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ON ϕ-RICCI SYMMETRIC (k, μ)-CONTACT METRIC MANIFOLDS

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ABSTRACT. The object of the present paper is to study globally and locally ϕ-Ricci symmetric (k, μ)-contact metric manifolds. Finally, an illustrative example is given to verify some results.

1. Introduction

The notion of locally ϕ-symmetric Sasakian manifolds was introduced by T. Takahashi [**8**]. He studied several interesting properties of such a manifold in the context of Sasakian geometry. U. C. De et al. [**9**] introduced the notion of ϕ-recurrent Sasakian manifolds which generalizes the notion of ϕ-symmetric Sasakian manifolds. Also in another paper, U. C. De and Aboul Kalam Gazi [**11**] introduced the notion of ϕ-recurrent $N(k)$-contact metric manifolds. In [**10**], U. C. De and Avijit Sarkar introduced the notion of ϕ-Ricci symmetric Sasakian manifolds. From the definitions it follows that every ϕ-symmetric Sasakian manifold is ϕ-Ricci symmetric, but the converse is not true, in general. Also a (k, μ)-contact metric manifold is Sasakian if $k = 1$. Considering the above facts in this paper we generalize, the notion of ϕ-symmetric Sasakian manifolds and study ϕ-Ricci symmetric (k, μ)-contact metric manifolds.

The paper is organized as follows:

In Section 2, we recall (k, μ)-contact metric manifolds. Globally ϕ-Ricci symmetric (k, μ)-contact metric manifolds studied in Section 3. We prove that a (k, μ)-contact metric manifold M^{2n+1} is globally ϕ-Ricci symmetric if and only if it is an Einstein manifold. Also we prove that a globally ϕ-Ricci symmetric (k, μ)-contact metric manifold is three-dimensional and flat. Section 4 is devoted to study, locally ϕ-Ricci symmetric 3-dimensional (k, μ)-contact metric manifold and we prove that such a manifold is locally ϕ-Ricci symmetric if and only if the scalar curvature is constant. Finally, in Section 5, we set an example of (k, μ)-contact metric manifolds which verifies the result of Section 4.

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2. Preliminaries

By a contact manifold we mean a $(2n+1)$-dimensional differentiable manifold M^{2n+1} which carries a global 1-form η exists a unique vector field ξ called the characteristic vector field such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$. A Riemannian metric g on M^{2n+1} is said to be an associated metric if there exists a $(1,1)$ tensor field ϕ such that

$$d\eta(X,Y) = g(X, \phi Y), \qquad \eta(X) = g(X, \xi), \qquad \phi^2 = -I + \eta \otimes \xi. \tag{2.1}$$

From these equations we have

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

The manifold M equipped with the contact structure (ϕ, ξ, η, g) is called a contact metric manifold [**2**], [**3**].

Given a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$, we define a $(1,1)$ tensor field h by $h = \pounds_\xi \phi$, where $\pounds$ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. Thus, if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is also an eigen value with eigen vector ϕX. Also we have $Tr \cdot h = Tr \cdot \phi h = 0$ and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds.

$$\nabla_X \xi = -\phi X - \phi h X. \tag{2.3}$$

A contact metric manifold is said to be Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant and η-Einstein if $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.4}$$

$X, Y \in TM$, where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However, a 3-dimensional K-contact manifold is Sasakian [**7**]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [**4**]. On the other hand, on a Sasakian manifold, the following relation holds.

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.5}$$

It is well known that there exist contact metric manifolds for which the curvature tensor R and the direction of the characteristic vector field ξ satisfy $R(X,Y)\xi = 0$ for any vector fields X and Y. For example, tangent sphere bundle of a flat Riemannian manifold admits such a structure.

As a generalization of $R(X,Y)\xi = 0$ and the Sasakian case, D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [**5**] considered the (k, μ)-nullity distribution on a contact metric manifold and gave several reasons for studying it. The

(k,μ)-nullity distribution $N(k,\mu)$ [**1**], [**5**] of a contact metric manifold is defined by

$$N(k,\mu)\colon p \to N_p(k,\mu)$$
$$N_p(k,\mu) = [W \in T_pM \mid R(X,Y)W = (kI + \mu h)(g(Y,W)X - g(X,W)Y)]$$

for all $X, Y \in TM$, where $(k,\mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold. Thus we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.6}$$

Applying a D-homothetic deformation to a contact metric manifold with $R(X,Y)\xi = 0$, we obtain a contact metric manifold satisfying (2.6). In [**5**], it is proved that the standard contact metric structure on the tangent sphere bundle $T_1(M)$ satisfies the condition that ξ belongs to the (k,μ)-nullity distribution if and only if the base manifold is the space of constant curvature. There exist examples in all dimensions and the condition that ξ belongs to the (k,μ)-nullity distribution is invariant under D-homothetic deformations; in dimension greater than 5, the condition determines the curvature completely; dimension 3 includes the 3-dimensional unimodular Lie groups with the left invariant metric.

On (k,μ)-contact metric manifold, $k \le 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, the (k,μ)-nullity condition completely determines the curvature of M^{2n+1} [**5**]. In fact, for a (k,μ)-contact metric manifold, the condition of being Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent. Again a (k,μ)-contact metric manifold reduces to an $N(k)$-contact metric manifold if and only if $\mu = 0$.

In a (k,μ)-contact metric manifold, the following relations hold [**5**], [6]:

$$h^2 = (k-1)\phi^2, \qquad k \le 1, \tag{2.7}$$
$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.8}$$
$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.9}$$
$$S(X,\xi) = 2nk\eta(X), \tag{2.10}$$
$$\begin{aligned} S(X,Y) &= [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ &\quad + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \qquad n \ge 1, \end{aligned} \tag{2.11}$$
$$r = 2n(2n - 2 + k - n\mu), \tag{2.12}$$
$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX,Y), \tag{2.13}$$

where S is the Ricci tensor of type $(0,2)$ and r is the scalar curvature of the manifold. From (2.3), it follows that

$$(\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{2.14}$$
$$\begin{aligned} (\nabla_X h)Y &= [(1-k)g(X,\phi Y) + g(X, h\phi Y)]\xi \\ &\quad + \eta(Y)[h(\phi X + \phi hX)] - \mu\eta(X)\phi hY. \end{aligned} \tag{2.15}$$

Also in a (k,μ)-contact manifold, the following holds

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)]. \tag{2.16}$$

Especially for the case $\mu = 2(1-n)$, from (2.11), it follows that the manifold is η-Einstein.

Now we prove the following lemma.

Lemma 2.1. *An Einstein (k,μ)-contact manifold is three dimensional and flat.*

Proof. For an Einstein manifold we have $S(X,Y) = \lambda g(X,Y)$, where λ is a constant. Comparing this value of $S(X,Y)$ with those given in (2.11), we have

$$\lambda g(X,Y) = [2(n-1) - n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y). \tag{2.17}$$

Putting $X = Y = \xi$ in (2.17) and applying $g(X,\xi) = \eta(X)$, $\eta(\xi) = 1$ and $h\xi = 0$, we obtain $\lambda = 2nk$. Therefore, the relation (2.17) becomes

$$2nkX = [2(n-1) - n\mu]X + [2(n-1)+\mu]hX + [2(1-n) + n(2k+\mu)]\eta(X)\xi, \tag{2.18}$$

i.e.,

$$[2(n-1) - n(2k+\mu)][X - \eta(X)\xi] + [2(n-1)+\mu]hX = 0. \tag{2.19}$$

Equating co-efficients of X and hX from both sides of (2.19), we obtain

$$2(n-1)+\mu = 0 \qquad \text{and} \qquad 2(n-1) - n(2k+\mu) = 0. \tag{2.20}$$

Using (2.20) in (2.18) we get

$$2nk = 2(n^2-1). \tag{2.21}$$

Therefore, $k = \frac{n^2-1}{n} \leq 1$, so $n = 1$ is the only case. This gives $\mu = 0$ which with $n = 1$ gives $k = 0$. Applying these in (2.6), we get $R(X,Y)\xi = 0$.

Now in [**4**], D. E. Blair proved that a $(2n+1)$-dimensional contact metric manifold satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat if $n = 1$.

Therefore, we conclude that the manifold of our consideration is three dimensional and flat. This proves the Lemma. □

3. Globally ϕ-Ricci symmetric (k,μ)-contact metric manifolds

Definition 3.1. A (k,μ)-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be globally ϕ-Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2(\nabla_X Q)(Y) = 0 \tag{3.1}$$

for all vector fields $X, Y \in \chi(M)$ and $S(X,Y) = g(QX,Y)$. In particular, if X, Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

Let us suppose that a $(2n+1)$-dimensional (k,μ)-contact manifold M^{2n+1} is globally ϕ-Ricci symmetric. Then by definition

$$\phi^2(\nabla_X Q)(Y) = 0.$$

Using (2.1),

$$-(\nabla_X Q)Y + \eta(\nabla_X Q)(Y)\xi = 0. \tag{3.2}$$

From (3.2), it follows that

$$-g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0, \tag{3.3}$$

i.e.,

$$-g(\nabla_X Q(Y) - Q(\nabla_X Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{3.4}$$

i.e.,

$$-g(\nabla_X Q(Y), Z) + g(Q\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{3.5}$$

Putting $Y = \xi$ in (3.5) and using (2.10), we obtain

$$-2nkg(\nabla_X \xi, Z) + g(Q(\nabla_X \xi), Z) + \eta((\nabla_X Q)\xi)\eta(Z) = 0. \tag{3.6}$$

Using (2.3) in (3.6), we have

$$\begin{aligned} 2nkg(\phi X, Z) + 2nkg(\phi hX, Z) - S(\phi hX, Z) - S(\phi X, Z) \\ +\eta((\nabla_X Q)\xi)\eta(Z) = 0. \end{aligned} \tag{3.7}$$

Replacing Z by ϕZ in (3.7) and applying (2.2), we get

$$\begin{aligned} 2nkg(\phi X, \phi Z) + 2nkg(\phi hX, \phi Z) - S(\phi hX, \phi Z) \\ -S(\phi X, \phi Z) = 0. \end{aligned} \tag{3.8}$$

Replacing X by hX in (3.8) and using (2.1), (2.2) and (2.7), we have

$$\begin{aligned} &2nkg(\phi hX, \phi Z) - S(\phi hX, \phi Z) \\ &= (k-1)S(\phi X, \phi Z) - 2nk(k-1)g(\phi X, \phi Z). \end{aligned} \tag{3.9}$$

Using (3.9) in (3.8), we obtain

$$(k-2)[S(\phi X, \phi Z) - 2nkg(\phi X, \phi Z)] = 0. \tag{3.10}$$

Since in (k,μ)-contact manifold $k \le 1$, we get from (3.10)

$$S(\phi X, \phi Z) = 2nkg(\phi X, \phi Z). \tag{3.11}$$

Replacing X and Z by ϕX and ϕZ, respectively, in (3.11) and using (2.1) and (2.10), we obtain

$$S(X, Z) = 2nkg(X, Z). \tag{3.12}$$

Hence the manifold is an Einstein manifold. Thus we state the following proposition.

Proposition 3.1. *A $(2n+1)$-dimensional globally ϕ-Ricci symmetric (k,μ)-contact metric manifold is an Einstein manifold.*

Conversely, suppose that the manifold is an Einstein manifold. Then

$$S(X,Y) = \lambda g(X,Y), \tag{3.13}$$

where $S(X,Y) = g(QX,Y)$ and λ is a constant. Therefore, we have

$$\phi^2(\nabla_X Q)(Y) = 0.$$

This helps us to conclude the following proposition.

Proposition 3.2. *If a $(2n+1)$-dimensional (k,μ)-contact metric manifold is Einstein, then the manifold is globally ϕ-Ricci symmetric.*

Combining Proposition 3.1 and 3.2, we can state the following theorem.

Theorem 3.1. *A $(2n+1)$-dimensional (k,μ)-contact metric manifold is globally ϕ-Ricci symmetric if and only if it is an Einstein manifold.*

Again in view of Lemma 2.1 we have next theorem.

Theorem 3.2. *If a $(2n+1)$-dimensional (k,μ)-contact metric manifold is globally ϕ-Ricci symmetric, then it is of dimension 3 and flat.*

Since a globally ϕ-Ricci symmetric $(2n+1)$-dimensional (k,μ)-contact metric manifold is three dimensional and flat, we therefore consider 3-dimensional locally ϕ-Ricci symmetric (k,μ)-contact metric manifolds in the next section.

4. Three dimensional locally ϕ-Ricci symmetric (k,μ)-contact metric manifolds

In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad - S(X,Z)Y + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X]. \end{aligned} \tag{4.1}$$

Putting $Z = \xi$ in (4.1) and using (2.10) for $n = 1$, we get

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)QX - \eta(X)QY \\ &\quad + (2k - \frac{r}{2})[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{4.2}$$

Using (2.6) in (4.2), we have

$$\begin{aligned} &\Big(k - \frac{r}{2}\Big)[\eta(Y)X - \eta(X)Y] - \mu[\eta(Y)hX - \eta(X)hY] \\ &\quad + \eta(Y)QX - \eta(X)QY = 0. \end{aligned} \tag{4.3}$$

Putting $Y = \xi$ in (4.3) and using $\eta(\xi) = 1$, $h\xi = 0$ and $Q\xi = 2k$, we obtain

$$QX = \Big(\frac{r}{2} - k\Big)X + \Big(3k - \frac{r}{2}\Big)\eta(X)\xi + \mu hX. \tag{4.4}$$

Differentiating (4.4) covariantly with respect to W, we obtain

$$\begin{aligned} (\nabla_W Q)(X) &= \frac{1}{2}dr(W)X - \frac{1}{2}dr(W)\eta(X)\xi + \Big(3k - \frac{r}{2}\Big)(\nabla_W \eta)(X)\xi \\ &\quad + \Big(3k - \frac{r}{2}\Big)\eta(X)\nabla_W \xi + \mu(\nabla_W h)(X). \end{aligned} \tag{4.5}$$

Using (2.15) in (4.5), we have

$$
\begin{aligned}
(\nabla_W Q)(X) = {} & \frac{1}{2}dr(W)X - \frac{1}{2}dr(W)\eta(X)\xi + \Big(3k - \frac{r}{2}\Big)(\nabla_W \eta)(X)\xi \\
& + \Big(3k - \frac{r}{2}\Big)\eta(X)\nabla_W \xi + \mu[(1-k)g(W,\phi X)\xi \\
& + g(W, h\phi X)\xi + \eta(X)h(\phi W + \phi h W) - \mu\eta(W)\phi h X].
\end{aligned}
\tag{4.6}
$$

Using (2.3), (2.7) and the relation $\phi h = -h\phi$ in (4.6), we get

$$
\begin{aligned}
(\nabla_W Q)(X) = {} & \frac{1}{2}dr(W)X - \frac{1}{2}dr(W)\eta(X)\xi + \Big(3k - \frac{r}{2}\Big)(\nabla_W \eta)(X)\xi \\
& - \Big(3k - \frac{r}{2}\Big)\eta(X)\phi W - \Big(3k - \frac{r}{2} + \mu\Big)\eta(X)\phi h W \\
& + \mu(1-k)g(W,\phi X)\xi + \mu g(W, h\phi X)\xi \\
& - \mu(1-k)\eta(X)\phi W - \mu^2\eta(W)\phi h X.
\end{aligned}
\tag{4.7}
$$

Now applying ϕ^2 on both sides of (4.7) and using (2.1) and (2.2), we obtain

$$
\begin{aligned}
\phi^2(\nabla_W Q)(X) = {} & -\frac{1}{2}dr(W)X + \frac{1}{2}dr(W)\eta(X)\xi + \Big[3k - \frac{r}{2} + \mu(1-k)\Big]\eta(X)\phi W \\
& + \Big(3k - \frac{r}{2} + \mu\Big)\eta(X)\phi h W + \mu^2\eta(W)\phi h X.
\end{aligned}
\tag{4.8}
$$

If we consider a locally ϕ-Ricci symmetric (k,μ)-contact manifold, we have $\eta(X) = \eta(W) = 0$ and using these into (4.8), we get

$$
\phi^2(\nabla_W Q)(X) = -\frac{1}{2}dr(W). \tag{4.9}
$$

In view of equation (4.9), we conclude that in the following theorem.

Theorem 4.1. *A 3-dimensional (k,μ)-contact manifold is locally ϕ-Ricci symmetric if and only if the scalar curvature r is constant.*

5. Example

In this section we, construct an example of a locally ϕ-Ricci symmetric 3-dimensional (k,μ)-contact metric manifold.

We consider 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let e_1, e_2, e_3 are three vector fields in $\mathbb{R}^3$ which satisfy

$$
[e_1,e_2] = (1+\lambda)e_3, \qquad [e_2,e_3] = 2e_1 \qquad \text{and} \qquad [e_3,e_1] = (1-\lambda)e_2,
$$

where λ is a real number.

Let g be the Riemannian metric defined by

$$
g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, \qquad g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.
$$

Let η be the 1-form defined by

$$
\eta(U) = g(U,e_1)
$$

for any $U \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = 0, \qquad \phi e_2 = e_3, \qquad \phi e_3 = -e_2.$$

Using the linearity of ϕ and g, we have

$$\eta(e_1) = 1,$$
$$\phi^2(U) = -U + \eta(U)e_1$$

and

$$g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Moreover,

$$he_1 = 0, \qquad he_2 = \lambda e_2 \qquad \text{and} \qquad he_3 = -\lambda e_3.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using Koszul's formula, we get the following:

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 = -(1+\lambda)e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = (1+\lambda)e_1, \\ \nabla_{e_3} e_1 = (1-\lambda)e_2, & \nabla_{e_3} e_2 = -(1-\lambda)e_1, & \nabla_{e_3} e_3 = 0. \end{array}$$

In view of the above relations, we have

$$\nabla_X \xi = -\phi X - \phi h X \qquad \text{for } e_1 = \xi$$

Therefore, the manifold is a contact metric manifold with the contact structure (ϕ, ξ, η, g).

Now, we find the curvature tensors as follows:

$$\begin{array}{ll} R(e_1,e_2)e_2 = (1-\lambda^2)e_1, & R(e_3,e_2)e_2 = -(1-\lambda^2)e_3, \\ R(e_1,e_3)e_3 = (1-\lambda^2)e_1, & R(e_2,e_3)e_3 = -(1-\lambda^2)e_2, \\ R(e_2,e_3)e_1 = 0, & R(e_1,e_2)e_1 = -(1-\lambda^2)e_2, \\ R(e_3,e_1)e_1 = (1-\lambda^2)e_3. & \end{array}$$

In view of the expressions of the curvature tensors, we conclude that the manifold is a $(1-\lambda^2, 0)$-contact metric manifold.

Using the expressions of the curvature tensor, we find the values of the Ricci tensors as follows:

$$S(e_1,e_1) = 2(1-\lambda^2), \qquad S(e_2,e_2) = 0, \qquad S(e_3,e_3) = 0.$$

Hence, $r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) = 2(1-\lambda^2)$.

Again we calculate the following:

$$S(e_1,e_2) = S(e_1,e_0) = 0, \quad S(e_2,e_1) = S(e_2,e_3) = 0, \quad S(e_3,e_1) = S(e_3,e_2) = 0.$$

Hence, we get he following:

$$Qe_1 = 2(1-\lambda^2)e_1, \qquad Qe_2 = 0 \qquad \text{and} \qquad Qe_3 = 0.$$

Let X and Y are any two vector fields given by

$$X = a_1e_1 + a_2e_2 + a_3e_3 \qquad \text{and} \qquad Y = b_1e_1 + b_2e_2 + b_3e_3.$$

Then we get

$$\phi^2(\nabla_Y Q)X = 2(1-\lambda^2)[(1+\lambda)a_1b_2e_3 - (1-\lambda)a_1b_2e_3]. \tag{5.1}$$

It is clear from (5.1) that for $\lambda = 1$, the manifold is ϕ-Ricci symmetric and also we see that for $\lambda = 1$, the scalar curvature $r = 2$, which is constant. Hence this example verifies Theorem 4.1.

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