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ON A QUARTER-SYMMETRIC METRIC CONNECTION IN AN ε-LORENTZIAN PARA-SASAKIAN MANIFOLD

A. HASEEB, A. PRAKASH AND M. D. SIDDIQI

Abstract. In this paper, we consider a quarter-symmetric metric connection in an ε-Lorentzian para-Sasakian manifold. We investigate the curvature tensor and the Ricci tensor of an ε-Lorentzian para-Sasakian manifold with a quarter-symmetric metric connection. Also we have shown that ε-Lorentzian para-Sasakian manifolds with a quarter-symmetric metric connection are η-Einstein manifolds if they are conformally flat, quasi conformally flat and ξ-conformally flat.

1. Introduction

In [**3**], A. Bejancu and K. L. Duggal introduced the concept of ε-Sasakian manifolds. Later, it was shown by X. Xufeng and C. Xiaoli [**16**] that these manifolds are real hypersurface of indefinite Kaehlerian manifolds. In 2007, R. Kumar, R. Rani and R. K. Nagaich studied some interesting properties of ε-Sasakian manifolds [**7**]. In 2010, Tripathi et al. studied ε-almost paracontact manifolds and in particular, ε-para Sasakian manifolds [**14**]. On the other hand, the concept of ε-Kenmotsu manifolds was introduced by U. C. De and A. Sarkar [**5**] who showed that the existence of new structure on an indefinite metrics influences the curvatures. K. Matsumoto introduced the notion of Lorentzian para-Sasakian manifolds [**8**] and this was further studied by I. Mihai et al. [**9**], C. Özgür [**10**], A. A. Shaikh et al. [**12**] and many others. Recently, LP-Sasakian manifolds with a quarter symmetric metric connection have been studied by M. Ahmad et al. [**1**], R. N. Singh and Shravan K. Pandey [**13**], Venkatesha et al. [**15**] and many others. U. C. De and A. K. Mondal discussed quarter symmetric metric connection in a 3-dimensional quasi-Sasakian manifold [**4**]. M. Ali and Z. Ahsan studied conformal curvature tensor for the space time of general relativity [**2**].

A linear connection $\bar{\nabla}$ in a Riemannian manifold M is said to be a quarter-symmetric connection [**6**] if the torsion tensor T of the connection $\bar{\nabla}$

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

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where η is a 1-form and ϕ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0$$

for all $X,Y,Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold M, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. If we put $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric metric connection reduces to a semi-symmetric metric connection [**17**]. Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

Motivated by the above studies, in this paper, we study some new results on a quarter-symmetric metric connection in an ε-Lorentzian para-Sasakian manifold. The paper is organized as follows: In Section 2, we give a brief account of an ε-LP-Sasakian manifold and define quarter-symmetric metric connection. In Section 3, we find the curvature tensor, the Ricci tensor and the scalar curvature in an ε-LP-Sasakian manifold with a quarter-symmetric metric connection. In Section 4, we show that the conformally flat ε-LP-Sasakian manifold with a quarter-symmetric metric connection is an η-Einstein manifold of a quasi constant curvature. Section 5 is devoted to the study of quasi conformally flat and ξ-conformally flat ε-LP-Sasakian manifold with a quarter-symmetric metric connection and in both cases we have shown that such manifolds are η-Einstein manifolds.

2. Preliminaries

A differentiable manifold of dimension n is called an ε-Lorentzian para-Sasakian (briefly, ε-LP-Sasakian), if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g(\xi,\xi) = -\varepsilon, \quad \eta(X) = \varepsilon g(X,\xi), \qquad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, $Y \in \chi(M)$, where ε is 1 or -1 according as ξ is a space like or time like vector field.

If an ε-contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \varepsilon\eta(Y)X + 2\varepsilon\eta(X)\eta(Y)\xi, \tag{2.4}$$

where ∇ denotes the Levi-Civita connection with respect to g, then M is called an ε-LP-Sasakian manifold.

An ε-almost contact metric manifold is an ε-LP-Sasakian manifold if and only if

$$\nabla_X \xi = \varepsilon\phi X. \tag{2.5}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in an ε-LP-Sasakian manifold M with respect to the Levi-Civita connection satisfy the following equations [**11**]:

$$(\nabla_X \eta)Y = g(\phi X, Y), \tag{2.6}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$R(\xi, X)Y = \varepsilon g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$R(\xi, X)\xi = -R(X,\xi)\xi = X + \eta(X)\xi, \tag{2.9}$$

$$\eta(R(X,Y)Z) = \varepsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.10}$$

$$S(X,\xi) = (n-1)\eta(X), \qquad Q\xi = \varepsilon(n-1)\xi, \tag{2.11}$$

where $X, Y, Z \in \chi(M)$ and $g(QX,Y) = S(X,Y)$.

We note that if $\varepsilon = 1$ and the structure vector field ξ is space like, then an ε-LP-Sasakian manifold is an usual LP-Sasakian manifold.

Definition 2.1. An ε-LP-Sasakian manifold called a manifold of quasi-constant curvature if the curvature tensor R' of type $(0,4)$ satisfies the condition

$$\begin{aligned} R^{'}(X,Y,Z,W) = {} & a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + b[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ & + g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)], \end{aligned} \tag{2.12}$$

where $R'(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type $(1,3)$; a, b are scalar functions and ρ is a unit vector field defined by

$$g(X,\rho) = T(X) \tag{2.13}$$

for any vector fields $X, Y, Z, W \in \chi(M)$.

Definition 2.2. An ε-LP-Sasakian manifold is said to be an η-Einstein manifold if its Ricci tensor S of type $(0,2)$ satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.14}$$

where a and b are scalar functions of ε.

Contracting (2.14), we have

$$r = na - b. \tag{2.15}$$

On the other hand, putting $X = Y = \xi$ in (2.14) and using (2.11), we also have

$$-(n-1) = -a\varepsilon + b. \tag{2.16}$$

Hence it follows from (2.15) and (2.16) that

$$a = \frac{r-(n-1)}{n-\varepsilon}, \qquad b = -\frac{n(n-1)-\varepsilon r}{n-\varepsilon}.$$

So the Ricci tensor S of an η-Einstein ε-LP-Sasakian manifold is given by

$$S(X,Y) = \frac{r-(n-1)}{n-\varepsilon} g(X,Y) - \frac{n(n-1)-\varepsilon r}{n-\varepsilon}\eta(X)\eta(Y). \tag{2.17}$$

Let M be an n-dimensional ε-LP-Sasakian manifold and ∇ be the Levi-Civita connection on M. The relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on M is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.18}$$

3. Curvature tensor on an ε-LP-Sasakian manifold with a quarter-symmetric metric connection

Let M be an n-dimensional ε-LP-Sasakian manifold. The curvature tensor $\bar{R}$ of M with respect to a quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.1}$$

From (2.6), (2.18) and (3.1), we have

$$\begin{aligned}\bar{R}(X,Y)Z = {} & (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \\ & - \nabla_{[X,Y]} Z) + \eta(Z)[(\nabla_X \phi)Y - (\nabla_Y \phi)X] \\ & + g[(\nabla_Y \phi)X - (\nabla_X \phi)Y, Z]\xi + [g(\phi X, Z)\nabla_Y \xi - g(\phi Y, Z)\nabla_X \xi].\end{aligned} \tag{3.2}$$

Using (2.4) and (2.5) in (3.2), we get

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + \varepsilon[\eta(Y)X - \eta(X)Y]\eta(Z) \\ & + \varepsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ & + \varepsilon[g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X],\end{aligned} \tag{3.3}$$

where $X, Y, Z \in \chi(M)$ and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the Riemannian curvature tensor of the connection ∇.

Now contracting X in (3.3), we get

$$\bar{S}(Y,Z) = S(Y,Z) + \varepsilon(n-1)\eta(Y)\eta(Z) - \varepsilon g(\phi Y, Z)\psi, \tag{3.4}$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively, on M and $\psi = \operatorname{trace}\phi$. This gives

$$\bar{Q}Y = QY + (n-1)\eta(Y)\xi - \varepsilon\phi Y\psi. \tag{3.5}$$

Contracting again Y and Z in (3.4), it follows that

$$\bar{r} = r - \varepsilon(n-1) - \varepsilon\psi^2, \tag{3.6}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively, on M.

Lemma 3.1. *Let M be an n-dimensional ε-LP-Sasakian manifold with a quarter-symmetric metric connection, then*

$$\bar{R}(X,Y)\xi = (1-\varepsilon)[\eta(Y)X - \eta(X)Y], \tag{3.7}$$

$$\bar{R}(\xi,X)Y = -\bar{R}(X,\xi)Y = -(1-\varepsilon)[X + \eta(X)\xi]\eta(Y), \tag{3.8}$$

$$\bar{R}(\xi,X)\xi = (1-\varepsilon)[X + \eta(X)\xi], \tag{3.9}$$

$$\bar{S}(X,\xi) = (1-\varepsilon)(n-1)\eta(X), \tag{3.10}$$

$$\bar{Q}\xi = -(1-\varepsilon)(n-1)\xi \tag{3.11}$$

for any vector fields $X, Y \in \chi(M)$.

Proof. By replacing $Z = \xi$ and using (2.1), (2.2) and (2.7) in (3.3), we get (3.7). (3.8) and (3.9) easily follow from (2.1), (2.2), (2.7) and (3.3). By taking $Y = \xi$ and using (2.1) and (2.2) in (3.4), (3.10) follows. By considering $Y = \xi$ and using (2.1), (2.2) and (2.11) in (3.5), we get (3.11). □

Lemma 3.2. *Let M be an n-dimensional ε-LP-Sasakian manifold with a quarter-symmetric metric connection, then*

$$(\bar{\nabla}_X\phi)(Y) = -(1-\varepsilon)[X+\eta(X)\xi]\eta(Y), \tag{3.12}$$

$$\bar{\nabla}_X\xi = -(1-\varepsilon)\phi X \tag{3.13}$$

for any vector fields $X, Y \in \chi(M)$.

Proof. By the covariant differentiation of ϕY with respect to X, we have

$$\bar{\nabla}_X\phi Y = (\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_X Y)$$

which by using (2.1), (2.2) and (2.18) takes the form

$$(\bar{\nabla}_X\phi)Y = (\nabla_X\phi)Y - g(\phi X,\phi Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi.$$

Using (2.3) and (2.4) in the last equation, we get

$$(\bar{\nabla}_X\phi)(Y) = -(1-\varepsilon)[X+\eta(X)\xi]\eta(Y).$$

To prove (3.13), we replace $Y = \xi$ in (2.18) and get

$$\bar{\nabla}_X\xi = \nabla_X\xi + \eta(\xi)X - g(\phi X,\xi)\xi$$

which by using (2.1), (2.2) and (2.5), reduces to

$$\bar{\nabla}_X\xi = -(1-\varepsilon)\phi X.$$

□

Now, let R and $\bar{R}$ be the curvature tensors of the connections ∇ and $\bar{\nabla}$, respectively, on M given by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W) \quad \text{and} \quad \bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W).$$

Therefore from (3.3), we have

$$\begin{aligned}\bar{R}(X,Y,Z,U) = {} & R(X,Y,Z,U) + \varepsilon[g(X,U)\eta(Y) - g(Y,U)\eta(X)]\eta(Z) \\ & + [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(U) \\ & + \varepsilon[g(\phi X,Z)g(\phi Y,U) - g(\phi Y,Z)g(\phi X,U)].\end{aligned} \tag{3.14}$$

Interchanging X and Y in (3.14), we have

$$\begin{aligned}\bar{R}(Y,X,Z,U) = {} & R(Y,X,Z,U) + \varepsilon[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(Z) \\ & + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\eta(U) \\ & + \varepsilon[g(\phi Y,Z)g(\phi X,U) - g(\phi X,Z)g(\phi Y,U)].\end{aligned} \tag{3.15}$$

By adding (3.14) and (3.15) and using the fact that $R(X,Y,Z,U)+R(Y,X,Z,U)=0$, we get

$$\bar{R}(X,Y,Z,U)+\bar{R}(Y,X,Z,U)=0. \tag{3.16}$$

Again interchanging U and Z in (3.14), we have

$$\begin{aligned}\bar{R}(X,Y,U,Z) = {} & R(X,Y,U,Z)+\varepsilon[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]\eta(U)\\ & +[g(Y,U)\eta(X)-g(X,U)\eta(Y)]\eta(Z)\\ & +\varepsilon[g(\phi X,U)g(\phi Y,Z)-g(\phi Y,U)g(\phi X,Z)].\end{aligned} \tag{3.17}$$

Now adding (3.14) and (3.17) and using the fact that $R(X,Y,Z,U)+R(X,Y,U,Z)=0$, we get

$$\begin{aligned}\bar{R}(X,Y,Z,U)+\bar{R}(X,Y,U,Z) = {} & (1-\varepsilon)[g(Y,U)\eta(X)-g(X,U)\eta(Y)]\eta(Z)\\ & +(1-\varepsilon)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\eta(U).\end{aligned} \tag{3.18}$$

Again interchanging pair of slots in (3.14), we have

$$\begin{aligned}\bar{R}(Z,U,X,Y) = {} & R(Z,U,X,Y)+\varepsilon[g(Z,Y)\eta(U)-g(U,Y)\eta(Z)]\eta(X)\\ & +[g(U,X)\eta(Z)-g(X,Z)\eta(U)]\eta(Y)\\ & +\varepsilon[g(\phi Z,X)g(\phi U,Y)-g(\phi U,X)g(\phi Z,Y)].\end{aligned} \tag{3.19}$$

Now, subtracting (3.19) from (3.14) and using the fact that $R(X,Y,Z,U)-R(Z,U,X,Y)=0$, we get

$$\begin{aligned}&\bar{R}(X,Y,Z,U)-\bar{R}(Z,U,X,Y)\\ &=(1-\varepsilon)[g(Y,Z)\eta(X)\eta(U)-g(X,U)\eta(Y)\eta(Z)].\end{aligned} \tag{3.20}$$

Thus in view of (3.16), (3.18) and (3.20), we can state the following theorem.

Theorem 3.1. *In an ε-LP-Sasakian manifold with a quarter-symmetric metric connection, we have:*

(i) $\bar{R}(X,Y,Z,U)+\bar{R}(Y,X,Z,U)=0$,

(ii) $\bar{R}(X,Y,Z,U)+\bar{R}(X,Y,U,Z)=(1-\varepsilon)[g(Y,U)\eta(X)-g(X,U)\eta(Y)]\eta(Z) + (1-\varepsilon)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\eta(U)$,

(iii) $\bar{R}(X,Y,Z,U)-\bar{R}(Z,U,X,Y)=(1-\varepsilon)[g(Y,Z)\eta(X)\eta(U)-g(X,U)\eta(Y)\eta(Z)]$

for any vector fields $X,Y,Z,U\in\chi(M)$.

Now, let $\bar{R}(X,Y)Z=0$, therefore from (3.3), we have

$$\begin{aligned}R(X,Y)Z = {} & \varepsilon[\eta(X)Y-\eta(Y)X]\eta(Z)-\varepsilon[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi\\ & +\varepsilon[g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y].\end{aligned} \tag{3.21}$$

Taking inner product of the above equation (3.21) with ξ, we have

$$g(R(X,Y)Z,\xi)=-[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \tag{3.22}$$

which can be written as

$$g(R(X,Y)Z,U)=-\varepsilon[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)]. \tag{3.23}$$

Thus we can state the following theorem.

Theorem 3.2. *If the curvature tensor of a quarter-symmetric metric connection in an ε-LP-Sasakian manifold M vanishes, then the manifold is of constant curvature $(-\varepsilon)$ and consequently it is locally isometric to the Hyperbolic space $H^n(-\varepsilon)$.*

4. Conformally flat ε-LP-Sasakian manifold with a quarter-symmetric metric connection

Definition 4.1. The conformal curvature tensor $\bar{C}$ of type $(1,3)$ of an n-dimensional ε-LP-Sasakian manifold with a quarter-symmetric metric connection $\bar{\nabla}$, is given by [**18**]

$$\begin{aligned}\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X \\ - g(X,Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{4.1}$$

where $\bar{Q}$ is the Ricci operator with respect to a quarter-symmetric metric connection related by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$ and $\bar{r}$ is the scalar curvature with respect to a quarter-symmetric metric connection.

Let us assume that the manifold M with respect to a quarter-symmetric metric connection is conformally flat, that is $\bar{C} = 0$. Then from (4.1), it follows that

$$\begin{aligned}\bar{R}(X,Y)Z = \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \\ - \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{4.2}$$

Taking inner product of (4.2) with ξ and using (2.2) and (3.10), we have

$$\begin{aligned}g(\bar{R}(X,Y)Z,\xi) = \frac{\varepsilon}{(n-2)}[\bar{S}(Y,Z)\eta(X) - \bar{S}(X,Z)\eta(Y) \\ - (1-\varepsilon)(n-1)g(Y,Z)\eta(X) + (1-\varepsilon)(n-1)g(X,Z)\eta(Y)] \\ - \frac{\varepsilon\bar{r}}{(n-1)(n-2)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].\end{aligned} \tag{4.3}$$

Putting $X = \xi$ and using (2.1), (2.2) and (3.8) in (4.3), we get

$$\begin{aligned}\bar{S}(Y,Z) = [(1-\varepsilon)(n-1) + \frac{\bar{r}}{n-1}]g(Y,Z) \\ + [-2(1-\varepsilon)(n-1) + \frac{\varepsilon\bar{r}}{n-1}]\eta(Y)\eta(Z).\end{aligned} \tag{4.4}$$

Therefore, (4.4) is of the form

$$\bar{S}(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $a = (1-\varepsilon)(n-1) + \frac{\bar{r}}{n-1}$ and $b = -2(1-\varepsilon)(n-1) + \frac{\varepsilon\bar{r}}{n-1}$.

Next, using (4.4) in (4.2), we have

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= \frac{1}{(n-2)}\Big[\Big(2(1-\varepsilon)(n-1)+\frac{2\bar{r}}{n-1}\Big) \\ &\cdot (g(Y,Z)g(X,W)-g(X,Z)g(Y,W)) \\ &+\Big(\frac{\varepsilon\bar{r}}{n-1}-2(1-\varepsilon)(n-1)\Big)(\eta(Y)\eta(Z)g(X,W) \\ &-\eta(Y)\eta(W)g(X,Z)+\eta(X)\eta(W)g(Y,Z)-\eta(X)\eta(Z)g(Y,W))\Big] \\ &-\frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)], \end{aligned} \tag{4.5}$$

which by simplifying takes the form

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= \frac{2(1-\varepsilon)(n-1)^2+\bar{r}}{(n-1)(n-2)} \\ &\cdot (g(Y,Z)g(X,W)-g(X,Z)g(Y,W)) \\ &+\frac{\varepsilon\bar{r}-2(1-\varepsilon)(n-1)^2}{(n-1)(n-2)}(\eta(Y)\eta(Z)g(X,W)-\eta(Y)\eta(W)g(X,Z) \\ &+\eta(X)\eta(W)g(Y,Z)-\eta(X)\eta(Z)g(Y,W)). \end{aligned} \tag{4.6}$$

Thus by virtue of (4.4) and (4.6), we can state the following theorem.

Theorem 4.2. *An n-dimensional conformally flat ε-LP-Sasakian manifold with a quarter-symmetric metric connection is an η-Einstein manifold of quasi constant curvature.*

5. Quasi conformally flat and ξ- conformally flat ε-LP-Sasakian manifold with a quarter-symmetric metric connection

Definition 5.1. An ε-LP-Sasakian manifold is said to be:

(i) quasi conformally flat with a quarter-symmetric metric connection if

$$g(\bar{C}(X,Y)Z,\phi W)=0, \qquad X,Y,Z,W\in\chi(M), \tag{5.1}$$

(ii) ξ-conformally flat with a quarter-symmetric metric connection if

$$\bar{C}(X,Y)\xi=0, \qquad X,Y\in\chi(M). \tag{5.2}$$

First we consider quasi conformally flat ε-LP-Sasakian manifold with a quarter-symmetric metric connection. Therefore from (4.1) and (5.1), we have

$$\begin{aligned} g(\bar{R}(X,Y)Z,\phi W) &- \frac{1}{(n-2)}[\bar{S}(Y,Z)g(X,\phi W)-\bar{S}(X,Z)g(Y,\phi W) \\ &+g(Y,Z)g(\bar{Q}X,\phi W)-g(X,Z)g(\bar{Q}Y,\phi W)] \\ &+\frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,\phi W)-g(X,Z)g(Y,\phi W)]=0, \end{aligned} \tag{5.3}$$

which by considering $Y = Z = \xi$ and using (2.2), (3.9)–(3.11) reduce to

$$\bar{S}(X, \phi W) = \Big[(1-\varepsilon) + \frac{\bar{r}}{(n-1)}\Big] g(X, \phi W). \tag{5.4}$$

Now replacing $W = \phi W$ and using (2.1), (2.2) and (3.10) in (5.4), we get

$$\bar{S}(X, W) = \Big[(1-\varepsilon) + \frac{\bar{r}}{(n-1)}\Big] g(X, W) + \Big[-n(1-\varepsilon) + \frac{\varepsilon\bar{r}}{(n-1)}\Big]\eta(X)\eta(W). \tag{5.5}$$

Thus we can state the following theorem.

Theorem 5.2. *An n-dimensional quasi conformally flat ε-LP-Sasakian manifold with a quarter-symmetric metric connection $\bar{\nabla}$ is an η-Einstein manifold.*

Next, by virtue of (4.1) and (5.2), we can write

(5.6)

$$g[\bar{R}(X,Y)\xi - \frac{1}{(n-2)}(\bar{S}(Y,\xi)X - \bar{S}(X,\xi)Y + g(Y,\xi)\bar{Q}X - g(X,\xi)\bar{Q}Y) + \frac{\bar{r}}{(n-1)(n-2)}(g(Y,\xi)X - g(X,\xi)Y), W] = 0.$$

Now using (2.2), (3.7) and (3.10) in (5.6), we have

$$\begin{aligned} &\Big[\frac{\bar{r} + (1-\varepsilon)(n-1)^2 - (1-\varepsilon)(n-1)(n-2)}{(n-1)}\Big] \\ &\cdot(\eta(Y)g(X,W) - \eta(X)g(Y,W)) + \eta(Y)\bar{S}(X,W) - \eta(X)\bar{S}(Y,W) = 0, \end{aligned} \tag{5.7}$$

which by taking $Y = \xi$ and using (2.1), takes the form

$$\begin{aligned} \bar{S}(X,W) = &-\Big[\frac{\bar{r} + (1-\varepsilon)(n-1)^2 - (1-\varepsilon)(n-1)(n-2)}{(n-1)}\Big] g(X,W) \\ &-\Big[\frac{\varepsilon\bar{r} - 2(1-\varepsilon)(n-1)^2 + (1-\varepsilon)(n-1)(n-2)}{(n-1)}\Big]\eta(X)\eta(W). \end{aligned} \tag{5.8}$$

Thus we can state the following theorem.

Theorem 5.3. *An n-dimensional ξ-conformally flat ε-LP-Sasakian manifold with a quarter-symmetric metric connection $\bar{\nabla}$ is an η-Einstein manifold.*

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A. Haseeb, Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, *e-mail*: malikhaseeb80@gmail.com

A. Prakash, Department of Mathematics, National Institute of Technology, Kurukshetra-136191, India, *e-mail*: amitmath@nitkkr.ac.in

M. D. Siddiqi, Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, *e-mail*: anallintegral@gmail.com