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UNIT FRACTIONS IN NORM-EUCLIDEAN RINGS OF INTEGERS

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ABSTRACT. In this article, we consider the Erdős-Straus conjecture in a more general setting. For instance, one can look at the diophantine equation

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

where n and a, b, c are Gaussian integers. We have considered this problem in the case of rings of integers of the norm-Euclidean quadratic fields. Without any other restrictions on a, b and c, we show that solutions exist except for a finite set, which is given explicitly in each particular case. The problem becomes as difficult as the original Erdős-Straus conjecture if we require that all variables are in the first or third quadrant, but numerical evidence shows a decomposition still exists. We formulate this new conjecture explicitly in the end of this article.

1. INTRODUCTION

The Erdős-Straus conjecture became a topic of interest in the late 1940s and early 1950s [**6, 15, 17**], and been the topic of many papers. Richard Guy has a wonderful account of the progress on this work (see [**7**]). In short, the conjecture asks to show that for every natural number $n \geq 2$, the Diophantine equation

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \tag{1}$$

has a solution a, b, $c \in \mathbb{N}$. There have been many partial results about the nature of solutions to this equation. Some people used algebraic geometry techniques to give structure this problem (see [**3**]). Many attempts use analytic number theory techniques to find mean and asymptotic results (see [**4, 5, 10, 18, 19, 24, 25, 29**]). Some people have tried to look at decompositions of related fractions, such as k/n for $k \geq 2$ (see [**1, 4, 12, 16, 26, 27**]). Some tried computational methods (see [**22**]). Many people have organized primes p into two classes based of the decompositions of $4/p$ in hopes to find a pattern within each class (see [**2, 5, 18, 19**]). Some people attempt to find patterns in the field of fractions of the polynomial ring $\mathbb{Z}[x]$ instead of $\mathbb{Q}$ (see [**21**]). A well known method was developed by Rosati [**17**]. Mordell [**13**] has a great description of this method and many attempts use the techniques also applied in his paper (see [**8, 20, 23, 28**]).

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We first note that if $n \in \mathbb{Z}$ such that $|n| \geq 2$, then (1) has a solution a, b, $c \in \mathbb{Z}$. The following decompositions render this problem trivial

$$\frac{4}{n} = \begin{cases} \frac{1}{k} + \frac{1}{k} & \text{if } n = 2k,\ k \in \mathbb{Z} \text{ with } k \neq 0 \\ \frac{1}{k+1} + \frac{1}{(k+1)(4k+1)} & \text{if } n = 4k+3,\ k \in \mathbb{Z} \text{ with } k \neq -1 \\ \frac{1}{k} - \frac{1}{k(4k+1)} & \text{if } n = 4k+1,\ k \in \mathbb{Z} \text{ with } k \neq 0. \end{cases} \tag{2}$$

We will call the collection of $n \in \mathbb{Z}$, where there is no solution, an exceptional set and denote it $\mathcal{E}$. Here $\mathcal{E} = \{-1, 0, 1\}$. Although it is obvious that (1) has a solution for all $n \in \mathbb{Z} \setminus \mathcal{E}$, where $a, b, c \in \mathbb{Z}$, it is less obvious that there exists a finite, exceptional set in a general ring with identity, $\mathcal{E} \subset \mathcal{R}$, so that (1) has a solution for all $n \in \mathcal{R} \setminus \mathcal{E}$, where $a, b, c \in \mathcal{R}$. Is the existence of a solution outside of a finite, exceptional set a consequence of unique factorization or is it necessary to require more structure? Finding solutions in general rings is difficult so we begin by considering the ring of integers for quadratic fields. It is still unclear which rings of integers have unique factorization, but the norm-Euclidean quadratic fields were fully classified [**9**]. These fields are $\mathbb{Q}(\sqrt{d})$, where d takes values

$$-11,\ -7,\ -3,\ -2,\ -1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 11,\ 13,\ 17,\ 19,\ 21,\ 29,\ 33,\ 37,\ 41,\ 57,\ 73.$$

The rings of integers for quadratic fields have been thoroughly studied. We will use the notation $\mathbb{D}[d]$ to represent the ring of integers for the quadratic field $\mathbb{Q}(\sqrt{d})$. We can cite [**11**] to argue that the proof of the following is an elementary homework problem in algebraic number theory

$$\mathbb{D}[d] = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2, 3 \pmod 4 \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \pmod 4). \end{cases} \tag{3}$$

The following theorem is the main result of this paper. This result will prove that a sufficient condition for (1) having a solution in $\mathbb{D}[d]$ has $\mathbb{Q}(\sqrt{d})$ being norm-Euclidean.

Theorem 1.1. *Let $\mathbb{Q}(\sqrt{d})$ be a norm-Euclidean quadratic field and let $\mathbb{D}[d]$ be its ring of integers. Let $\mathcal{E}_d$ be a finite exceptional set,* (1) *has a solution $a, b, c \in \mathbb{D}[d]$ for every $n \in \mathbb{D}[d] \setminus \mathcal{E}_d$.*

This is not to say that (1) does not have solutions in general for the rings of integers of quadratic fields that are not norm-Euclidean. We can highlight this with the following decomposition in $\mathbb{Z}[\omega]$, where $\omega = (1/2) + (\sqrt{69}/2)$:

$$4 = \frac{1}{1710 + 468\omega} + \frac{1}{2178 - 468\omega}. \tag{4}$$

It is well-known that the ring of integers for a quadratic field $\mathbb{Q}(\sqrt{d})$ will be a unique factorization domain if it has class number 1. Determining the values of $d \geq 0$ so that $\mathbb{D}[d]$ has class number 1 is an open problem whereas it is well-established that the only possible values of $d \leq 0$ are those mentioned already for norm-Euclidean quadratic fields as well as the following:

$$-19,\ -43,\ -67,\ -163.$$

We also want to suggest that a sufficient condition for (1) having a solution in the ring of integers for a quadratic field has $\mathbb{D}[d]$ being a unique factorization domain.

The rest of the paper is organized as follows. In Section 2, we find decompositions for the ring of integers for norm-Euclidean quadratic fields when $d \geq 0$, in Section 3 decompositions when $d \leq 0$, and in Section 4, we provide the motivation behind our main theorem by making an insightful conjecture similar to the Erdős-Straus conjecture.

2. Positive values

In this section, we are interested in finding solutions to (1) for the rings of integers $\mathbb{D}[d]$, where

$$d \in \{2,\ 3,\ 5,\ 6,\ 7,\ 11,\ 13,\ 17,\ 19,\ 21,\ 29,\ 33,\ 37,\ 41,\ 57,\ 73\}. \tag{5}$$

It is quite interesting and somewhat unexpected that we have a rather trivial situation in each of these cases.

Theorem 2.1. *For every $n \in \mathbb{D}[d] \setminus \{0\}$, there exist a, b in $\mathbb{D}[d]$ such that*

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b}. \tag{6}$$

Proof. The proof of this statement follows from the following identity

$$4 = \begin{cases} \dfrac{1}{a+b\sqrt{d}} + \dfrac{1}{a-b\sqrt{d}} & \text{if } d \equiv 2,3 \pmod 4, \\ \dfrac{1}{a+b\omega} + \dfrac{1}{(a+b)-b\omega} & \text{if } d \equiv 1 \pmod 4, \end{cases}$$

where $\omega = (1/2) + (\sqrt{d}/2)$.

We summarize this information in the following table.

d	a	b	d	a	b	d	a	b	d	a	b
2	−4	3	7	32	12	19	14450	3315	37	−21	6
3	2	1	11	50	15	21	11	6	41	−592	160
5	−3	2	13	−207	90	29	−2905	910	57	33	10
6	−12	−5	17	−10	4	33	5	2	73	*	*

for $d = 73$ has $a = -637062$ and $b = 133500$. □

We point out that for $d \geq 0$ mentioned above, the pattern appears to be such that there exists $a + b\sqrt{d} \in \mathbb{D}[d] \setminus \{0\}$ and it holds

$$4 = \frac{1}{a+b\sqrt{d}} + \frac{1}{a-b\sqrt{d}}. \tag{7}$$

This can be rewritten to suggest that for the given $d \geq 0$, there exist $a, b \in \mathbb{Z}$ such that

$$(4a-1)^2 - d(4b)^2 = 1. \tag{8}$$

If we relabel $x = 4a-1$ and $y = 4b$, we can see that we look for specific solutions to Pell's equation (see [**14**])

$$x^2 - dy^2 = 1. \tag{9}$$

Using this method mentioned above, it is not difficult to show that 4 can be decomposed as in (6) for all quadratic fields $\mathbb{D}[d]$ for which d is a squarefree, positive integer.

3. Negative values

In this section, we are interested in solving (1) for rings of integers of norm-Euclidean quadratic fields $\mathbb{Q}(\sqrt{d})$ for which

$$d \in \{-1, -2, -3, -7, -11\}. \tag{10}$$

Notice that all these fields are subsets of $\mathbb{C}$. Much of the methodology in finding decompositions for the rings in this section is the same, as in (1) however, each ring brings its own complications. To simplify this as much as possible, we introduce some propositions that will be used in every scenario. We also define some functions that will simplify our notation and make it easy to identify the general pattern to the decompositions.

The first step in every possible scenario of d will be the same. If we take any number $n \in \mathbb{D}[d]$ and divide it by 4, we can consider the remainder and find our first unit fraction. For example, if there exist $m, r \in \mathbb{D}[d]$ with $m \neq 0$ so that $n = 4m + r$, then we can write

$$\frac{4}{n} = \frac{1}{m} - \frac{r}{nm}. \tag{11}$$

If we take any number in $\mathbb{D}[d]$ and divide it by 4, we have sixteen possible remainders. Expressing $\mathbb{D}[d] = \mathbb{Z}[\omega]$, where ω is defined as in (3), we see that the remainders will be $m + n\omega$, where $m, n \in \{-1, 0, 1, 2\}$. Letting $x + y\omega = 4(a + b\omega) + (m + n\omega)$ with $m, n \in \{-1, 0, 1, 2\}$ it is clear that if $m + n\omega$ is a multiple of a prime divisor of 2, we have that $x + y\omega$ is not a prime number with the only exception where $a + b\omega = 0$ and $m + n$ is an associate of a prime divisor of 2. An important aspect of the Erdős-Straus conjecture is that reduces to primes. That is to say that solving (1) for all primes in the ring outside of an exceptional set is sufficient to solve (1) for all numbers in the ring outside of an exceptional set. It is clear that we do not have to derive decompositions for some remainder scenarios.

To make our decomposition equations easier to read, we define a function p: $\mathbb{D}[d] \times \mathbb{D}[d] \to \mathbb{D}[d]$ by $p(a, b) = 4a + b$. This function also helps us account for the 16 remainder scenarios. The value of b will tell us which coset we are using and the different remainders require different techniques to find the decomposition as in (1). There are some remainders that use the same method for finding a decomposition. The following two propositions reduce the number of remainder scenarios to consider by using some symmetry within the rings $\mathbb{D}[d]$. Because they describe the action of units and Galois automorphisms (conjugation in our case) on residue classes, which are clearly understood, we will omit the proofs.

Proposition 3.1. *Suppose that $b \in \mathbb{D}[d] \setminus \{0\}$. If there exists a decomposition as in (1) for $\frac{4}{p(a,b)}$ for all $a \in \mathbb{D}[d]$, then there exists a decomposition as in (1) for $\frac{4}{p(a,ub)}$ for all $a \in \mathbb{D}[d]$ and units $u \in \mathbb{D}[d]$.*

Proposition 3.2. *Suppose that $b \in \mathbb{D}[d] \setminus \{0\}$. If there exists a decomposition as in (1) for $\frac{4}{p(a,b)}$ for all $a \in \mathbb{D}[d]$, then there exists a decomposition as in (1) for $\frac{4}{p(a,\bar{b})}$ for all $a \in \mathbb{D}[d]$.*

At this point we consider the remainder scenarios that exist after reductions through symmetry. Some scenarios are shown to have decompositions rather easily while other scenarios require a more advanced method to find the decompositions. For every ring $\mathbb{D}[d]$ in this setion, the methods used in the more complicated scenarios are roughly the same. For example, we argued that finding an initial decomposition as in (11) would be the first step for finding the decomposition as in (1). For each remainder scenario, after the first division by 4, the next step is to divide m by r and consider the possible remainders. The following proposition tells us the nature of these remainders.

Proposition 3.3. *Let $x, n \in \mathbb{D}[d]$ be any numbers such that $|n|^2$ is odd, $|n| \neq 1$ and the nonreal component of n is relatively prime from $|n|^2$, then there exist numbers $q \in \mathbb{D}[d]$ and $r \in \mathbb{Z}$ such that $|r| \leq \frac{|n|^2}{2}$ and $x = nq + r$.*

Proof. Note that $|n|^2$ will be an integer because $\mathbb{D}[d]$ is the ring of integers of a norm-Euclidean quadratic field.

Our first goal is to show that there exists $m \in \mathbb{Z}$ so that $x + mn$ has a nonreal component that is a multiple of $|n|^2$.
Let the nonreal component of x be $a \in \mathbb{Z}$ and the nonreal component of n be $b \in \mathbb{Z}$. Because b and $|n|^2$ are relatively prime, we see that there exist $s, t \in \mathbb{Z}$ such that $1 = sb + t|n|^2$.
If we let $m = -as$, we see that

$$a + mb = a - asb = a - a(1 - t|n|^2) = at|n|^2.$$

This shows that the nonreal component of $x + mn$ is a multiple of $|n|^2$. Regardless of the value of the real component of $x + mn$ when $m = -as$, we can express it as $r + m'|n|^2$ where $r, m' \in \mathbb{Z}$ and $|r| < |n|^2$. Define $k \in \mathbb{D}[d]$ as a number with real component m' and nonreal component at.
We see now that

$$x = -mn + (x + mn) = -mn + (r + k|n|^2) = n(-m + k\bar{n}) + r.$$

If we let $q = (-m + k\bar{n})$, then we see that $x = nq + r$, where r is an integer such that $|r| < |n|^2$.

If $|r| > \frac{|n|^2}{2}$, then $|n|^2 - |r| \leq \frac{|n|^2}{2}$. We can let $q = (-m + k\bar{n}) \pm \bar{n}$ in the appropriate scenario and rename r, so without loss of generality, we can assume that $|r| \leq \frac{|n|^2}{2}$. □

Again, to make the decompositions able to be read with a terse notation, we define another function for $n \in \mathbb{D}[d]$, $q_n \colon \mathbb{D}[d] \times \mathbb{D}[d] \to \mathbb{D}[d]$ such that $q_n(a,b) = na + b$. We can also use this function to reduce the amount of work in our method further through symmetry. The decompositions for some of these possible scenarios after the second division are redundant. The following proposition accounts for the redundancies and again makes a simple statement about the action of units on the residue classes.

Proposition 3.4. *Suppose that $n, r \in \mathbb{D}[d] \setminus \{0\}$. If there exists a decomposition as in (1) for $\frac{4}{p(q_n(b,r),-n)}$ for all $b \in \mathbb{D}[d]$, then there exists a decomposition as in (1) for $\frac{4}{p(q_n(b,ur),-n)}$ for all $b \in \mathbb{D}[d]$ and units $u \in \mathbb{D}[d]$.*

Proof. Let $n, r \in \mathbb{D}[d] \setminus \{0\}$. Let $u_1, u_2 \in \mathbb{D}[d]$ be units such that $u_1 u_2 = 1$. For all $b \in \mathbb{D}[d]$, suppose that there exist $x, y, z \in \mathbb{D}[d]$ such that

$$\frac{4}{p(q_n(b,r),-n)} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Proposition 3.1 tells us that for all $b \in \mathbb{D}[d]$ there exist $x', y', z' \in \mathbb{D}[d]$ such that

$$\frac{4}{p(q_n(b,r),-u_2 n)} = \frac{1}{x'} + \frac{1}{y'} + \frac{1}{z'}.$$

This implies that there exists $x'', y'', z'' \in \mathbb{D}[d]$ for any $b \in \mathbb{D}[d]$ so that

$$\frac{4}{p(q_n(u_2 b,r),-u_2 n)} = \frac{1}{x''} + \frac{1}{y''} + \frac{1}{z''}.$$

Notice then that

$$\begin{aligned} \frac{4}{p(q_n(b,u_1 r),-n)} &= \frac{4}{p(q_n(u_1 u_2 b, u_1 r), -u_1 u_2 n)} \\ &= \frac{4}{u_1 p(q_n(u_2 b, r), -u_2 n)} = \frac{1}{u_1 x''} + \frac{1}{u_1 y''} + \frac{1}{u_1 z''}. \end{aligned}$$

□

At this point each ring will use the properties intrinsic to the ring to find the decompositions. We mentioned earlier that some decompositions were quite simple and the decompositions in the following proposition show that there are some simple decompositions that have the same basic pattern across all the possible rings in this section.

Proposition 3.5. *For $a, b \in \mathbb{D}[d] \setminus \{0\}$ and any $n \in \mathbb{D}[d]$,*

$$\frac{4}{p(a,-1)} = \frac{1}{a} + \frac{1}{a \cdot p(a,-1)} \tag{12}$$

(13)
$$\frac{4}{p(q_n(b,1),-n)} = \frac{1}{q_n(b,1)} + \frac{1}{p(q_n(b,1),-n)\cdot b} - \frac{1}{p(q_n(b,1),-n)\cdot q_n(b,1)\cdot b}.$$

Proof. First we see that
$$\frac{4}{p(a,-1)} = \frac{4}{4a-1} = \frac{1}{a} + \frac{1}{a\cdot(4a-1)} = \frac{1}{a} + \frac{1}{a\cdot p(a,-1)}.$$
Next we see that
$$\begin{aligned}\frac{4}{p(q_n(b,1),-n)} &= \frac{4}{4(nb+1)-n} = \frac{1}{nb+1} + \frac{n}{(4(nb+1)-n)\cdot(nb+1)}\\ &= \frac{1}{nb+1} + \frac{nb+1-1}{(4(nb+1)-n)\cdot(nb+1)\cdot b}\\ &= \frac{1}{nb+1} + \frac{1}{(4(nb+1)-n)\cdot b} - \frac{1}{(4(nb+1)-n)\cdot(nb+1)\cdot b}\\ &= \frac{1}{q_n(b,1)} + \frac{1}{p(q_n(b,1),-n)\cdot b} - \frac{1}{p(q_n(b,1),-n)\cdot q_n(b,1)\cdot b}.\end{aligned}$$
□

Now we will divide the difficult decompositions into three types. For the first type of decomposition we define the function s so that for $m,n\in\mathbb{D}[d]$ and $r\in\mathbb{Z}$, there exists a unit $u\in\mathbb{D}[d]$ such that

$$s_{n,r}(q_n(m,r)) = \frac{q_n(m,r)+u}{n} \in \mathbb{D}[d]. \tag{14}$$

For the second type of decomposition, we define the function s so that for $m,n\in\mathbb{D}[d]$ and $r\in\mathbb{Z}$, there exists a unit $u\in\mathbb{D}[d]$ such that

$$s_{n,r}(q_n(m,r)) = \frac{p(q_n(m,r),-n)+u}{n} \in \mathbb{D}[d]. \tag{15}$$

For the third type of decomposition, we define the function s so that for $m,n\in\mathbb{D}[d]$ and $r\in\mathbb{Z}$, there exists a unit $u\in\mathbb{D}[d]$ such that

$$s_{n,r}(q_n(m,r)) = \frac{p(q_n(m,r),-n)\cdot q_n(m,r)+u}{n} \in \mathbb{D}[d]. \tag{16}$$

If we write
$$\begin{aligned}p &= p(q_n(m,r),-n)\\ q_n &= q_n(m,r)\\ s_{n,r} &= s_{n,r}(q_n(m,r)),\end{aligned}$$
then the following proposition tells us how to decompose the fraction $4/p$.

Proposition 3.6. *If there exists a function $s_{n,r}$ as in the first type of decomposition, we see that there exists a unit $u'\in\mathbb{D}[d]$ such that*
$$\frac{4}{p} = \frac{1}{q_n} + \frac{1}{p\cdot s_{n,r}} + \frac{1}{u'\cdot p\cdot q_n\cdot s_{n,r}}.$$

If there exists a function $s_{n,r}$ as in the second type of decomposition, we see that there exists a unit $u' \in \mathbb{Z}$ such that

$$\frac{4}{p} = \frac{1}{q_n} + \frac{1}{q_n \cdot s_{n,r}} + \frac{1}{u' \cdot p \cdot q_n \cdot s_{n,r}}.$$

If there exists a function $s_{n,r}$ as in the third type of decomposition, we see that there exists a unit $u' \in \mathbb{Z}$ such that

$$\frac{4}{p} = \frac{1}{q_n} + \frac{1}{s_{n,r}} + \frac{1}{u' \cdot p \cdot q_n \cdot s_{n,r}}.$$

Proof. First notice that for all three types of functions we have that

$$\frac{4}{p} = \frac{1}{q_n} + \frac{n}{p \cdot q_n}.$$

In all three situation, we always multiply the second fraction by $s_{n,r}$ and use the identities (14), (15) and (16) to manipulate the equation.

For all three situation, we have that

$$\frac{4}{p} = \frac{1}{q_n} + \frac{n}{p \cdot q_n} = \frac{1}{q_n} + \frac{n \cdot s_{n,r}}{p \cdot q_n \cdot s_{n,r}}.$$

For the first situation we see that there exists $u' \in \mathbb{D}[d]$ such that $u' \cdot u = 1$ so that

$$\frac{4}{p} = \frac{1}{q_n} + \frac{q_n + u}{p \cdot q_n \cdot s_{n,r}} = \frac{1}{q_n} + \frac{1}{p \cdot s_{n,r}} + \frac{1}{u' \cdot p \cdot q_n \cdot s_{n,r}}.$$

For the second situation we see that there exists $u' \in \mathbb{D}[d]$ such that $u' \cdot u = 1$ so that

$$\frac{4}{p} = \frac{1}{q_n} + \frac{p + u}{p \cdot q_n \cdot s_{n,r}} = \frac{1}{q_n} + \frac{1}{q_n \cdot s_{n,r}} + \frac{1}{u' \cdot p \cdot q_n \cdot s_{n,r}}.$$

For the third situation we see that there exists $u' \in \mathbb{D}[d]$ such that $u' \cdot u = 1$ so that

$$\frac{4}{p} = \frac{1}{q_n} + \frac{p \cdot q_n + u}{p \cdot q_n \cdot s_{n,r}} = \frac{1}{q_n} + \frac{1}{s_{n,r}} + \frac{1}{u' \cdot p \cdot q_n \cdot s_{n,r}}.$$

□

At this point, we put into action our general methodology for finding decompositions. While this method is similar for each value in (10), we will see that the decompositions for each ring is unique. We begin by considering the Gaussian integers. To make the notation similar for each ring, we let $\omega = \mathrm{i} = \sqrt{-1}$. Let $\mathcal{E}_{-1} = \{n \in \mathbb{Z}[\omega] : |n|^2 \leq 2\}$.

Theorem 3.1. *There exists a decomposition similar to* (1) *for every element in* $\mathbb{D}[-1] \smallsetminus \mathcal{E}_{-1}$.

Proof. After dividing by 4, considering Propositions 3.1 and 3.2 and accounting for nonprime remainder scenarios, it suffices to find a decomposition for remainders $-1, -(1-2\omega)$. Proposition 3.5 finds a decomposition when the remainder is -1.

Because $|1-2\omega|^2 = 5$ and $(2,5) = 1$, we can use Proposition 3.3 to suggest that for $x + y\omega \in \mathbb{Z}[\omega]$, there exist $c + d\omega \in \mathbb{Z}[\omega]$ and $r \in \{-2,-1,0,1,2\}$ so that $x + y\omega = (1-2\omega)(c+d\omega) + r$. Let $q_{1-2\omega}\left((c+d\omega), r\right) = (1-2\omega)(c+d\omega) + r$.

Notice that $p\left(q_{1-2\omega}\left((c+d\omega),0\right),-(1-2\omega)\right)$ is not a prime number except when $c+d\omega=0$. Proposition 3.4 tells us that it suffices to find a solution for $r\in\{1,2\}$ and Proposition 3.5 finds a decomposition when $r=1$.

Define

$$s_{1-2\omega,2}\left(q_{1-2\omega}\left((c+d\omega),2\right)\right)=\frac{q_{1-2\omega}\left((c+d\omega),2\right)+\omega}{1-2\omega}=c+(d+1)\omega.$$

Notice that $s_{1-2\omega,2}$ is a function of the first type from Proposition 3.6, so we see it has an appropriate decomposition. We only need to mention the following decompositions because either the prime of which it is a power was is $\mathcal{E}_{-1}$ or because $p\left(q_{1-2\omega}\left(0,0\right),-(1-2\omega)\right)$ is prime:

$$\frac{4}{(1+\omega)^2}=\frac{1}{\omega}+\frac{1}{2\omega}+\frac{1}{2\omega},\qquad\qquad \frac{4}{-1+2\omega}=\frac{1}{\omega}+\frac{1}{-1+\omega}+\frac{1}{-3+\omega}.$$

□

Next we denote $\omega=\sqrt{-2}$. Let $\mathcal{E}_{-2}=\{n\in\mathbb{Z}[\omega]:|n|^2\leq 3\}$.

Theorem 3.2. *There exists a decomposition similar to* (1) *for every element in* $\mathbb{D}[-2]\smallsetminus\mathcal{E}_{-2}$.

Proof. After dividing by 4, considering Propositions 3.1 and 3.2, and accounting for nonprime remainder scenarios, it suffices to find a decomposition for remainders $-1,-(1+\omega),-(1+2\omega)$. Proposition 3.5 finds a decomposition when the remainder is -1.

We first find a decomposition for remainder $-(1+\omega)$. Because $|1+\omega|^2=3$ and $(1,3)=1$, we can use Proposition 3.3 to suggest that for $x+y\omega\in\mathbb{Z}[\omega]$, there exist $c+d\omega\in\mathbb{Z}[\omega]$ and $r\in\{-1,0,1\}$ so that $x+y\omega=(1+\omega)(c+d\omega)+r$. Let $q_{1+\omega}\left((c+d\omega),r\right)=(1+\omega)(c+d\omega)+r$. Notice that $p\left(q_{1+\omega}\left((c+d\omega),0\right),-(1+\omega)\right)$ is not a prime number except when $c+d\omega=0$, but this prime number is an element of $\mathcal{E}_{-2}$. Proposition 3.4 tells us that it suffices to find a solution for $r=1$ and Proposition 3.5 finds a decomposition when $r=1$.

Next we find a decomposition for remainder $-(1+2\omega)$. Because $|1+2\omega|^2=9$ and $(2,9)=1$, we can use Proposition 3.3 again to suggest that for $x+y\omega\in\mathbb{Z}[\omega]$, there exist $c+d\omega\in\mathbb{Z}[\omega]$ and $r\in\{-4,-3,-2,-1,0,1,2,3,4\}$ so that $x+y\omega=(1+2\omega)(c+d\omega)+r$. Let $q_{1+2\omega}\left((c+d\omega),r\right)=(1+2\omega)(c+d\omega)+r$. Because $(1+2\omega)=-1\cdot(1-\omega)^2$ and $3=(1-\omega)\cdot(1+\omega)$ we see $p\left(q_{1+2\omega}\left((c+d\omega),0\right),-(1+2\omega)\right)$ and $p\left(q_{1+2\omega}\left((c+d\omega),\pm3\right),-(1+2\omega)\right)$ are not prime numbers. Proposition 3.4 tells us that it suffices to find a solution for $r\in\{1,2,4\}$ and Proposition 3.5 finds a decomposition when $r=1$.

Define

$$\begin{aligned}s_{1+2\omega,2}\left(q_{1+2\omega}\left((c+d\omega),2\right)\right)&=\frac{p\left(q_{1+2\omega}\left((c+d\omega),2\right),-(1+2\omega)\right)+1}{1+2\omega}\\&=4(c+d\omega)-2\omega\end{aligned}$$

$$
\begin{aligned}
&s_{1+2\omega,4}\left(q_{1+2\omega}\left((c+d\omega),4\right)\right)\\
&\qquad=\frac{p\left(q_{1+2\omega}\left((c+d\omega),4\right),-(1+2\omega)\right)\cdot q_{1+2\omega}\left((c+d\omega),4\right)-1}{1+2\omega}\\
&\qquad=(c+d\omega)\left(4\left((1+2\omega)(c+d\omega)+8\right)-(1+2\omega)\right)+(3-14\omega).
\end{aligned}
$$

Notice that $s_{1+2\omega,2}$ is a function of the second type from Proposition 3.6 and $s_{1+2\omega,4}$ is a function of the third type from Proposition 3.6, so we see that they both have appropriate decompositions. We only need to mention the following decompositions because they are the product of primes in $\mathcal{E}_{-2}$:

$$
\frac{4}{\omega^2}=\frac{1}{-1}+\frac{1}{-2}+\frac{1}{-2},\qquad \frac{4}{(1+\omega)^2}=\frac{1}{\omega}+\frac{1}{-2+\omega}+\frac{1}{-1+2\omega},
$$

$$
\frac{4}{(1-\omega)(1+\omega)}=\frac{1}{2}+\frac{1}{2}+\frac{1}{3},\qquad \frac{4}{\omega(1+\omega)}=\frac{1}{-1}+\frac{1}{\omega}+\frac{1}{-2+\omega}.
$$

□

Next we denote $\omega=(1/2)+(\sqrt{-3}/2)$. Let $\mathcal{E}_{-3}=\{n\in\mathbb{Z}[\omega]:|n|^2\le 1\}$.

Theorem 3.3. *There exists a decomposition similar to* (1) *for every element in* $\mathbb{D}[-3]\setminus\mathcal{E}_{-3}$.

Proof. After dividing by 4, considering Propositions 3.1 and 3.2, and accounting for nonprime remainder scenarios, it suffices to find a decomposition for remainders -1 and $-(1+\omega)$ along with all associates of the prime number 2. Proposition 3.5 finds a decomposition when the remainder is -1.

We have to find a decomposition for remainder $-(1+\omega)$. Because $|1+\omega|^2=3$ and $(1,3)=1$, we can use Proposition 3.3 to suggest that for $x+y\omega\in\mathbb{Z}[\omega]$, there exist $c+d\omega\in\mathbb{Z}[\omega]$ and $r\in\{-1,0,1\}$ so that $x+y\omega=(1+\omega)(c+d\omega)+r$. Let $q_{1+\omega}\left((c+d\omega),r\right)=(1+\omega)(c+d\omega)+r$. Notice that $p\left(q_{1+\omega}\left((c+d\omega),0\right),-(1+\omega)\right)$ is not a prime number except when $c+d\omega=0$. Proposition 3.4 tells us that it suffices to find a solution for $r=1$ and Proposition 3.5 finds a decomposition when $r=1$. We only need to mention the following decompositions because either they lead to decompositions of associates of the prime number 2 or because $p\left(q_{1+\omega}\left(0,0\right),-(1+\omega)\right)$ is prime:

$$
\frac{4}{2}=\frac{1}{1}+\frac{1}{2}+\frac{1}{2},\qquad \frac{4}{1+\omega}=\frac{1}{1}+\frac{1}{\omega}+\frac{1}{1+\omega}.
$$

□

We now denote $\omega=\frac{1}{2}+\frac{\sqrt{-7}}{2}$. Let $\mathcal{E}_{-7}=\{n\in\mathbb{Z}[\omega]:|n|^2\le 2\}$.

Theorem 3.4. *There exists a decomposition similar to* (1) *for every element in* $\mathbb{D}[-7]\setminus\mathcal{E}_{-7}$.

Proof. After dividing by 4, considering Propositions 3.1 and 3.2, and accounting for nonprime remainder scenarios, it suffices to find a decomposition for remainders -1 and $-(1-2\omega)$. Proposition 3.5 finds a decomposition when the remainder is -1.

We only have to find a decomposition for remainder $-(1-2\omega)$. Because $|1-2\omega|^2=7$ and $(2,7)=1$, we can use Proposition 3.3 to suggest that for

$x + y\omega \in \mathbb{Z}[\omega]$, there exist $c + d\omega \in \mathbb{Z}[\omega]$ and $r \in \{-3, -2, -1, 0, 1, 2, 3\}$ so that $x + y\omega = (1 - 2\omega)(c + d\omega) + r$. Let $q_{1-2\omega}((c + d\omega), r) = (1 - 2\omega)(c + d\omega) + r$. Notice that $p(q_{1-2\omega}((c + d\omega), 0), -(1 - 2\omega))$ is not a prime number except when $c+d\omega = 0$. Proposition 3.4 tells us that it suffices to find a solution for $r \in \{1, 2, 3\}$ and Proposition 3.5 finds a decomposition when $r = 1$.

Define

$$\begin{aligned} s_{1-2\omega,2}(q_{1-2\omega}((c + d\omega), 2)) &= \frac{p(q_\omega((c + d\omega), 2), -(1 - 2\omega)) - 1}{1 - 2\omega} \\ &= 4(c + d\omega) + (-2 + 2\omega) \end{aligned}$$

$$\begin{aligned} &s_{1-2\omega,3}(q_{1-2\omega}((c + d\omega), 3)) \\ &\quad = \frac{p(q_{1-2\omega}((c + d\omega), 3), -(1 - 2\omega)) \cdot q_{1-2\omega}((c + d\omega), 3) - 1}{1 - 2\omega} \\ &\quad = (c + d\omega)(4((1 - 2\omega)(c + d\omega) + 6) - (1 - 2\omega)) - (8 - 10\omega). \end{aligned}$$

Notice that $s_{1-2\omega,2}$ is a function of the second type from Proposition 3.6 and $s_{1-2\omega,3}$ is a function of the third type from Proposition 3.6, so we see that they both have appropriate decompositions. We only need to mention the following decompositions because they are either products of the primes in $\mathcal{E}_{-7}$ or because $p(q_{1-2\omega}(0, 0), -(1 - 2\omega))$ is prime:

$$\frac{4}{\omega^2} = \frac{1}{-1} + \frac{1}{-1+\omega} + \frac{1}{-1+\omega}, \qquad \frac{4}{(1-\omega)^2} = \frac{1}{-1} + \frac{1}{-\omega} + \frac{1}{-\omega},$$

$$\frac{4}{\omega(1-\omega)} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2}, \qquad \frac{4}{-1+2\omega} = \frac{1}{\omega} + \frac{1}{-1+\omega} + \frac{1}{-2+4\omega}.$$

□

We now denote $\omega = \frac{1}{2} + \frac{\sqrt{-11}}{2}$. Let $\mathcal{E}_{-11} = \{n \in \mathbb{Z}[\omega] : |n|^2 \leq 5\} \setminus \{2, -2\}$.

Theorem 3.5. *There exists a decomposition similar to* (1) *for every element in* $\mathbb{D}[-11] \setminus \mathcal{E}_{-11}$.

Proof. After dividing by 4, considering Propositions 3.1 and 3.2, and accounting for nonprime remainder scenarios, it suffices to find a decomposition for remainders $-1, -\omega, -(1 + \omega), -(1 + 2\omega)$ along with all associates of the prime number 2. Proposition 3.5 finds a decomposition when the remainder is -1.

We first find a decomposition for remainder $-\omega$. Because $|\omega|^2 = 3$ and $(1, 3) = 1$, we can use Proposition 3.3 to suggest that for $x+y\omega \in \mathbb{Z}[\omega]$, there exist $c+d\omega \in \mathbb{Z}[\omega]$ and $r \in \{-1, 0, 1\}$ so that $x+y\omega = \omega(c+d\omega)+r$. Let $q_\omega((c + d\omega), r) = \omega(c+d\omega)+r$. Notice that $p(q_\omega((c + d\omega), 0), -\omega)$ is not a prime number except when $c+d\omega = 0$, but this prime number is an element of $\mathcal{E}_{-11}$. Proposition 3.4 tells us that it suffices to find a solution for $r = 1$ and Proposition 3.5 finds a decomposition when $r = 1$.

Next we find a decomposition for remainder $-(1+\omega)$. Because $|1+\omega|^2 = 5$ and $(1, 5) = 1$, we can use Proposition 3.3 again to suggest that for $x+y\omega \in \mathbb{Z}[\omega]$, there exist $c+d\omega \in \mathbb{Z}[\omega]$ and $r \in \{-2, -1, 0, 1, 2\}$ so that $x+y\omega = (1+\omega)(c+d\omega)+r$. Let $q_{1+\omega}((c + d\omega), r) = (1+\omega)(c+d\omega)+r$. We see that $p(q_{1+\omega}((c + d\omega), 0), -(1 + \omega))$ is not a prime number, except when $c+d\omega = 0$, but this prime number is an element of $\mathcal{E}_{-11}$. Proposition 3.4 tells us that it suffices to find a solution when $r \in \{1, 2\}$ and Proposition 3.5 finds a decomposition when $r = 1$.

Define

$$\begin{aligned} & s_{1+\omega,2}\left(q_{1+\omega}\left((c+d\omega),2\right)\right) \\ &\qquad = \frac{p\left(q_{1+\omega}\left((c+d\omega),2\right),-(1+\omega)\right)\cdot q_{1+\omega}\left((c+d\omega),2\right)-1}{1+\omega} \\ &\qquad = (c+d\omega)\left(4\left((1+\omega)(c+d\omega)+4\right)-(1+\omega)\right)+(4+3\omega). \end{aligned}$$

Notice that $s_{1+\omega,2}$ is a function of the third type from Proposition 3.6, so we see it has an appropriate decomposition.

Finally we find a decomposition for $-(1 + 2\omega)$. Because $|1 + 2\omega|^2 = 15$ and $(2, 15) = 1$, we can use Proposition 3.3 to suggest that for $x + y\omega \in \mathbb{Z}[\omega]$, there exist $c + d\omega \in \mathbb{Z}[\omega]$ and $r \in \{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$ so that $x + y\omega = (1 + 2\omega)(c + d\omega) + r$. Let $q_{1+2\omega}((c + d\omega), r) = (1 + 2\omega)(c + d\omega) + r$. Because $(1 + 2\omega) = -1 \cdot (1 - \omega)(2 - \omega)$, $3 = \omega(1 - \omega)$ and $5 = (1 + \omega)(2 - \omega)$, we see that $p\left(q_{1+2\omega}((c+d\omega), 0), -(1+2\omega)\right)$, $p\left(q_{1+2\omega}((c+d\omega), \pm 3), -(1+2\omega)\right)$, $p\left(q_{1+2\omega}((c+d\omega), \pm 5), -(1+2\omega)\right)$ and $p\left(q_{1+2\omega}((c+d\omega), \pm 6), -(1+2\omega)\right)$ are not prime numbers. Proposition 3.4 tells us that it suffices to find a solution for $r \in \{1, 2, 4, 7\}$ and Proposition 3.5 finds a decomposition when $r = 1$.

Define

$$\begin{aligned} & s_{1+2\omega,2}\left(q_{1+2\omega}\left((c+d\omega),2\right)\right) \\ &\qquad = \frac{p\left(q_{1+2\omega}\left((c+d\omega),2\right),-(1+2\omega)\right)\cdot q_{1+2\omega}\left((c+d\omega),2\right)-1}{1+2\omega} \\ &\qquad = (c+d\omega)\left(4\left((1+2\omega)(c+d\omega)+4\right)-(1+2\omega)\right)+(1-2\omega) \\ & s_{1+2\omega,4}\left(q_{1+2\omega}\left((c+d\omega),4\right)\right) \\ &\qquad = \frac{p\left(q_{1+2\omega}\left((c+d\omega),4\right),-(1+2\omega)\right)-1}{1+2\omega} \\ &\qquad = 4(c+d\omega)+(2-2\omega) \\ & s_{1+2\omega,7}\left(q_{1+2\omega}\left((c+d\omega),7\right)\right) \\ &\qquad = \frac{p\left(q_{1+2\omega}\left((c+d\omega),7\right),-(1+2\omega)\right)\cdot q_{1+2\omega}\left((c+d\omega),7\right)-1}{1+2\omega} \\ &\qquad = (c+d\omega)\left(4\left((1+2\omega)(c+d\omega)+8\right)-(1+2\omega)\right)+(35-26\omega). \end{aligned}$$

Notice that $s_{1+2\omega,2}$ is a function of the third type from Proposition 3.6, $s_{1+2\omega,4}$ is a function of the second type from Proposition 3.6 and $s_{1+2\omega,7}$ is a function of the third type from Proposition 3.6, so we see that they both have appropriate decompositions. We only mention the following decompositions because either they are products of primes in $\mathcal{E}_{-11}$ or because they lead to decompositions of associates of the prime number 2:

$$\frac{4}{2} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2}, \qquad \frac{4}{\omega^2} = \frac{1}{-1} + \frac{1}{\omega^2} + \frac{1}{\omega},$$

$$\frac{4}{(1+\omega)^2} = \frac{1}{-1+\omega} + \frac{1}{2} + \frac{1}{12-18\omega}, \qquad \frac{4}{\omega(1-\omega)} = \frac{1}{2} + \frac{1}{2} + \frac{1}{3},$$
$$\frac{4}{\omega(1+\omega)} = \frac{1}{-1} + \frac{1}{\omega} + \frac{1}{1+\omega}, \qquad \frac{4}{(1-\omega)(1+\omega)} = \frac{1}{1} + \frac{1}{-\omega} + \frac{1}{3+3\omega},$$
$$\frac{4}{(1+\omega)(2-\omega)} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}.$$

□

The theorems in Section 2 and Section 3 combine to prove Theorem 1.1. In the next section, we will use our main theorem as a foundation for a conjecture that is very similar to the Erdős-Straus conjecture. This conjecture is the focus of our current research.

4. Conjecture

Notice that for (2), we relaxed the restriction of having the values $a, b, c \in \mathbb{N}$, which is a specific cone within the integers. With the Erdős-Straus conjecture being unsolved for decades and this integer version easily solved as in (2), this illuminates a stark contrast in difficulty. We would like to create a version of this conjecture in different number fields. For example, the Gaussian integers $\mathbb{Z}[\mathrm{i}]$ form a $\mathbb{Z}$-module with basis $\{1, \mathrm{i}\}$. After reducing through the symmetries of associates, it suffices to consider primes where both the real and imaginary parts are positive or, in other words, $n \in \mathbb{Z}[\mathrm{i}]$ within the positive cone generated by the $\mathbb{Z}$-module basis $\{1, \mathrm{i}\}$. If we wanted to restrict the possible solutions to a specific cone within $\mathbb{Z}[\mathrm{i}]$, then we would need to find $a, b, c \in \mathbb{Z}[\mathrm{i}]$ within the positive and negative cone, or simply cone, generated by the $\mathbb{Z}$-module basis $\{1, \mathrm{i}\}$. The following conjecture is the analogue of the natural number Erdős-Straus conjecture.

Conjecture 4.1 (Bradford-Ionascu)**.** *Let $\mathcal{E} := \{0, 1, \mathrm{i}, 1+\mathrm{i}\}$. For $n \in \mathbb{Z}[\mathrm{i}] \setminus \mathcal{E}$ with the real and imaginary part of n nonnegative,* (1) *has a solution $a, b, c \in \mathbb{Z}[\mathrm{i}]$ such that the real and imaginary parts of a, b and c are either both nonnegative or both nonpositive.*

For example, we see that

$$\frac{4}{1+2\,\mathrm{i}} = \frac{1}{\mathrm{i}} + \frac{1}{1+\mathrm{i}} + \frac{1}{3+\mathrm{i}}. \tag{17}$$

Notice that all of the Gaussian integers in the denominators of the unit fractions in (17) are in the appropriate region for our conjecture. Although our conjecture is similar in many ways to the natural number scenario, the restriction of the solution to (1) for $a, b, c \in \mathbb{Z}[\mathrm{i}]$ to this cone introduces complications. For example, the Erdős-Straus conjecture reduces to finding a solution to (1) for prime natural numbers. This is not the case for our conjecture. For example $1+2\,\mathrm{i}$ is prime in $\mathbb{Z}[\mathrm{i}]$, which has a unique solution outlined in (17), and $1+\mathrm{i}$ is a prime number in

$\mathcal{E}$. We see that $3+\mathrm{i} = (-\mathrm{i})(1+\mathrm{i})(1+2\,\mathrm{i})$, yet we see that

$$\begin{aligned}\frac{4}{3+\mathrm{i}} &= \frac{1}{(-\mathrm{i})(1+\mathrm{i})}\cdot\frac{4}{1+2\,\mathrm{i}}\\ &= \frac{1}{(-\mathrm{i})(1+\mathrm{i})}\cdot\left(\frac{1}{\mathrm{i}}+\frac{1}{1+\mathrm{i}}+\frac{1}{3+\mathrm{i}}\right)\\ &= \frac{1}{1+\mathrm{i}}+\frac{1}{2}+\frac{1}{4-2\,\mathrm{i}}\end{aligned} \tag{18}$$

is the only decomposition possible when decomposing one of the prime factors of $3+\mathrm{i}$. However, this does not imply that our conjecture does not hold. It means that our conjecture does not reduce to finding a solution to (1) for prime Gaussian integers outside of $\mathcal{E}$. We see that

$$\frac{4}{3+\mathrm{i}} = \frac{1}{1}+\frac{1}{1+3\,\mathrm{i}}+\frac{1}{5+5\,\mathrm{i}}. \tag{19}$$

This conjecture introduces a new approach that one can take to find results for problems similar to Erdős-Straus. Finding a solution of (1) for all Gaussian integers adds a foundation to our conjecture, but the difficultly of finding solutions for these two could be vastly different as in the relaxation of finding integer solutions to (1) rather than natural number solutions.

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