ON A QUARTER-SYMMETRIC METRIC CONNECTION IN AN $\varepsilon$-LORENTZIAN PARA-SASAKIAN MANIFOLD

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Abstract. In this paper, we consider a quarter-symmetric metric connection in an $\varepsilon$-Lorentzian para-Sasakian manifold. We investigate the curvature tensor and the Ricci tensor of an $\varepsilon$-Lorentzian para-Sasakian manifold with a quarter-symmetric metric connection. Also we have shown that $\varepsilon$-Lorentzian para-Sasakian manifolds with a quarter-symmetric metric connection are $\eta$-Einstein manifolds if they are conformally flat, quasi conformally flat and $\xi$-conformally flat.

1. Introduction

In [3], A. Bejancu and K. L. Duggal introduced the concept of $\varepsilon$-Sasakian manifolds. Later, it was shown by X. Xufeng and C. Xiaoli [16] that these manifolds are real hypersurface of indefinite Kaehlerian manifolds. In 2007, R. Kumar, R. Rani and R. K. Nagaich studied some interesting properties of $\varepsilon$-Sasakian manifolds [7]. In 2010, Tripathi et al. studied $\varepsilon$-almost paracontact manifolds and in particular, $\varepsilon$-para Sasakian manifolds [14]. On the other hand, the concept of $\varepsilon$-Kenmotsu manifolds was introduced by U. C. De and A. Sarkar [5] who showed that the existence of new structure on an indefinite metrics influences the curvatures. K. Matsumoto introduced the notion of Lorentzian para-Sasakian manifolds [8] and this was further studied by I. Mihai et al. [9], C. Özgür [10], A. A. Shaikh et al. [12] and many others. Recently, LP-Sasakian manifolds with a quarter symmetric metric connection have been studied by M. Ahmad et al. [1], R. N. Singh and Shravan K. Pandey [13], Venkatesha et al. [15] and many others. U. C. De and A. K. Mondal discussed quarter symmetric metric connection in a 3-dimensional quasi-Sasakian manifold [4]. M. Ali and Z. Ahsan studied conformal curvature tensor for the space time of general relativity [2].

A linear connection $\nabla$ in a Riemannian manifold $M$ is said to be a quarter-symmetric connection [6] if the torsion tensor $T$ of the connection $\nabla$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

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where $\eta$ is a 1-form and $\phi$ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\nabla$ satisfies the condition

$$(\nabla_X g)(Y,Z) = 0$$

for all $X,Y,Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold $M$, then $\nabla$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. If we put $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric metric connection reduces to a semi-symmetric metric connection \cite{17}. Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

Motivated by the above studies, in this paper, we study some new results on a quarter-symmetric metric connection in an $\varepsilon$-Lorentzian para-Sasakian manifold.

The paper is organized as follows: In Section 2, we give a brief account of an $\varepsilon$-LP-Sasakian manifold and define quarter-symmetric metric connection. In Section 3, we find the curvature tensor, the Ricci tensor and the scalar curvature in an $\varepsilon$-LP-Sasakian manifold with a quarter-symmetric metric connection. In Section 4, we show that the conformally flat $\varepsilon$-LP-Sasakian manifold with a quarter-symmetric metric connection is an $\eta$-Einstein manifold of a quasi constant curvature. Section 5 is devoted to the study of quasi conformally flat and $\xi$-conformally flat $\varepsilon$-LP-Sasakian manifold with a quarter-symmetric metric connection and in both cases we have shown that such manifolds are $\eta$-Einstein manifolds.

2. Preliminaries

A differentiable manifold of dimension $n$ is called an $\varepsilon$-Lorentzian para-Sasakian (briefly, $\varepsilon$-LP-Sasakian), if it admits a $(1,1)$-tensor field $\phi$, a contravariant vector field $\xi$, a 1-form $\eta$ and a Lorentzian metric $g$ which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1,$$

(2.1)

$$g(\xi,\xi) = -\varepsilon, \quad \eta(X) = \varepsilon g(X,\xi), \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

(2.2)

$$g(\phi X,\phi Y) = g(X,Y) - \varepsilon \eta(X)\eta(Y)$$

(2.3)

for all vector fields $X, Y \in \chi(M)$, where $\varepsilon$ is 1 or $-1$ according as $\xi$ is a space like or time like vector field.

If an $\varepsilon$-contact metric manifold satisfies

$$\nabla_X (\phi) (Y) = g(X,Y)\xi + \varepsilon \eta(Y)X + 2\varepsilon \eta(X)\eta(Y)\xi,$$

(2.4)

where $\nabla$ denotes the Levi-Civita connection with respect to $g$, then $M$ is called an $\varepsilon$-LP-Sasakian manifold.

An $\varepsilon$-almost contact metric manifold is an $\varepsilon$-LP-Sasakian manifold if and only if

$$\nabla_X \xi = \varepsilon \phi X.$$  

(2.5)

Moreover, the curvature tensor $R$, the Ricci tensor $S$ and the Ricci operator $Q$ in an $\varepsilon$-LP-Sasakian manifold $M$ with respect to the Levi-Civita connection satisfy
the following equations [11]:

(2.6) \((\nabla_X \eta)Y = g(\phi X, Y)\),

(2.7) \(R(X, Y)\xi = \eta(Y)X - \eta(X)Y\),

(2.8) \(R(\xi, X)Y = \varepsilon g(X, Y)\xi - \eta(Y)X\),

(2.9) \(R(\xi, X)\xi = -R(X, \xi)\xi = X + \eta(X)\xi\),

(2.10) \(\eta(R(X, Y)Z) = \varepsilon g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\),

(2.11) \(S(X, \xi) = (n - 1)\eta(X), \quad Q\xi = \varepsilon(n - 1)\xi\),

where \(X, Y, Z \in \chi(M)\) and \(g(QX, Y) = S(X, Y)\).

We note that if \(\varepsilon = 1\) and the structure vector field \(\xi\) is space like, then an \(\varepsilon\)-LP-Sasakian manifold is an usual LP-Sasakian manifold.

**Definition 2.1.** An \(\varepsilon\)-LP-Sasakian manifold called a manifold of quasi-constant curvature if the curvature tensor \(R'\) of type \((0, 4)\) satisfies the condition

\[
R'(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
+ b[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) \\
+ g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)],
\]

where \(R'(X, Y, Z, W) = g(R(X, Y)Z, W)\), \(R\) is the curvature tensor of type \((1, 3)\); \(a, b\) are scalar functions and \(\rho\) is a unit vector field defined by

\[
g(X, \rho) = T(X)
\]

for any vector fields \(X, Y, Z, W \in \chi(M)\).

**Definition 2.2.** An \(\varepsilon\)-LP-Sasakian manifold is said to be an \(\eta\)-Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) satisfies

\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]

where \(a\) and \(b\) are scalar functions of \(\varepsilon\).

Contracting (2.14), we have

\[
r = na - b.
\]

On the other hand, putting \(X = Y = \xi\) in (2.14) and using (2.11), we also have

\[
-(n - 1) = -a\varepsilon + b.
\]

Hence it follows from (2.15) and (2.16) that

\[
a = \frac{r - (n - 1)}{n - \varepsilon}, \quad b = -\frac{n(n - 1) - \varepsilon r}{n - \varepsilon}.
\]

So the Ricci tensor \(S\) of an \(\eta\)-Einstein \(\varepsilon\)-LP-Sasakian manifold is given by

\[
S(X, Y) = \frac{r - (n - 1)}{n - \varepsilon}g(X, Y) - \frac{n(n - 1) - \varepsilon r}{n - \varepsilon}\eta(X)\eta(Y).
\]
Let $M$ be an $n$-dimensional $\varepsilon$-LP-Sasakian manifold and $\nabla$ be the Levi-Civita connection on $M$. The relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection $\nabla$ on $M$ is given by
\begin{equation}
(2.18) \; \bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi.
\end{equation}

3. Curvature tensor on an $\varepsilon$-LP-Sasakian manifold with a quarter-symmetric metric connection

Let $M$ be an $n$-dimensional $\varepsilon$-LP-Sasakian manifold. The curvature tensor $\bar{R}$ of $M$ with respect to a quarter-symmetric metric connection $\bar{\nabla}$ is defined by
\begin{equation}
(3.1) \; \bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z.
\end{equation}

From (2.6), (2.18) and (3.1), we have
\begin{equation}
(3.2) \; \bar{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\end{equation}

Using (2.4) and (2.5) in (3.2), we get
\begin{equation}
(3.3) \; \bar{R}(X,Y)Z = R(X,Y)Z + \varepsilon(n - 1)\eta(Y)X - \varepsilon g(\phi Y, Z)\psi,
\end{equation}
where $X,Y,Z \in \chi(M)$ and $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ is the Riemannian curvature tensor of the connection $\nabla$.

Now contracting $X$ in (3.3), we get
\begin{equation}
(3.4) \; \bar{S}(Y,Z) = S(Y,Z) + \varepsilon(n - 1)\eta(Y)\eta(Z) - \varepsilon g(\phi Y, Z)\psi,
\end{equation}
where $\bar{S}$ and $S$ are the Ricci tensors of the connections $\bar{\nabla}$ and $\nabla$, respectively, on $M$ and $\psi = \text{trace} \phi$. This gives
\begin{equation}
(3.5) \; \bar{Q}Y = QY + (n - 1)\eta(Y)\xi - \varepsilon\phi Y\psi.
\end{equation}

Contracting again $Y$ and $Z$ in (3.4), it follows that
\begin{equation}
(3.6) \; \bar{r} = r - \varepsilon(n - 1) - \varepsilon\psi^2,
\end{equation}
where $\bar{r}$ and $r$ are the scalar curvatures of the connections $\bar{\nabla}$ and $\nabla$, respectively, on $M$.

Lemma 3.1. Let $M$ be an $n$-dimensional $\varepsilon$-LP-Sasakian manifold with a quarter-symmetric metric connection, then
\begin{align}
(3.7) \; \bar{R}(X,Y)\xi &= (1 - \varepsilon)[\eta(Y)X - \eta(X)Y], \\
(3.8) \; \bar{R}(\xi,X)Y &= -\bar{R}(X,\xi)Y = -(1 - \varepsilon)[X + \eta(X)\xi][\eta(Y)], \\
(3.9) \; \bar{R}(\xi,X)\xi &= (1 - \varepsilon)[X + \eta(X)\xi],
\end{align}
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\[ \bar{S}(X, \xi) = (1 - \varepsilon)(n - 1)\eta(X), \]  
\[ \bar{Q}\xi = -(1 - \varepsilon)(n - 1)\xi \]

for any vector fields \( X, Y \in \chi(M) \).

**Proof.** By replacing \( Z = \xi \) and using (2.1), (2.2) and (2.7) in (3.3), we get (3.7). (3.8) and (3.9) easily follow from (2.1), (2.2), (2.7) and (3.3). By taking \( Y = \xi \) and using (2.1) and (2.2) in (3.4), (3.10) follows. By considering \( Y = \xi \) and using (2.1), (2.2) and (2.11) in (3.5), we get (3.11). \( \Box \)

**Lemma 3.2.** Let \( M \) be an \( n \)-dimensional \( \varepsilon \)-LP-Sasakian manifold with a quarter-symmetric metric connection, then

\[ \bar{\nabla}_X \phi(Y) = -(1 - \varepsilon)[X + \eta(X)\xi]\eta(Y), \]
\[ \bar{\nabla}_X \xi = -(1 - \varepsilon)\phi X \]

for any vector fields \( X, Y \in \chi(M) \).

**Proof.** By the covariant differentiation of \( \phi Y \) with respect to \( X \), we have

\[ \nabla_X \phi Y = (\nabla_X \phi)Y + \phi(\nabla_X Y) \]

which by using (2.1), (2.2) and (2.18) takes the form

\[ (\nabla_X \phi)Y = (\nabla_X \phi)Y - g(\phi X, \phi Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi. \]

Using (2.3) and (2.4) in the last equation, we get

\[ (\nabla_X \phi)(Y) = -(1 - \varepsilon)[X + \eta(X)\xi]\eta(Y). \]

To prove (3.13), we replace \( Y = \xi \) in (2.18) and get

\[ \nabla_X \xi = \nabla_X \xi + \eta(\xi)X - g(\phi X, \xi)\xi \]

which by using (2.1), (2.2) and (2.5), reduces to

\[ \nabla_X \xi = -(1 - \varepsilon)\phi X. \]

\( \Box \)

Now, let \( R \) and \( \bar{R} \) be the curvature tensors of the connections \( \nabla \) and \( \bar{\nabla} \), respectively, on \( M \) given by

\[ R(X, Y, Z, W) = g(R(X, Y)Z, W) \quad \text{and} \quad \bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W). \]

Therefore from (3.3), we have

\[ \bar{R}(X, Y, Z, U) = R(X, Y, Z, U) + \varepsilon[g(X, U)\eta(Y) - g(Y, U)\eta(X)]\eta(Z) \]
\[ + [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\eta(U) \]
\[ + \varepsilon[g(\phi X, Z)g(\phi Y, U) - g(\phi Y, Z)g(\phi X, U)]. \]

Interchanging \( X \) and \( Y \) in (3.14), we have

\[ \bar{R}(Y, X, Z, U) = R(Y, X, Z, U) + \varepsilon[g(Y, U)\eta(X) - g(X, U)\eta(Y)]\eta(Z) \]
\[ + [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\eta(U) \]
\[ + \varepsilon[g(\phi Y, Z)g(\phi X, U) - g(\phi X, Z)g(\phi Y, U)]. \]
By adding (3.14) and (3.15) and using the fact that 
\[ R(X,Y,Z,U) + R(Y,X,Z,U) = 0, \]
we get
\[ (3.16) \]
\[ \bar{R}(X,Y,Z,U) + \bar{R}(Y,X,Z,U) = 0. \]
Again interchanging \( U \) and \( Z \) in (3.14), we have
\[ (3.17) \]
\[ \bar{R}(X,Y,U,Z) = R(X,Y,U,Z) + \varepsilon[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\eta(U) \]
\[ + [g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(Z) \]
\[ + \varepsilon[g(\phi X,U)g(\phi Y,Z) - g(\phi Y,U)g(\phi X,Z)]. \]
Now adding (3.14) and (3.17) and using the fact that 
\[ R(X,Y,Z,U) + R(Y,X,U,Z) = 0, \]
we get
\[ (3.18) \]
\[ \bar{R}(X,Y,Z,U) + \bar{R}(X,Y,U,Z) = (1 - \varepsilon)[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(Z) \]
\[ + (1 - \varepsilon)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(U). \]
Again interchanging pair of slots in (3.14), we have
\[ (3.19) \]
\[ \bar{R}(Z,U,X,Y) = R(Z,U,X,Y) + \varepsilon[g(Z,Y)\eta(U) - g(U,Y)\eta(Z)]\eta(X) \]
\[ + [g(U,X)\eta(Z) - g(X,Z)\eta(U)]\eta(Y) \]
\[ + \varepsilon[g(\phi Z,X)g(\phi Y,U) - g(\phi U,X)g(\phi Z,Y)]. \]
Now, subtracting (3.19) from (3.14) and using the fact that 
\[ R(X,Y,Z,U) - R(Z,U,X,Y) = 0, \]
we get
\[ (3.20) \]
\[ \bar{R}(X,Y,Z,U) - \bar{R}(Z,U,X,Y) \]
\[ = (1 - \varepsilon)[g(Y,Z)\eta(X)\eta(U) - g(X,U)\eta(Y)\eta(Z)]. \]
Thus in view of (3.16), (3.18) and (3.20), we can state the following theorem.

**Theorem 3.1.** In an \( \varepsilon \)-LP-Sasakian manifold with a quarter-symmetric metric connection, we have:

(i) \( \bar{R}(X,Y,Z,U) + \bar{R}(Y,X,Z,U) = 0, \)

(ii) \( \bar{R}(X,Y,Z,U) + \bar{R}(Y,X,U,Z) = (1 - \varepsilon)[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(Z) \]
\[ + (1 - \varepsilon)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(U), \]

(iii) \( \bar{R}(X,Y,Z,U) - \bar{R}(Z,U,X,Y) = (1 - \varepsilon)[g(Y,Z)\eta(X)\eta(U) - g(X,U)\eta(Y)\eta(Z)] \]
for any vector fields \( X,Y,Z,U \in \chi(M). \)

Now, let \( \bar{R}(X,Y)Z = 0 \), therefore from (3.3), we have
\[ R(X,Y)Z = \varepsilon[\eta(X)Y - \eta(Y)X]\eta(Z) - \varepsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \]
\[ + \varepsilon[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y]. \]
Taking inner product of the above equation (3.21) with \( \xi \), we have
\[ (3.22) \]
\[ g(R(X,Y)Z,\xi) = -[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \]
which can be written as
\[ (3.23) \]
\[ g(R(X,Y)Z,U) = -\varepsilon[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \]
Thus we can state the following theorem.
Theorem 3.2. If the curvature tensor of a quarter-symmetric metric connection in an $\varepsilon$-LP-Sasakian manifold $M$ vanishes, then the manifold is of constant curvature $(-\varepsilon)$ and consequently it is locally isometric to the Hyperbolic space $H^n(-\varepsilon)$.

4. Conformally flat $\varepsilon$-LP-Sasakian manifold with a quarter-symmetric metric connection

Definition 4.1. The conformal curvature tensor $\tilde{C}$ of type $(1,3)$ of an $n$-dimensional $\varepsilon$-LP-Sasakian manifold with a quarter-symmetric metric connection $\tilde{\nabla}$, is given by

\[
\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-2)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X
\]

\[
- g(X,Z)\tilde{Q}Y] + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],
\]

where $\tilde{Q}$ is the Ricci operator with respect to a quarter-symmetric metric connection related by $g(\tilde{Q}X,Y) = \tilde{S}(X,Y)$ and $\tilde{r}$ is the scalar curvature with respect to a quarter-symmetric metric connection.

Let us assume that the manifold $M$ with respect to a quarter-symmetric metric connection is conformally flat, that is $\tilde{C} = 0$. Then from (4.1), it follows that

\[
\tilde{R}(X,Y)Z = \frac{1}{(n-2)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y]
\]

\[
- g(X,Z)\tilde{Q}Y] + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].
\]

Taking inner product of (4.2) with $\xi$ and using (2.2) and (3.10), we have

\[
g(\tilde{R}(X,Y)Z,\xi) = \frac{\varepsilon}{(n-2)}[\tilde{S}(Y,Z)\eta(X) - \tilde{S}(X,Z)\eta(Y)
\]

\[
- (1 - \varepsilon)(n-1)g(Y,Z)\eta(X) + (1 - \varepsilon)(n-1)g(X,Z)\eta(Y)]
\]

\[
- \frac{\varepsilon\tilde{r}}{(n-1)(n-2)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].
\]

Putting $X = \xi$ and using (2.1), (2.2) and (3.8) in (4.3), we get

\[
\tilde{S}(Y,Z) = [(1 - \varepsilon)(n-1) + \frac{\tilde{r}}{n-1}]g(Y,Z)
\]

\[
+ [-2(1 - \varepsilon)(n-1) + \frac{\varepsilon\tilde{r}}{n-1}]\eta(Y)\eta(Z).
\]

Therefore, (4.4) is of the form

\[
\tilde{S}(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),
\]

where $a = (1 - \varepsilon)(n-1) + \frac{\tilde{r}}{n-1}$ and $b = -2(1 - \varepsilon)(n-1) + \frac{\varepsilon\tilde{r}}{n-1}$. 


Next, using (4.4) in (4.2), we have

\[
\begin{align*}
(4.5) \quad g(\overline{R}(X,Y)Z&W) & = \frac{1}{(n-2)} \left[ (2(1-\varepsilon)(n-1) + \frac{2\bar{r}}{n-1} \right) \\
& \cdot (g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \\
& + \left( \frac{\varepsilon\bar{r}}{n-1} - 2(1-\varepsilon)(n-1) \right) \left( \eta(Y)\eta(Z)g(X,W) \\
& - \eta(Y)\eta(W)g(X,Z) + \eta(X)\eta(W)g(Y,Z) - \eta(X)\eta(Z)g(Y,W) \right) \\
& - \frac{\bar{r}}{(n-1)(n-2)} \left[ g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \right].
\end{align*}
\]

which by simplifying takes the form

\[
(4.6) \quad g(\overline{R}(X,Y)Z&W) = \frac{2(1-\varepsilon)(n-1)^2 + \bar{r}}{(n-1)(n-2)} \\
\cdot (g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \\
+ \frac{\varepsilon\bar{r} - 2(1-\varepsilon)(n-1)^2}{(n-1)(n-2)} \left( \eta(Y)\eta(Z)g(X,W) - \eta(Y)\eta(W)g(X,Z) \\
+ \eta(X)\eta(W)g(Y,Z) - \eta(X)\eta(Z)g(Y,W) \right). 
\]

Thus by virtue of (4.4) and (4.6), we can state the following theorem.

**Theorem 4.2.** An \(n\)-dimensional conformally flat \(\varepsilon\)-LP-Sasakian manifold with a quarter-symmetric metric connection is an \(\eta\)-Einstein manifold of quasi constant curvature.

5. **Quasi conformally flat and \(\xi\)- conformally flat \(\varepsilon\)-LP-Sasakian manifold with a quarter-symmetric metric connection**

**Definition 5.1.** An \(\varepsilon\)-LP-Sasakian manifold is said to be:

(i) quasi conformally flat with a quarter-symmetric metric connection if

\[
(5.1) \quad g(\overline{C}(X,Y)Z&W) = 0, \quad X,Y,Z,W \in \chi(M),
\]

(ii) \(\xi\)-conformally flat with a quarter-symmetric metric connection if

\[
(5.2) \quad \overline{C}(X,Y)\xi = 0, \quad X,Y \in \chi(M).
\]

First we consider quasi conformally flat \(\varepsilon\)-LP-Sasakian manifold with a quarter-symmetric metric connection. Therefore from (4.1) and (5.1), we have

\[
\begin{align*}
(5.3) \quad g(\overline{R}(X,Y)Z&W) & = \frac{1}{(n-2)} [\overline{S}(Y,Z)g(X,\phi W) - \overline{S}(X,Z)g(Y,\phi W) \\
& + g(Y,Z)g(\overline{Q}X,\phi W) - g(X,Z)g(\overline{Q}Y,\phi W)] \\
& + \frac{\bar{r}}{(n-1)(n-2)} [g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)] = 0,
\end{align*}
\]

\[
\begin{align*}
& + \left( \frac{\varepsilon\bar{r}}{n-1} - 2(1-\varepsilon)(n-1) \right) \left( \eta(Y)\eta(Z)g(X,W) \\
& - \eta(Y)\eta(W)g(X,Z) + \eta(X)\eta(W)g(Y,Z) - \eta(X)\eta(Z)g(Y,W) \right) \\
& - \frac{\bar{r}}{(n-1)(n-2)} \left[ g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \right].
\end{align*}
\]
which by considering \( Y = Z = \xi \) and using (2.2), (3.9)–(3.11) reduce to

\[
\bar{S}(X, \phi W) = \left[ (1 - \epsilon) + \frac{\bar{r}}{n - 1} \right] g(X, \phi W).
\]

Now replacing \( W = \phi W \) and using (2.1), (2.2) and (3.10) in (5.4), we get

\[
\bar{S}(X, W) = \left[ (1 - \epsilon) + \frac{\bar{r}}{n - 1} \right] g(X, W) + \left[ -n(1 - \epsilon) + \frac{\epsilon \bar{r}}{n - 1} \right] \eta(X)\eta(W).
\]

Thus we can state the following theorem.

**Theorem 5.2.** An \( n \)-dimensional quasi conformally flat \( \epsilon \)-LP-Sasakian manifold with a quarter-symmetric metric connection \( \bar{\nabla} \) is an \( \eta \)-Einstein manifold.

Next, by virtue of (4.1) and (5.2), we can write

\[
g[\bar{R}(X, Y)\xi - \frac{1}{n - 2} (\bar{S}(Y, \xi)X - \bar{S}(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY) + \frac{\bar{r}}{(n - 1)(n - 2)} (g(Y, \xi)X - g(X, \xi)Y), W] = 0.
\]

Now using (2.2), (3.7) and (3.10) in (5.6), we have

\[
\left[ \frac{\bar{r} + (1 - \epsilon)(n - 1)^2 - (1 - \epsilon)(n - 1)(n - 2)}{n - 1} \right] (\eta(Y)g(X, W) - \eta(X)g(Y, W)) + \eta(Y)\bar{S}(X, W) - \eta(X)\bar{S}(Y, W) = 0,
\]

which by taking \( Y = \xi \) and using (2.1), takes the form

\[
\bar{S}(X, W) = -\left[ \frac{\bar{r} + (1 - \epsilon)(n - 1)^2 - (1 - \epsilon)(n - 1)(n - 2)}{n - 1} \right] g(X, W)
- \left[ \frac{\epsilon \bar{r} - 2(1 - \epsilon)(n - 1)^2 + (1 - \epsilon)(n - 1)(n - 2)}{n - 1} \right] \eta(X)\eta(W).
\]

Thus we can state the following theorem.

**Theorem 5.3.** An \( n \)-dimensional \( \xi \)-conformally flat \( \epsilon \)-LP-Sasakian manifold with a quarter-symmetric metric connection \( \bar{\nabla} \) is an \( \eta \)-Einstein manifold.

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