ON 3-DIMENSIONAL ALMOST KENMOTSU MANIFOLDS
ADMITTING CERTAIN NULLITY DISTRIBUTION

K. MANDAL AND U. C. DE

ABSTRACT. The aim of this paper is to characterize 3-dimensional almost Kenmotsu
manifolds with $\xi$ belonging to the $(k,\mu)'$-nullity distribution and $\mu' \neq 0$ satisfying
certain geometric conditions. Finally, we give an example to verify some results.

1. Introduction

The conformal curvature tensor $C$ is invariant under conformal transformations
and vanishes identically for 3-dimensional manifolds. Using this fact, many authors
[4, 6, 7, 14] studied several types of 3-dimensional manifolds.

A Riemannian manifold is called semisymmetric (resp., Ricci semisymmetric)
if $R(X,Y) \cdot R = 0$ (resp. $R(X,Y) \cdot S = 0$) [19], where $R(X,Y)$ is considered as a
field of linear operators acting on $R$ (resp., $S$).

The notion of $k$-nullity distribution ($k \in \mathbb{R}$) was introduced by Gray [11] and
Tanno [21] in the study of Riemannian manifolds $(M,g)$, which is defined for any
$p \in M$ and $k \in \mathbb{R}$, as follows:

\[(1.1) \quad N_p(k) = \{ Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \} \]

for any $X, Y \in T_pM$, where $T_pM$ denotes the tangent vector space of $M$ at any
point $p \in M$ and $R$ denotes the Riemannian curvature tensor of type $(1,3)$.

Recently, Blair, Koufogiorgos and Papantoniou [3] introduced the $(k,\mu)$-nullity
distribution which is a generalized notion of the $k$-nullity distribution on a contact
metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$,
as follows:

\[(1.2) \quad N_p(k,\mu) = \{ Z \in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY] \} \]

for any $X, Y \in T_pM$ and $h = \frac{1}{2} L \xi \phi$, where $L$ denotes the Lie differentiation.

Next, Dileo and Pastore [9] introduced another generalized notion of the $k$-nullity
distribution which is named the $(k,\mu)'$-nullity distribution on an almost Kenmotsu

Received December 15, 2015; revised February 20, 2017.
2010 Mathematics Subject Classification. Primary 53C25, 53D15.
Key words and phrases. Almost Kenmotsu manifold; Ricci semisymmetric; Codazzi type Ricci
tensor; Cyclic parallel Ricci tensor; $\eta$-parallel Ricci tensor; Locally $\phi$-Ricci symmetry; Einstein
manifold.
manifold \( (M^{2n+1}, \phi, \xi, \eta, g) \) and is defined for any \( p \in M^{2n+1} \) and \( k, \mu \in \mathbb{R} \), as follows:
\[
N_p(k, \mu)' = \{ Z \in T_pM^{2n+1} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu [g(Y, Z)h'X - g(X, Z)h'Y] \},
\]
for any \( X, Y \in T_pM \) and \( h' = h \circ \phi \).

On the other hand, in 1972, Kenmotsu [15] introduced a special class of almost contact metric manifolds known as Kenmotsu manifolds nowadays. Recently, Dileo and Pastore ([8], [9], [10]) and Wang et al. ([22], [23], [24], [25], [26]) studied almost Kenmotsu manifolds with some nullity distributions and obtained some classification theorems. In [9], Dileo and Pastore gave some classifications on 3-dimensional almost Kenmotsu manifolds assuming \( \xi \) belongs to the \((k, \mu)\)'-nullity distribution. Later, Wang and Liu [26] obtained some theorems on 3-dimensional almost Kenmotsu manifolds.

Motivated by these circumstances, in this paper, we study some meaningful geometric conditions in 3-dimensional almost Kenmotsu manifolds such that \( \xi \) belongs to the \((k, \mu)\)'-nullity distribution and \( h' \neq 0 \).

The present paper is organized as follows: In Section 2, we give some basic results on almost Kenmotsu manifolds with \( \xi \) belonging to the \((k, \mu)\)'-nullity distribution. Section 3 is devoted to study 3-dimensional Ricci semisymmetric almost Kenmotsu manifolds with \( \xi \) belonging to the \((k, \mu)\)'-nullity distribution. Section 4 deals with Codazzi type Ricci tensor with \( \xi \) belonging to the \((k, \mu)\)'-nullity distribution. Cyclic parallel Ricci tensor with \( \xi \) belonging to the \((k, \mu)\)'-nullity distribution is studied in Section 5. In the next two sections, we consider \( \eta \)-parallel Ricci tensor and locally \( \phi \)-Ricci symmetric almost Kenmotsu manifolds of dimension 3 assuming \( \xi \) belongs to the \((k, \mu)\)'-nullity distribution. Finally, we give an example to verify some results.

2. Almost Kenmotsu Manifolds

Let \( M \) be a \((2n+1)\)-dimensional differentiable manifold endowed with an almost contact metric structure \((\phi, \xi, \eta, g)\), where \( \phi, \xi, \eta \) are tensor fields on \( M \) of types \((1,1),(1,0),(0,1)\), respectively, and a Riemannian metric \( g \) such that
\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,
\]
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]
where \( I \) denotes the identity endomorphism ([1], [2]). Then also \( \phi \xi = 0 \) and \( \eta \circ \phi = 0 \); both can be derived from (2.1).

The fundamental 2-form \( \Phi \) on an almost contact metric manifold is defined by \( \Phi(X, Y) = g(X, \phi Y) \) for any vector fields \( X, Y \) of \( T_pM^{2n+1} \). An almost Kenmotsu manifold is defined as an almost contact metric manifold such that \( d\eta = 0 \) and \( d\Phi = 2\eta \wedge \Phi \). An almost contact metric manifold is said to be normal if \((1,2)\)-type torsion tensor \( N_\phi \) vanishes, where \( N_\phi = [\phi, \phi] + 2d\eta \otimes \xi \) and \([\phi, \phi] \) is the Nijenhuis torsion of \( \phi \) [1]. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by \((\nabla X)\phi Y = g(\phi X, Y)\xi - \eta(Y)\phi X \) for any vector fields \( X, Y \). It is well known [15] that a Kenmotsu manifold \( M^{2n+1} \) is locally a warped product \( I \times N^{2n} \), where \( N^{2n} \) is a
Kähler manifold, $I$ is an open interval with coordinate $t$ and the warping function $f$, defined by $f = ce^t$ for some positive constant $c$. Let $\mathcal{D}$ be the distribution orthogonal to $\xi$ and defined by $\mathcal{D} = \text{Ker}(\eta) = \text{Im}(\phi)$. In an almost Kenmotsu manifold $\mathcal{D}$ is an integrable distribution as $\eta$ is closed. Further, on an almost Kenmotsu manifold $M^{2n+1}$, we let the two tensor fields $h = \frac{1}{2} L \xi \phi$ and $l = R(\cdot, \xi)\xi$, which are symmetric and satisfy the following relations [9, 23]:

\begin{equation}
\nabla_X \xi = -\phi^2 X + h' X \quad (\Rightarrow \nabla_\xi \xi = 0),
\end{equation}

\begin{equation}
\phi \xi - l = 2(h^2 - \phi^2),
\end{equation}

\begin{equation}
R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_X \phi h)X - (\nabla_X \phi)h Y.
\end{equation}

Now we provide some basic results on almost Kenmotsu manifolds with $\xi$ belonging to the $(k, \mu)$-nullity distribution. The $(1, 1)$-type symmetric tensor field $h'$ satisfies $h'\phi + \phi h' = 0$ and $h'\xi = 0$. Also it is clear that

\begin{equation}
h = 0 \iff h' = 0, \quad h'^2 = (k + 1)\phi^2 \quad (\iff h^2 = (k + 1)\phi^2).
\end{equation}

For an almost Kenmotsu manifold, we have from (1.3)

\begin{equation}
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y],
\end{equation}

\begin{equation}
R(\xi, X)Y = k[g(\xi, X)Y - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X],
\end{equation}

where $k, \mu \in \mathbb{R}$. Contracting $Y$ in (2.8), we have

\begin{equation}
S(X, \xi) = 2k\eta(X).
\end{equation}

Let $X \in \mathcal{D}$ be the eigenvector of $h'$ corresponding to the eigen value $\lambda$. It follows from (2.6) that $\lambda^2 = -(k + 1)$, a constant. Therefore, $k \leq -1$ and $\lambda = \pm \sqrt{-k - 1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces associated with $h'$ corresponding to the non-zero eigen value $\lambda$ and $-\lambda$, respectively. We have the following lemmas.

**Lemma 2.1.** ([9, Proposition 4.1]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $\xi$ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and $\text{Spec}(h') = \{0, \lambda, -\lambda\}$ with 0 as simple eigen value and $\lambda = \sqrt{-k - 1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves.

**Lemma 2.2.** ([9, Lemma 4.1]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and $\xi$ belonging to the $(k, -2)'$-nullity distribution. Then for any $X, Y \in T_p M$,

\begin{equation}
(\nabla_X h')Y = -g(h'X + h'^2 X, Y)\xi - \eta(Y)(h'X + h'^2 X).
\end{equation}

Takahashi [20] introduced the notion of $\phi$-symmetry in the study of Sasakian manifolds. Then De and Sarkar [5] introduced a generalized notion of $\phi$-symmetry called $\phi$-Ricci symmetry in the study of Sasakian manifolds.
**Definition 2.1.** An almost Kenmotsu manifold is said to be $\phi$-Ricci symmetric if it satisfies
\begin{equation}
\phi^2(\nabla_{W} Q)Y = 0
\end{equation}
for any vector fields $W, Y \in T_p M$, where $Q$ is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. In addition, if the vector fields $W, Y$ are orthogonal to $\xi$, then the manifold is called locally $\phi$-Ricci symmetric manifold.

3. Ricci semisymmetric almost Kenmotsu manifolds

In a 3-dimensional Riemannian manifold, we have \[ R(X, Y)Z = S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY \] (3.1)
\[ - \frac{r}{2} [g(Y, Z)X - g(X, Z)Y], \]
where $Q$ is the Ricci operator defined by $g(QX, Y) = S(X, Y)$ for all $X, Y \in T_p M$ and $r$ is the scalar curvature of the manifold.

Putting $Y = Z = \xi$ in (3.1) and using Lemma 2.1 and (2.9), we obtain
\begin{equation}
QX = \left( \frac{r}{2} - k \right) X - \left( \frac{r}{2} - 3k \right) \eta(X)\xi - 2h'X,
\end{equation}
which is equivalent to
\begin{equation}
S(X, Y) = \left( \frac{r}{2} - k \right) g(X, Y) - \left( \frac{r}{2} - 3k \right) \eta(X)\eta(Y) - 2g(h'X, Y)
\end{equation}
for any $X, Y \in T_p M$.

With the help of (3.2) and (3.3), it follows from (3.1) that
\begin{equation}
R(X, Y)Z = \left( \frac{r}{2} - k \right) [g(Y, Z)X - g(X, Z)Y] - \left( \frac{r}{2} - 3k \right) [g(Y, Z)\eta(X)\xi
-g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] - 2g(h'X, Z)X + 2g(h'Y, Z)X + 2g(h'X, Z)Y
\end{equation}
for any $X, Y, Z \in T_p M$.

Now we suppose that the manifold $M^3$ is Ricci semisymmetric, that is,
\begin{equation}
(R(X, Y) - S(U, V)) = 0
\end{equation}
(3.5)
for all vector fields $X, Y, U, V$, which implies
\begin{equation}
S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0.
\end{equation}
(3.6)
Substituting $X = U = \xi$ in (3.6), we get
\begin{equation}
S(R(\xi, Y)\xi, V) + S(\xi, R(\xi, Y)V) = 0.
\end{equation}
(3.7)
Using (2.9), it follows from (3.7) that
\begin{equation}
S(R(\xi, Y)\xi, V) + 2k\eta(R(\xi, Y)V) = 0.
\end{equation}
(3.8)
Making use of (2.8) and (3.8), we have
\begin{equation}
2k^2\eta(Y)\eta(V) - kS(Y, V) + 2S(h'Y, V) + 2k^2g(Y, V) - 2k^2\eta(Y)\eta(V) - 4kg(h'Y, V) = 0,
\end{equation}
which implies
\begin{equation}
kS(Y, V) - 2S(h'Y, V) - 2k^2g(Y, V) + 4kg(h'Y, V) = 0.
\end{equation}
Replacing \(Y\) by \(h'Y\) in (3.10) and using the fact \(h'^2 = (k + 1)\phi^2\), it yields to
\begin{equation}
kS(h'Y, V) + 2(k + 1)S(Y, V) - 2k^2g(h'Y, V) - 4k(k + 1)g(Y, V) = 0.
\end{equation}
Adding \(k\) times of (3.10) and two times of (3.11), we have
\begin{equation}
(k + 2)^2[S(Y, V) - 2kg(Y, V)] = 0.
\end{equation}
Now we consider the following two cases:

Case 1. \(k \neq -2\). It follows from (3.12) that
\[S(Y, V) = 2kg(Y, V),\]
which implies that the manifold is an Einstein manifold.

Case 2. \(k = -2\). Then by [9, Corollary 4.1], the manifold is an \(CR\)-manifold.

From the above discussions, we have the following theorem.

**Theorem 3.1.** Let \((M^3, \phi, \xi, \eta, g)\) be an almost Kenmotsu manifold such that \(\xi\) belongs to the \((k, \mu)'\)-nullity distribution with \(h' \neq 0\). If \(M^3\) is Ricci semisymmetric, then, either the manifold is
1. an Einstein manifold, or
2. a \(CR\)-manifold.

Also Ricci symmetric manifold \((\nabla S = 0)\) implies Ricci semisymmetric \((R \cdot S = 0)\), therefore we can state the following:

**Corollary 3.1.** Let \((M^3, \phi, \xi, \eta, g)\) be an almost Kenmotsu manifold such that \(\xi\) belongs to the \((k, \mu)'\)-nullity distribution with \(h' \neq 0\). If \(M^3\) is Ricci symmetric, then either the manifold is
1. an Einstein manifold or
2. a \(CR\)-manifold.

A Riemannian manifold is said to be Ricci-recurrent [18] if the Ricci tensor \(S\) is non-zero and satisfies the condition
\[(\nabla_X S)(Y, Z) = A(X)S(Y, Z),\]
where \(X, Y, Z \in T_pM\) and \(A\) is a non-zero 1-form.

In [13], Jun et al proved that a Ricci-recurrent Riemannian manifold is Ricci semisymmetric.

Hence we can state the following corollary.
In this section, we assume that the manifold under consideration satisfies Codazzi type Ricci tensor \([\nabla X S](Y, Z) = (\nabla Y S)(X, Z)\). From \(\nabla S\) of Ricci tensor, then the Ricci tensor \(\nabla Y S\) along arbitrary vector field \(Y\) and using \(\nabla X Z\), we have
\[
(\nabla Y S)(X, Z) = \frac{dr(Y)}{2}[g(Y, Z) - \eta(\eta(Z))] - \frac{dr(Y)}{2}[g(X, Z) - \eta(\eta(Z))]
\]
Making use of \(\nabla X Z\) and \(\nabla Y S\) in \(\nabla S\) with \(\nabla Y S\) in \(\nabla X Z\), we get
\[
-2g((\nabla X h')Y, Z) + 2g((\nabla Y h')X, Z) - \left(\frac{r}{2} - 3k\right) [g(X, Z)\eta(Y) - g(Y, Z)\eta(\eta(X) + g(h', Y)Z, \eta(\eta(X)) = 0.
\]
It is known [16] that Cartan hypersurfaces are manifolds with non-parallel Ricci tensor satisfying \(\nabla S\). From \(\nabla S\), it follows that \(r = \text{constant}\). Then \(\nabla S\) implies
\[
\left(\frac{r}{2} - 3k\right) [g(X, Z)\eta(Y) + g(h', X, Z)\eta(Y) - g(Y, Z)\eta(\eta(X)) - g(h', Y)Z, \eta(\eta(X)) = 0.
\]
Making use of \(\nabla S\) and \(\nabla Y S\), we have
\[
(\nabla Y h')X - (\nabla X h')Y = \eta(Y)h'X - \eta(X)h'Y - (k + 1)\eta(\eta(X)X + (k + 1)\eta(\eta(X)Y.
\]
In view of \(\nabla S\) and \(\nabla Y S\), it follows that
\[
\left(\frac{r}{2} - 3k\right) [g(X, Z)\eta(Y) + g(h', X, Z)\eta(Y) - g(Y, Z)\eta(\eta(X)) - g(h', Y)Z, \eta(\eta(X)) = 0.
\]

**Corollary 3.2.** Let \((M^3, \phi, \xi, \eta, g)\) be an almost Kenmotsu manifold such that \(\xi\) belongs to the \((k, \mu)\)-nullity distribution with \(h' \neq 0\). If \(M^3\) is Ricci-recurrent, then either the manifold is
1. an Einstein manifold or
2. a CR-manifold.

### 4. Codazzi type Ricci tensor

In this section, we assume that the manifold under consideration satisfies Codazzi type \([12]\) of Ricci tensor, then the Ricci tensor \(S\) satisfies
\[
\]
Taking the covariant derivative of \(3.3\) along arbitrary vector field \(Y\) and using \(2.3\), we have
\[
(\nabla Y S)(X, Z) = \frac{dr(Y)}{2}[g(Y, Z) - \eta(\eta(Z))] - \frac{dr(Y)}{2}[g(X, Z) - \eta(\eta(Z))]
\]
Interchanging \(X\) and \(Y\) in \(4.2\), we get
\[
(\nabla X S)(Y, Z) = \frac{dr(X)}{2}[g(Y, Z) - \eta(\eta(Z))] - \frac{dr(X)}{2}[g(Y, Z) - \eta(\eta(Z))]
\]
Making use of \(4.2\) and \(4.3\) in \(4.1\) yields to
\[
\left(\frac{r}{2} - 3k\right) [g(X, Z)\eta(Y) + g(h', X, Z)\eta(Y) - g(Y, Z)\eta(\eta(X)) - g(h', Y)Z, \eta(\eta(X)) = 0.
\]

Substituting $X = \xi$ in (4.7) gives

\[(r - 6k)[g(Y, Z) + g(h'Y, Z) - \eta(Y)\eta(Z)] + 4[(k + 1)g(Y, Z) - (k + 1)\eta(Y)\eta(Z) - g(h'Y, Z)] = 0.\]

Putting $Y = h'Y$ in (4.8) and applying $h'^2 = (k + 1)\phi^2$ yield to

\[(r - 6k)[g(h'Y, Z) - (k + 1)g(Y, Z) + (k + 1)\eta(Y)\eta(Z)] + 4(k + 1)[g(Y, Z) - \eta(Y)\eta(Z) + g(h'Y, Z)] = 0.\]

Subtracting (4.9) from (4.8), we have

\[(r - 6k)(k + 2)[g(Y, Z) - \eta(Y)\eta(Z)] - 4(k + 2)g(h'Y, Z) = 0.\]

From (4.10), it follows that either $k = -2$ or

\[g(h'Y, Z) = \frac{r - 6k}{4}[g(Y, Z) - \eta(Y)\eta(Z)].\]

Making use of (3.3) and (4.11), we obtain

\[S(Y, Z) = 2kg(Y, Z),\]

that is, the manifold is an Einstein manifold.

Hence by the similar argument as in Section 3, we can state the following.

**Theorem 4.1.** Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $\xi$ belongs to the $(k, \mu)$-nullity distribution with $h' \neq 0$. If $M^3$ admits Codazzi type Ricci tensor, then either the manifold is

1. an Einstein manifold or
2. a CR-manifold.

5. **Cyclic parallel Ricci tensor**

This section is devoted to study cyclic parallel Ricci tensor in almost Kenmotsu manifolds with $\xi$ belonging to the $(k, \mu)$-nullity distribution and $h' \neq 0$ of dimension 3. Suppose the manifold under consideration satisfies cyclic parallel Ricci tensor [12], then the Ricci tensor $S$ satisfies

\[(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.\]

Taking the covariant derivative of (3.3) along arbitrary vector field $Z$ and using (2.3), we have

\[(\nabla_Z S)(X, Y) = \frac{dr(Z)}{2}[g(X, Y) - \eta(X)\eta(Y)] - \left(\frac{r}{2} - 3k\right)[g(X, Z)\eta(Y)] + g(Y, Z)\eta(X) + g(h'X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] - 2g((\nabla_Z h'), X, Y).\]

Similarly,

\[(\nabla_X S)(Y, Z) = \frac{dr(X)}{2}[g(Y, Z) - \eta(Y)\eta(Z)] - \left(\frac{r}{2} - 3k\right)[g(X, Y)\eta(Z)] + g(X, Z)\eta(Y) + g(h'X, Y)\eta(Z) + g(h'X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] - 2g((\nabla_X h'), Y, Z),\]
and

\[(\nabla_Y S)(Z, X) = \frac{dr(Y)}{2} [g(Z, X) - \eta(Z)\eta(X)] - \left(\frac{r}{2} - 3k\right) [g(Y, Z)\eta(X)] + g(Y, X)\eta(Z) + g(h'Y, Z)\eta(X) + 3g(h'Y, X)\eta(Z)
- 2\eta(X)\eta(Y)\eta(Z)] - 2g((\nabla_Y h')Z, X).
\]

(5.4)

With the help of (3.3) and (5.11), we get tensor satisfying (5.1). From (5.1), it follows that

\[(5.5)
(5.2) \rightarrow (5.4) \text{ in (5.1), we have}
\]

\[\text{It is known [16] that Cartan hypersurfaces are manifolds with non-parallel Ricci}
\]

tensor satisfying (5.1). From (5.1), it follows that \(r = \text{constant}.\) Making use of

(5.2)\text{–(5.4) in (5.1), we have}

\[(5.6)
(5.10, \text{ we see that either}
\]

\[g(\nabla z h', X, Y) + g((\nabla_X h')Y, Z) + g((\nabla_Y h')Z, X)
\]

(5.9)

\[= 2[(k + 1)[g(X, Y)\eta(Z) + g(Y, Z)\eta(X) + g(X, Z)\eta(Y)]
- 3\eta(X)\eta(Y)\eta(Z) - g(h'X, Y)\eta(Z) - g(h'X, Z)\eta(Y)] - 0.
\]

In account of (5.5) and (5.6), we get

(5.7)

\[\text{Setting } Z = \xi \text{ in (5.7) yields to}
\]

\[(r - 6k)[g(X, Y) + g(h'X, Y) - \eta(X)\eta(Y)]
+ 4[(k + 1)[g(X, Y) - (k + 1)\eta(X)\eta(Y) - g(h'X, Y)] = 0.
\]

Replacing \(X \text{ by } h'X \text{ in (5.8) and applying } h^2 = (k + 1)g^2, \text{ it implies}

(5.9)

\[(r - 6k)[g(h'X, Y) - (k + 1)g(X, Y) + (k + 1)\eta(X)\eta(Y)]
+ 4(k + 1)\{g(X, Y) - \eta(X)\eta(Y) + g(h'X, Y)\} = 0.
\]

Subtracting (5.9) from (5.8), we have

(5.10)

\[(r - 6k)(k + 2)[g(X, Y) - \eta(X)\eta(Y)] - 4(k + 2)g(h'X, Y) = 0.
\]

From (5.10), we see that either \(k = -2 \text{ or}

(5.11)

\[g(h'X, Y) = \frac{r - 6k}{4}[g(X, Y) - \eta(X)\eta(Y)].
\]

With the help of (3.3) and (5.11), we get

\[S(X, Y) = 2kg(X, Y),
\]

that is, the manifold is an Einstein manifold.

Therefore, by the similar argument as in Section 3, we have the following theo-
rem.
Theorem 5.1. Let \((M^3, \phi, \xi, \eta, g)\) be an almost Kenmotsu manifold such that \(\xi\) belongs to the \((k, \mu)\)'-nullity distribution with \(h' \neq 0\). If \(M^3\) admits cyclic parallel Ricci tensor, then, either the manifold is

1. an Einstein manifold, or
2. a CR-manifold.

6. \(\eta\)-parallel Ricci tensor

Definition 6.1. The Ricci tensor \(S\) of an almost Kenmotsu manifold \(M\) is called \(\eta\)-parallel if it satisfies

\[
(\nabla_X S)(\phi Y, \phi Z) = 0
\]

for all vector fields \(X, Y\) and \(Z\).

The notion of \(\eta\)-parallel Ricci tensor for Sasakian manifolds was given by Kon [17]. From (3.3), we have

\[
S(\phi X, \phi Y) = \left(\frac{r}{2} - k\right) g(\phi X, \phi Y) - 2g(h' \phi X, \phi Y).
\]

Taking covariant derivative of (6.2) along any vector field \(Z\) we get

\[
(\nabla_Z S)(\phi X, \phi Y) = \frac{\text{d}r(Z)}{2} g(\phi X, \phi Y) - 2g((\nabla_Z h') \phi X, \phi Y).
\]

Using (2.10), we obtain

\[
g((\nabla_Z h') \phi X, \phi Y) = 0.
\]

Taking account of (6.4), from (6.3), we get

\[
(\nabla_Z S)(\phi X, \phi Y) = \frac{\text{d}r(Z)}{2} g(\phi X, \phi Y).
\]

In view of (6.1) and (6.5), we have

\[
\frac{\text{d}r(Z)}{2} g(\phi X, \phi Y) = 0,
\]

that is, \(r = \text{constant}\).

Conversely, if \(r = \text{constant}\), then it can be easily shown that

\[
(\nabla_X S)(\phi Y, \phi Z) = 0
\]

for all vector fields \(X, Y\) and \(Z\).

Hence we can state the following theorem.

Theorem 6.1. The Ricci tensor of an almost Kenmotsu manifold \(M\) of dimension 3 with \(\xi\) belonging to the \((k, \mu)'\)-nullity distribution and \(h' \neq 0\) is \(\eta\)-parallel if and only if the scalar curvature \(r\) is constant.

7. Locally \(\phi\)-Ricci symmetric almost Kenmotsu manifolds

In this section, we study locally \(\phi\)-Ricci symmetric almost Kenmotsu manifolds of dimension 3 with \(\xi\) belonging to the \((k, \mu)'\)-nullity distribution and \(h' \neq 0\).
Taking covariant derivative of (3.2) along any vector field $X$, we have
\begin{equation}
(\nabla_X Q) Y = \frac{dr(X)}{2} [Y - \eta(Y)\xi] - \left(\frac{r}{2} - 3k\right) \left[\eta(Y)\nabla_X \xi + \eta(Y)\nabla_X \xi\right] - 2\nabla_X h' Y.
\end{equation}

Applying $\phi^2$ on both sides of (7.1) and using (2.3) yield to
\begin{equation}
\phi^2((\nabla_X Q) Y) = \frac{dr(X)}{2} [-Y + \eta(Y)\xi] - \left(\frac{r}{2} - 3k\right) \eta(Y)\phi^2(\nabla_X \xi) - 2\phi^2(\nabla_X h') Y.
\end{equation}

Making use of (2.10), the above equation implies
\begin{equation}
\phi^2((\nabla_X Q) Y) = \frac{dr(X)}{2} [-Y + \eta(Y)\xi] - \left(\frac{r}{2} - 3k\right) \eta(Y)\phi^2(\nabla_X \xi) + 2\eta(Y)\phi^2(h' X + h'^2 X).
\end{equation}

In view of (2.11) and (7.3), we have
\begin{equation}
\frac{dr(X)}{2} Y = 0,
\end{equation}
that is, $r$ = constant.

Conversely, if $r$ is constant, then the manifold is locally $\phi$-Ricci symmetric. Thus we have the following theorem.

**Theorem 7.1.** An almost Kenmotsu manifold $M$ of dimension 3 with $\xi$ belonging to the $(k, \mu)'$-nullity distribution and $h' \neq 0$ is locally $\phi$-Ricci symmetric if and only if the scalar curvature $r$ is a constant, provided the scalar curvature $r$ is invariant under $\xi$.

Hence from Theorem 6.1 and Theorem 7.1, we have the following corollary.

**Corollary 7.1.** In an almost Kenmotsu manifold $M$ of dimension 3 with $\xi$ belonging to the $(k, \mu)'$-nullity distribution and $h' \neq 0$, the following statements are equivalent:
1. Ricci tensor is $\eta$-parallel;
2. manifold is locally $\phi$-Ricci symmetric;
3. scalar curvature $r$ is a constant, provided the scalar curvature $r$ is invariant under $\xi$.

8. Example of a 3-dimensional almost Kenmotsu manifold

We consider 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Let $\xi, e_2, e_3$ are three vector fields in $\mathbb{R}^3$ which satisfy [9]
\begin{align*}
[e_2, e_3] = 0, \\
[\xi, e_2] = -e_2 - e_3, \\
\end{align*}

Let $g$ be the Riemannian metric defined by
\begin{align*}
g(\xi, \xi) &= g(e_2, e_2) = g(e_3, e_3) = 1, \\
g(\xi, e_2) &= g(\xi, e_3) = g(e_2, e_3) = 0.
\end{align*}
Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, \xi)$ for any $Z \in T(M)$. Let $\phi$ be the $(1,1)$ tensor field defined by

$$\phi(\xi) = 0, \quad \phi(e_2) = e_3, \quad \phi(e_3) = -e_2.$$ 

Using the linearity of $\phi$ and $g$, we have $\eta(\xi) = 1$, $\phi^2 X = -X + \eta(X)\xi$, and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any $X, Y \in \chi(M)$. Thus the structure $(\phi, \xi, \eta, g)$ is an almost contact structure. Also we have

$$h'\xi = 0, \quad h'(e_2) = e_3, \quad h'(e_3) = e_2.$$ 

The Riemannian connection $\nabla$ of the metric $g$ is given by the Koszul’s formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using the Koszul’s formula, we obtain

$$\nabla_\xi \eta = 0, \quad \nabla_\xi e_2 = 0, \quad \nabla_\xi e_3 = 0,$$

$$\nabla_{e_2} \xi = e_2 + e_3, \quad \nabla_{e_2} e_2 = -\xi, \quad \nabla_{e_2} e_3 = -\xi,$$

$$\nabla_{e_3} \xi = e_2 + e_3, \quad \nabla_{e_3} e_2 = -\xi, \quad \nabla_{e_3} e_3 = -\xi.$$ 

In view of the above relations, we get

$$\nabla_X \xi = -\phi^2 X + h'X$$

for any $X \in \chi(M)$. Therefore, the structure $(\phi, \xi, \eta, g)$ is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that $M$ is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor $R$ as follows:

$$R(\xi, e_2)\xi = 2(e_2 + e_3), \quad R(\xi, e_2)e_2 = -2\xi, \quad R(\xi, e_2)e_3 = -2\xi,$$

$$R(e_2, e_3)\xi = R(e_2, e_3)e_2 = R(e_2, e_3)e_3 = 0,$$

$$R(\xi, e_3)\xi = 2(e_2 + e_3), \quad R(\xi, e_3)e_2 = -2\xi, \quad R(\xi, e_3)e_3 = -2\xi.$$ 

With the help of the expressions of the curvature tensor, we conclude that the characteristic vector field $\xi$ belongs to the $(k, \mu)$-nullity distribution with $k = -2$ and $\mu = -2$.

Using the expressions of the curvature tensor, we find the values of the Ricci tensor $S$ as follows:

$$S(\xi, \xi) = -4, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2.$$ 

Therefore, the scalar curvature $r = S(\xi, \xi) + S(e_2, e_2) + S(e_3, e_3) = -8$, a constant. Hence, Theorem 6.1 and Theorem 7.1 are verified.

**Acknowledgment.** The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

**References**


K. Mandal and U. C. De, Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol-700019, West Bengal, India, e-mail: krishanu.mandal1013@gmail.com; uc_de@yahoo.com