b]$, we have*

$$\left|\frac{f(x)}{\exp(x)} - \frac{f(t)}{\exp(t)}\right|$$

$$\le \begin{cases} \|f' - f\|_\infty \left|\frac{1}{\exp(t)} - \frac{1}{\exp(x)}\right| & \text{if } f' - f \in L_\infty[a,b], \\ q^{1/q}\|f' - f\|_p \left|\frac{1}{\exp(tq)} - \frac{1}{\exp(xq)}\right|^{1/q} & \text{if } f' - f \in L_p[a,b], \\ & \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - f\|_1 \frac{1}{\min\{\exp(t), \exp(x)\}}, & \end{cases} \tag{2.6}$$

or equivalently,

$$|\exp(t)f(x) - f(t)\exp(x)|$$

$$\leq \begin{cases} \|f' - f\|_{\infty} |\exp(x) - \exp(t)| & \text{if } f' - f \in L_{\infty}[a,b], \\ q^{1/q} \|f' - f\|_{p} |\exp(xq) - \exp(tq)|^{1/q} & \text{if } f' - f \in L_{p}[a,b] \\ & p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - f\|_{1} \max\{\exp(t), \exp(x)\}. \end{cases} \tag{2.7}$$

Remark 1. If $\mathrm{Re}(\alpha) = 0$, then the inequality (2.5) becomes

$$I \leq \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s) - \mathrm{i}\,\mathrm{Im}(\alpha)f(s)| \left|\int_t^x \frac{1}{|\exp(\mathrm{i}\,\mathrm{Im}(\alpha)s)|}\mathrm{d}s\right|, \\ \left|\int_t^x |f'(s) - \mathrm{i}\,\mathrm{Im}(\alpha)f(s)|^p \,\mathrm{d}s\right|^{1/p} \left|\int_t^x \frac{1}{|\exp(\mathrm{i}\,\mathrm{Im}(\alpha)s)|^q}\mathrm{d}s\right|^{1/q}, \\ \left|\int_t^x |f'(s) - \mathrm{i}\,\mathrm{Im}(\alpha)f(s)| \,\mathrm{d}s\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(i\,\mathrm{Im}(\alpha)s)|}\right\}, \end{cases}$$

$$\leq \begin{cases} \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{\infty} \left|\int_t^x \mathrm{d}s\right|, \\ \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{p} \left|\int_t^x \mathrm{d}s\right|^{1/q}, \\ \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{1}, \end{cases} = \begin{cases} \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{\infty} |x - t|, \\ \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{p} |x - t|^{1/q}, \\ \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{1}. \end{cases}$$

Therefore, we have

$$\left|\frac{f(x)}{\exp(\mathrm{i}\,\mathrm{Im}(\alpha)x)} - \frac{f(t)}{\exp(\mathrm{i}\,\mathrm{Im}(\alpha)t)}\right| \leq \begin{cases} \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{\infty} |x - t|, \\ \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{p} |x - t|^{1/q}, \\ \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{1}, \end{cases} \tag{2.8}$$

or equivalently,

$$|\exp(\mathrm{i}\,\mathrm{Im}(\alpha)t)f(x) - f(t)\exp(\mathrm{i}\,\mathrm{Im}(\alpha)x)|$$

$$\leq \begin{cases} \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{\infty} |x - t|, \\ \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{p} |x - t|^{1/q}, \\ \|f' - \mathrm{i}\,\mathrm{Im}(\alpha)f\|_{1} \end{cases} \tag{2.9}$$

for any $t, x \in [a,b]$.

In particular, we have

$$\left|\frac{f(x)}{\exp(\mathrm{i}\,x)} - \frac{f(t)}{\exp(\mathrm{i}\,t)}\right| \leq \begin{cases} \|f' - \mathrm{i}\,f\|_{\infty} |x - t|, \\ \|f' - \mathrm{i}\,f\|_{p} |x - t|^{1/q}, \\ \|f' - \mathrm{i}\,f\|_{1}, \end{cases} \tag{2.10}$$

or equivalently,

$$|\exp(\mathrm{i}\,t)\,f(x)-f(t)\exp(\mathrm{i}\,x)|\le\begin{cases}\|f'-\mathrm{i}\,f\|_\infty\,|x-t|\,,\\[2mm] \|f'-\mathrm{i}\,f\|_p\,|x-t|^{1/q}\,,\\[2mm] \|f'-\mathrm{i}\,f\|_1\end{cases}\tag{2.11}$$

for any $t,x\in[a,b]$.

3. Inequalities of Ostrowski Type

The following result holds.

Theorem 5. *Let $f\colon[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$. Then for any $x\in[a,b]$ we have*

$$\left|f(x)\frac{\exp(\alpha b)-\exp(\alpha a)}{\alpha}-\exp(\alpha x)\int_a^b f(t)\mathrm{d}t\right|$$

$$\le\begin{cases}|\mathrm{Re}(\alpha)|\,\|f'-\alpha f\|_\infty\,B_1(a,b,x,\alpha) & \text{if } f'-\alpha f\in L_\infty[a,b],\\[2mm] q^{1/q}\,|\mathrm{Re}(\alpha)|^{1/q}\,(b-a)^{1/p} & \text{if } f'-\alpha f\in L_p[a,b]\\ \times\|f'-\alpha f\|_p\,|B_q(a,b,x,\alpha)|^{1/q} & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2mm] \|f'-\alpha f\|_1\,B_\infty(a,b,x,\alpha),\end{cases}\tag{3.1}$$

where

$$B_q(a,b,x,\alpha):=2\left[\exp(xq\,\mathrm{Re}(\alpha))\left(x-\frac{a+b}{2}\right)+\frac{1}{q\,\mathrm{Re}(\alpha)}\left(\frac{\exp(bq\,\mathrm{Re}(\alpha))+\exp(aq\,\mathrm{Re}(\alpha))}{2}-\exp(xq\,\mathrm{Re}(\alpha))\right)\right]$$

for $q\ge1$ and

$$B_\infty(a,b,x,\alpha):=\exp(x\,\mathrm{Re}(\alpha))\,(x-a)+\frac{\exp(b\,\mathrm{Re}(\alpha))-\exp(x\,\mathrm{Re}(\alpha))}{\mathrm{Re}(\alpha)}.$$

Proof. Utilising the first inequality in (2.2), we have

$$\begin{aligned}&\left|f(x)\int_a^b\exp(\alpha t)\,\mathrm{d}t-\exp(\alpha x)\int_a^b f(t)\mathrm{d}t\right|\\ &\le\int_a^b|\exp(\alpha t)\,f(x)-f(t)\exp(\alpha x)|\,\mathrm{d}t\\ &\le|\mathrm{Re}(\alpha)|\,\|f'-\alpha f\|_\infty\int_a^b|\exp(x\,\mathrm{Re}(\alpha))-\exp(t\,\mathrm{Re}(\alpha))|\,\mathrm{d}t\end{aligned}\tag{3.2}$$

for any $x\in[a,b]$.

Observe that since $\operatorname{Re}(\alpha) > 0$, then

$$\begin{aligned}
&\int_a^b |\exp(x\operatorname{Re}(\alpha)) - \exp(t\operatorname{Re}(\alpha))|\,\mathrm{d}t \\
&= \int_a^x (\exp(x\operatorname{Re}(\alpha)) - \exp(t\operatorname{Re}(\alpha)))\,\mathrm{d}t \\
&\quad + \int_x^b (\exp(t\operatorname{Re}(\alpha)) - \exp(x\operatorname{Re}(\alpha)))\,\mathrm{d}t \\
&= \exp(x\operatorname{Re}(\alpha))(x-a) - \left.\frac{\exp(t\operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)}\right|_a^x \\
&\quad + \left.\frac{\exp(t\operatorname{Re}(\alpha))}{\operatorname{Re}}(\alpha)\right|_x^b - (b-x)\exp(x\operatorname{Re}(\alpha)) \\
&= \exp(x\operatorname{Re}(\alpha))(2x-a-b) - \frac{1}{\operatorname{Re}(\alpha)}(\exp(x\operatorname{Re}(\alpha)) - \exp(a\operatorname{Re}(\alpha))) \\
&\quad + \frac{1}{\operatorname{Re}(\alpha)}(\exp(b\operatorname{Re}(\alpha)) - \exp(x\operatorname{Re}(\alpha))) \\
&= \exp(x\operatorname{Re}(\alpha))(2x-a-b) \\
&\quad + \frac{1}{\operatorname{Re}(\alpha)}(\exp(b\operatorname{Re}(\alpha)) + \exp(a\operatorname{Re}(\alpha)) - 2\exp(x\operatorname{Re}(\alpha))) \\
&= 2\left[\exp(x\operatorname{Re}(\alpha))\left(x - \frac{a+b}{2}\right)\right. \\
&\quad \left. + \frac{1}{\operatorname{Re}(\alpha)}\left(\frac{\exp(b\operatorname{Re}(\alpha)) + \exp(a\operatorname{Re}(\alpha))}{2} - \exp(x\operatorname{Re}(\alpha))\right)\right]
\end{aligned}$$

for any $x \in [a,b]$.

Also

$$\begin{aligned}
f(x)\int_a^b \exp(\alpha t)\,\mathrm{d}t - \exp(\alpha x)\int_a^b f(t)\mathrm{d}t \\
= f(x)\frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} - \exp(\alpha x)\int_a^b f(t)\mathrm{d}t
\end{aligned}$$

for any $x \in [a,b]$ and by (3.2), we get the first inequality in (3.1).

Using the second inequality in (2.2), we have

$$\begin{aligned}
&\left| f(x)\int_a^b \exp(\alpha t)\,\mathrm{d}t - \exp(\alpha x)\int_a^b f(t)\mathrm{d}t \right| \\
&\le \int_a^b |\exp(\alpha t) f(x) - f(t)\exp(\alpha x)|\,\mathrm{d}t \qquad (3.3)\\
&\le q^{1/q}|\operatorname{Re}(\alpha)|^{1/q}\|f' - \alpha f\|_p \int_a^b |\exp(xq\operatorname{Re}(\alpha)) - \exp(tq\operatorname{Re}(\alpha))|^{1/q}\,\mathrm{d}t
\end{aligned}$$

for any $x \in [a,b]$.

By Hölder's integral inequality, we also have

$$\begin{aligned}
&\int_a^b |\exp(xq\operatorname{Re}(\alpha)) - \exp(tq\operatorname{Re}(\alpha))|^{1/q}\,\mathrm{d}t \\
&\quad \le \left(\int_a^b \mathrm{d}t\right)^{1/p} \left[\int_a^b \left(|\exp(xq\operatorname{Re}(\alpha)) - \exp(tq\operatorname{Re}(\alpha))|^{1/q}\right)^q \mathrm{d}t\right]^{1/q} \\
&\quad = (b-a)^{1/p} \left[\int_a^b |\exp(xq\operatorname{Re}(\alpha)) - \exp(tq\operatorname{Re}(\alpha))|\,\mathrm{d}t\right]^{1/q},
\end{aligned}$$

for any $x \in [a,b]$.

Observe that as above we have

$$\begin{aligned}
&\int_a^b |\exp(xq\operatorname{Re}(\alpha)) - \exp(tq\operatorname{Re}(\alpha))|\,\mathrm{d}t \\
&= 2\left[\exp(xq\operatorname{Re}(\alpha))\left(x - \frac{a+b}{2}\right)\right. \\
&\quad \left. + \frac{1}{q\operatorname{Re}(\alpha)}\left(\frac{\exp(bq\operatorname{Re}(\alpha)) + \exp(aq\operatorname{Re}(\alpha))}{2} - \exp(xq\operatorname{Re}(\alpha))\right)\right] \\
&= B_q(a,b,x,\alpha)
\end{aligned}$$

for any $x \in [a,b]$ and by (3.3), we get the second part of (3.1).

Using the third inequality in (2.2), we have

$$\begin{aligned}
&\left| f(x)\int_a^b \exp(\alpha t)\,\mathrm{d}t - \exp(\alpha x)\int_a^b f(t)\mathrm{d}t \right| \\
&\quad \le \int_a^b |\exp(\alpha t) f(x) - f(t)\exp(\alpha x)|\,\mathrm{d}t \qquad (3.4)\\
&\quad \le \|f' - \alpha f\|_1 \int_a^b \max\{\exp(t\operatorname{Re}(\alpha)), \exp(x\operatorname{Re}(\alpha))\}\,\mathrm{d}t
\end{aligned}$$

for any $x \in [a,b]$.

Observe that

$$\begin{aligned}
&\int_a^b \max\{\exp(t\operatorname{Re}(\alpha)), \exp(x\operatorname{Re}(\alpha))\}\,\mathrm{d}t \\
&\quad = \int_a^x \max\{\exp(t\operatorname{Re}(\alpha)), \exp(x\operatorname{Re}(\alpha))\}\,\mathrm{d}t \\
&\qquad + \int_x^b \max\{\exp(t\operatorname{Re}(\alpha)), \exp(x\operatorname{Re}(\alpha))\}\,\mathrm{d}t \\
&\quad = \int_a^x \exp(x\operatorname{Re}(\alpha))\,\mathrm{d}t + \int_x^b \exp(t\operatorname{Re}(\alpha))\,\mathrm{d}t \\
&\quad = \exp(x\operatorname{Re}(\alpha))(x-a) + \frac{\exp(b\operatorname{Re}(\alpha)) - \exp(x\operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)}
\end{aligned}$$

and by (3.4), we get the third part of (3.1). □

Remark 2. If $\mathrm{Re}(\alpha) < 0$, then a similar result may be stated. However, the details are left to the interested reader.

Corollary 3. *Let $f\colon [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$. Then for any $x \in [a,b]$, we have*

$$\left| f(x)[\exp(b) - \exp(a)] - \exp(x) \int_a^b f(t)\mathrm{d}t \right|$$

$$(3.5)\qquad \leq \begin{cases} \|f' - f\|_\infty B_1(a,b,x) & \text{if } f'-f \in L_\infty[a,b], \\ q^{1/q}(b-a)^{1/p} \|f' - f\|_p & \text{if } f'-f \in L_p[a,b], \\ \times |B_q(a,b,x)|^{1/q} & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f' - f\|_1 B_\infty(a,b,x), \end{cases}$$

where

$$B_q(a,b,x) := 2\left[\left(x - \frac{a+b}{2}\right)\exp(xq) + \frac{1}{q}\left(\frac{\exp(bq)+\exp(aq)}{2} - \exp(xq)\right)\right]$$

for $q \geq 1$ and

$$B_\infty(a,b,x) := (x-a)\exp(x) + \exp(b) - \exp(x).$$

Remark 3. The midpoint case is as follows:

$$\left| f\left(\frac{a+b}{2}\right)[\exp(b) - \exp(a)] - \exp\left(\frac{a+b}{2}\right) \int_a^b f(t)\mathrm{d}t \right|$$

$$(3.6)\qquad \leq \begin{cases} \|f' - f\|_\infty B_1(a,b) & \text{if } f'-f \in L_\infty[a,b], \\ q^{1/q}(b-a)^{1/p} \|f' - f\|_p & \text{if } f'-f \in L_p[a,b], \\ \times |B_q(a,b)|^{1/q} & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f' - f\|_1 B_\infty(a,b), \end{cases}$$

where

$$B_q(a,b,x) := \frac{2}{q}\left(\frac{\exp(bq)+\exp(aq)}{2} - \exp\left(\frac{a+b}{2}q\right)\right)$$

for $q \geq 1$ and

$$B_\infty(a,b) := \frac{b-a}{2}\exp\left(\frac{a+b}{2}\right) + \exp(b) - \exp\left(\frac{a+b}{2}\right).$$

The case $\mathrm{Re}(\alpha) = 0$ is different and may be stated as follows.

Theorem 6. *Let $f\colon [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) = 0$ and $\operatorname{Im}(\alpha) \neq 0$. Then for any $x \in [a,b]$, we have*

$$\left| f(x)\frac{\exp(\mathrm{i}\operatorname{Im}(\alpha)b) - \exp(\mathrm{i}\operatorname{Im}(\alpha)a)}{\mathrm{i}\operatorname{Im}(\alpha)} - \exp(\mathrm{i}\operatorname{Im}(\alpha)x)\int_a^b f(t)\mathrm{d}t\right|$$

$$\leq \begin{cases} \|f' - \mathrm{i}\operatorname{Im}(\alpha)f\|_\infty \\ \times\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 & \text{if } f' - \mathrm{i}\operatorname{Im}(\alpha)f \in L_\infty[a,b], \\ \frac{q}{q+1}\|f' - \mathrm{i}\operatorname{Im}(\alpha)f\|_p & \text{if } f' - \mathrm{i}\operatorname{Im}(\alpha)f \in L_p[a,b], \\ \times\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{\frac{q+1}{q}} & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f' - \mathrm{i}\operatorname{Im}(\alpha)f\|_1 (b-a). \end{cases} \tag{3.7}$$

Proof. Utilizing the inequality (2.9), we have

$$\begin{aligned} &\left| f(x)\frac{\exp(\mathrm{i}\operatorname{Im}(\alpha)b) - \exp(\mathrm{i}\operatorname{Im}(\alpha)a)}{\mathrm{i}\operatorname{Im}(\alpha)} - \exp(\mathrm{i}\operatorname{Im}(\alpha)x)\int_a^b f(t)\mathrm{d}t\right| \\ &\leq \int_a^b |\exp(\mathrm{i}\operatorname{Im}(\alpha)t)f(x) - f(t)\exp(\mathrm{i}\operatorname{Im}(\alpha)x)|\,\mathrm{d}t \\ &\leq \begin{cases} \|f' - \mathrm{i}\operatorname{Im}(\alpha)f\|_\infty \int_a^b |x-t|\,\mathrm{d}t, \\ \|f' - \mathrm{i}\operatorname{Im}(\alpha)f\|_p \int_a^b |x-t|^{1/q}\,\mathrm{d}t, \\ \|f' - \mathrm{i}\operatorname{Im}(\alpha)f\|_1 \int_a^b \mathrm{d}t. \end{cases} \end{aligned} \tag{3.8}$$

Since

$$\int_a^b |x-t|\,\mathrm{d}t = \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2$$

and

$$\int_a^b |x-t|^{1/q}\,\mathrm{d}t = \frac{q}{q+1}\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{\frac{q+1}{q}},$$

then from (3.8), we get the desired result (3.7). □

Corollary 4. *Let $f\colon [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$. Then for any $x \in [a,b]$, we have*

$$\left| f(x) \frac{\exp(ib) - \exp(\mathrm{i}\,a)}{\mathrm{i}} - \exp(\mathrm{i}\,x) \int_a^b f(t) \mathrm{d}t \right|$$

$$\leq \begin{cases} \|f' - \mathrm{i}\,f\|_\infty \\ \times \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)^2 & \textit{if } f' - \mathrm{i}\,f \in L_\infty[a,b], \\ \frac{q}{q+1} \|f' - \mathrm{i}\,f\|_p & \textit{if } f' - \mathrm{i}\,f \in L_p[a,b], \\ \times \left[\left(\frac{b-x}{b-a} \right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a} \right)^{\frac{q+1}{q}} \right] (b-a)^{\frac{q+1}{q}} & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - if\|_1 (b-a). \end{cases} \tag{3.9}$$

Remark 4. The midpoint case is as follows

$$\left| f\left(\frac{a+b}{2}\right) \frac{\exp(\mathrm{i}\,b) - \exp(\mathrm{i}\,a)}{\mathrm{i}} - \exp\left(\mathrm{i}\,\frac{a+b}{2}\right) \int_a^b f(t)\,\mathrm{d}t \right|$$

$$\leq \begin{cases} \frac{1}{4} \|f' - \mathrm{i}\,f\|_\infty (b-a)^2, & \text{if } f' - \mathrm{i}\,f \in L_\infty[a,b], \\ \frac{q}{(q+1)2^{1/q}} \|f' - \mathrm{i}\,f\|_p (b-a)^{\frac{q+1}{q}}, & \text{if } f' - \mathrm{i}\,f \in L_p[a,b]. \end{cases} \tag{3.10}$$

Similar inequalities may be stated if one uses (2.1) and integrates over t on $[a,b]$. The details are left to the interested reader.

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