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NEW RESULTS ON THE SEQUENCE SPACES EQUATIONS USING THE OPERATOR OF THE FIRST DIFFERENCE

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Abstract. Given any sequence $z = (z_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_z for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/z = (y_n/z_n)_{n\geq 1} \in E$; in particular, c_z denotes the set of all sequences y such that y/z converges. By w_∞, we denote the set of all sequences y such that $\sup_{n\geq 1}(n^{-1}\sum_{k=1}^{n}|y_k|) < \infty$. By Δ we denote the operator of the first difference defined by $\Delta_n y = y_n - y_{n-1}$ for all sequences y and all $n \geq 1$, with the convention $y_0 = 0$. In this paper, we state some results on the (SSE) $(E_a)_\Delta + F_x = F_b$, where $c_0 \subset E \subset \ell_\infty$ and $F \subset \ell_\infty$. Then for r, $u > 0$, we deal with the solvability of the (SSE) $(E_r)_\Delta + F_x = F_u$, where E, $F \in \{c_0, c, \ell_\infty\}$ and on the (SSE), $(W_r)_\Delta + c_x = c_u$. For instance, the solvability of the (SSE) $(W_r)_\Delta + c_x = c_u$ consists in determining the set of all positive sequences x, for which the next statement holds. The condition $y_n/u^n \to l_1$ holds if and only if there are two sequences α and β with $y = \alpha + \beta$, for which $\sup_{n\geq 1}(n^{-1}\sum_{k=1}^{n}|\Delta_k\alpha|r^{-k}) < \infty$ and $\beta_n/x_n \to l_2$ $(n \to \infty)$ for all sequences y and for some scalars l_1 and l_2.

1. Introduction

For any given set of sequences E and any positive sequence a, we write $E_a = (1/a)^{-1} * E$ for the set of all sequences y for which $y/a = (y_n/a_n)_{n\geq 1} \in E$. In [**3**], s_a, s_a^0 and $s_a^{(c)}$, we defined by the sets E_a for $E = \ell_\infty$, c_0, or c, respectively. Then in [**4**] we defined the sum $E_a + F_b$ and the product $E_a * F_b$, where E and F are any of the sets ℓ_∞, c_0, or c. Then in [**7**], we gave a solvability of sequences spaces inclusions $G_b \subset E_a + F_x$, where E, F, $G \in \{\ell_\infty, c_0, c\}$, and some applications to sequence spaces inclusions with operators. In the same way recall that the spaces w_∞ and w_0 of strongly bounded and summable sequences are the sets of all y such that $(n^{-1}\sum_{k=1}^{n}|y_k|)_{n\geq 1}$ is bounded and tend to zero, respectively. These spaces were studied by Maddox [**1**] and Malkowsky [**19**]. In [**16, 13**], some properties were given of well known operators defined on the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$. The sets of analytic and entire sequences denoted by $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ are defined by $\sup_{n\geq 1}(|y_n|^{1/n}) < \infty$ and $\lim_{n\to\infty}(|y_n|^{1/n}) = 0$, respectively.

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In this paper, we deal with special *sequence spaces inclusion equations (SSIE), (resp., sequence spaces equations (SSE)),* which are determined by an inclusion, (resp. identity), for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $(E_{f(x)})_T$, where f maps U^+ to itself, E is any linear space of sequences and T is a triangle. Some results on (SSE) were stated in [**7, 5, 6, 8, 10, 14, 15, 17, 18**].

In [**14**], we determined the set of all positive sequences x for which the *(SSIE)* $(s_x^{(c)})_{B(r,s)} \subset (s_x^{(c)})_{B(r',s')}$ holds, where r, r', s', and s are real numbers, and $B(r,s)$ is the generalized operator of the first difference defined by $(B(r,s)y)_n = ry_n + sy_{n-1}$ for all $n \geq 2$ and $(B(r,s)y)_1 = ry_1$. In this the set of all positive sequences x for which $(ry_n + sy_{n-1})/x_n \to l$ implies $(r'y_n + s'y_{n-1})/x_n \to l$ $(n \to \infty)$ for all y and some scalar l, way was determined.

In this paper we extend in a certain sense some results given in [**5, 6, 7, 8, 15, 17, 18**]. In [**17**], it was shown that for any given sequences a and b, the solutions of the equations $\chi_a + s_x^0 = s_b^0$ where χ is any of the symbols s, or $s^{(c)}$, are given by $s_x = s_b$ if $a/b \in c_0$, and if $a/b \notin c_0$, each of these equations has no solution. We also determined the set of all positive sequences x, for which $y_n/b_n \to l$ if and only if there are sequences u and v, for which $y = u + v$ and $u_n/a_n \to 0$, $v_n/x_n \to l'$ $(n \to \infty)$ for all y and some scalars l and l'. This statement is equivalent to the equation $s_a^0 + s_x^{(c)} = s_b^{(c)}$. In [**8**], we gave some properties of the sets of a-analytic and a-entire sequences denoted by $\mathbf{\Lambda}_a$ and $\mathbf{\Gamma}_a$ and defined by $\sup_{n\geq 1}\left\{(|y_n|/a_n)^{1/n}\right\} < \infty$ and $\lim_{n\to\infty}\left\{(|y_n|/a_n)^{1/n}\right\} = 0$, respectively. Then we determined the set of all $x \in U^+$ such that for every sequence y, we have $y_n/b_n \to l$ if and only if there are sequences u and v with $y = u + v$, $(|u_n|/a_n)^{1/n} \to 0$, and $v_n/x_n \to l'$ $(n \to \infty)$ for some scalars l and l'. This statement means $\mathbf{\Gamma}_a + s_x^{(c)} = s_b^{(c)}$. In [**6**], can be found a solvability of the (SSE) $\chi_a + (s_x^{(c)})_{B(r,s)} = s_x^{(c)}$ where $\chi = s$, s^0, or $s^{(c)}$ and x is the unknown. In [**5**], under some conditions, we determined the solutions of (SSE) with operators of the form $(\chi_a * \chi_x + \chi_b)_\Delta = \chi_\eta$ and $(\chi_a * (\chi_x)^2 + \chi_b * \chi_x)_\Delta = \chi_\eta$, and $\chi_a + (\chi_x)_\Delta = \chi_x$, where χ is any of the symbols s, or s^0. In [**17**], we determined the sets of all positive sequences x that satisfy the systems $s_a^0 + (s_x)_\Delta = s_b$, $s_x \supset s_b$ and $s_a + (s_x^{(c)})_\Delta = s_b^{(c)}$, $s_x^{(c)} \supset s_b^{(c)}$. There is a study of the (SSE) with operators defined by $(\chi_a)_{C(\lambda)D_\tau} + (s_x^{(c)})_{C(\mu)D_\tau} = s_b^{(c)}$, where χ is either s^0 or s. In [**15**], we dealt with the (SSE) $E_a + s_x = s_b$, where $E \in \{w_\infty, w_0, \ell_p\}$ and ℓ_p is the set of all sequences of p-absolute type. Then there is a solvability of the (SSE) $E_a + s_x^{(c)} = s_b^{(c)}$, where $E \in \{w_0, \ell_p\}$ and a solvability of the equation $E_a + s_x = s_b$, where $E \in \{c, \ell_\infty\}$. In [**9**] a study dicussed the (SSE) with operators $(E_a)_{C(\lambda)C(\mu)} + (E_x)_{C(\lambda\sigma)C(\mu)} = E_b$, where $b \in \widehat{C}_1$ and E is any of the sets ℓ_∞, or c_0. Recently in [**10**], we dealt with the solvability of (SSE) of the form $E_T + F_x = F_b$, where T is either one of the triangles Δ or Σ, where Δ is the operator of the first difference and Σ is the operator defined by $\Sigma_n y = \sum_{k=1}^n y_k$ for all sequences y. More precisely, we gave a solvability of the (SSE) $E_\Delta + F_x = F_b$, where E is any of the sets c_0, ℓ_p, $(p > 1)$, w_0, or $\mathbf{\Lambda}$ and $F = c$, or ℓ_∞. Then there is a solvability of the (SSE) $E_\Sigma + F_x = F_b$ where E is any of the sets c_0, c, ℓ_∞, ℓ_p, $(p > 1)$, w_0, $\mathbf{\Gamma}$, $\mathbf{\Lambda}$, and

$F = c$, or ℓ_∞. Finally, there is a solvability of the (SSE) with operator defined by $E_\Sigma + F_x = F_b$, where $E = \Gamma$, or $\mathbf{\Lambda}$, and $F = c$, or ℓ_∞, and a solvability of the (SSE) $\Gamma_\Sigma + \mathbf{\Lambda}_x = \mathbf{\Lambda}_b$. In [**11**], for any given positive sequence a, we solved the (SSE) defined by $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$, where $E = c_0$, or ℓ_p, $(p > 1)$, and the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ for $E = c$, or s_1, and we gave applications to particular classes of (SSE). In this paper, we extend some of the previous results and obtain a resolution of the (SSE) $(E_r)_\Delta + F_x = F_u$ for r, $u > 0$, where E, F are any of the spaces c_0, c, or ℓ_∞, and of the (SSE) $(W_r)_\Delta + c_x = c_u$.

This paper is organized as follows. In Section 2, we recall some definitions and results on sequence spaces and matrix transformations. In Section 3, are given some results on the multiplier $M(E,F)$ of classical spaces. Then in Section 4 we recall some results on the solvability of some sequence spaces equations of the form $E_a + F_x = F_b$, where E and F are any of the sets c_0, c, or ℓ_∞. In Section 5, we recall some results on the sets $\widehat{\Gamma}$, $\widehat{C}$, Γ, $\widehat{C_1}$, and G_1. In Section 6, we state some results on the (SSE) $(E_a)_\Delta + F_x = F_b$, where $c_0 \subset E \subset \ell_\infty$ and $F \subset \ell_\infty$. In Section 7, we determine the solutions of the (SSE) defined by $(E_r)_\Delta + F_x = F_u$ for r, $u > 0$, where E, F are any of the spaces c_0, c, or ℓ_∞. Finally, in Section 8, we solve the (SSE) $(W_r)_\Delta + c_x = c_u$.

2. Premiliminary results

An FK space is a *complete metric space* for which convergence implies *coordinatewise convergence.* A *BK space* is a Banach space of sequences that is an *FK space.* A BK *space E is said to have AK if for every sequence $y = (y_n)_{n\geq 1} \in E$, $y = \lim_{n\to\infty} \sum_{k=1}^n y_k e^{(k)}$, where $e^{(k)} = (0, \dots, 0, 1, 0, \dots)$, 1 being in the* k-th *position.*

For a given infinite matrix $\Lambda = (\lambda_{nk})_{n,k\geq 1}$, we define the operators Λ_n for any integer $n \geq 1$, by $\Lambda_n y = \sum_{k=1}^\infty \lambda_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator Λ defined by $\Lambda y = (\Lambda_n y)_{n\geq 1}$ mapping between sequence spaces. When Λ *maps E into F, where E and F are any sets of sequences, we write that* $\Lambda \in (E,F)$, (cf. [**1**]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators* L mapping in E, with norm $\|L\| = \sup_{y\neq 0}(\|L(y)\|_E / \|y\|_E)$, satisfies the identity $\mathcal{B}(E) = (E,E)$. For the sets of all sequences, by ω, c_0, c and ℓ_∞, we denote the sets of null, convergent and bounded sequences. Let $U^+ \subset \omega$ be the set of all sequences $\mathbf{u} = (u_n)_{n\geq 1}$ with $u_n > 0$ for all n. Then for any given sequence $\mathbf{u} = (u_n)_{n\geq 1} \in \omega$, we define the infinite diagonal matrix $D_\mathbf{u}$ with $[D_\mathbf{u}]_{nn} = u_n$ for all n. For $\mathbf{u} = (r^n)_{n\geq 1}$, we write D_r for $D_\mathbf{u}$. Let E be any subset of ω and $\mathbf{u}$ be any sequence with $u_n \neq 0$ for all n, using Wilansky's notations [**22**], we have $(1/\mathbf{u})^{-1} * E = D_\mathbf{u} E = \{y = (y_n)_{n\geq 1} \in \omega : y/\mathbf{u} \in E\}$. By $E_\mathbf{u}$, we can also denote the set $D_\mathbf{u} E$. We use the sets s_a^0, $s_a^{(c)}$, s_a, and ℓ_a^p defined as follows, (cf. [**3**]). For given $a \in U^+$ and $p \geq 1$, we put $D_a c_0 = s_a^0$, $D_a c = s_a^{(c)}$, also denoted by c_a, and $D_a \ell_\infty = s_a$. Each of the spaces $D_a E$, where $E \in \{c_0, c, \ell_\infty\}$ is a *BK space normed* by $\|y\|_{s_a} = \sup_{n\geq 1}(\|y_n\|/a_n)$, and s_a^0 *has AK.* We use the set $W_a = (w_\infty)_a$, where w_∞ is the set of all sequences

y such that $\sup_{n\geq 1}(n^{-1}\sum_{k=1}^{n}|y_k|)<\infty$. If $a=(r^n)_{n\geq 1}$ with $r>0$, we write s_r, s_r^0, $s_r^{(c)}$, and W_r for the sets s_a, s_a^0, $s_a^{(c)}$ and W_a, respectively. When $r=1$, we obtain $s_1=\ell_\infty$, $s_1^0=c_0$, $s_1^{(c)}=c$ and $W_1=w_\infty$. Recall that $S_1=(s_1,s_1)$ is a Banach algebra (cf. [**2**]) and $(c_0,s_1)=(c,s_1)=(s_1,s_1)=S_1$. We have $A\in S_1$ if and only if $\sup_{n\geq 1}(\sum_{k=1}^{\infty}|\lambda_{nk}|)<\infty$. Recall the next Schur's result on the class (s_1,c). We have $\Lambda\in(s_1,c)$ if and only if $\lim_{n\to\infty}\lambda_{nk}=l_k$ for some scalar l_k, $k=1,2,\ldots$, and $\lim_{n\to\infty}\sum_{k=1}^{\infty}|\lambda_{nk}|=\sum_{k=1}^{\infty}|l_k|$, the series being convergent. For any subset F of ω, we write $F(\Lambda)=F_\Lambda=\{y\in\omega:\ \Lambda y\in F\}$ for the matrix domain of Λ in F. The infinite matrix $T=(t_{nk})_{n,k\geq 1}$ is said to be a triangle if $t_{nk}=0$ for $k>n$ and $t_{nn}\neq 0$ for all n. Throughout this paper, we use the next well known statement. If T, T', and T'' are triangles, E and F are any sets of sequences, then we have

$$T\in(E_{T'},F_{T''})\quad\Longleftrightarrow\quad T''TT'^{-1}\in(E,F),$$

(cf. [**5**, Lemma 9, p. 45]). Finally, for any given set E of sequences, we write $\Lambda E=\{y\in\omega: y=\Lambda x \text{ for some } x\in E\}$.

3. The multipliers of some sets of sequences

First we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω, we then write $yz=(y_nz_n)_{n\geq 1}$. Then by

$$M(E,F)=\{y\in\omega: yz\in F \text{ for all } z\in E\},$$

we denote the *multiplier space of E and F*. In this way we recall the well known result.

Lemma 1. *Let E, $\widetilde{E}$, F and $\widetilde{F}$ be arbitrary subsets of ω. Then*
(i) *$M(E,F)\subset M(\widetilde{E},F)$ for all $\widetilde{E}\subset E$.*
(ii) *$M(E,F)\subset M(E,\widetilde{F})$ for all $F\subset\widetilde{F}$.*

Lemma 2. *Let $a\in\omega$ and b be a nonzero sequence and E, $F\subset\omega$. Then $\Lambda\in(D_aE,D_bF)$ if and only if $D_{1/b}\Lambda D_a=(\lambda_{nk}a_k/b_n)_{n,k\geq 1}\in(E,F)$.*

We deduce the next lemma.

Lemma 3. *Let a, $b\in U^+$ and let E and F be two subsets of ω. Then $D_aE\subset D_bF$ if and only if $a/b\in M(E,F)$.*

Proof. We have $D_aE\subset D_bF$ if and only if $I\in(D_aE,D_bF)$, which is equivalent to $D_{a/b}\in(E,F)$ and to $a/b\in M(E,F)$. □

In a similar way we obtain the following lemma.

Lemma 4. *Let a, $b\in U^+$ and let E, F, and G be subsets of ω that satisfy the condition $M(E,F)=G$. Then the next statements are equivalent:*
i) *$a\in D_bG$,*
ii) *$a/b\in M(E,F)$,*
iii) *$D_aE\subset D_bF$.*

By [**20**, Lemma 3.1, p. 648] and [**21**, Example 1.28, p. 157], we obtain the next result.

Lemma 5. *We have:*
i) $M(c, c_0) = M(\ell_\infty, c) = M(\ell_\infty, c_0) = c_0$ *and* $M(c, c) = c$.
ii) $M(E, \ell_\infty) = M(c_0, F) = \ell_\infty$ *for* $E,\ F = c_0,\ c,$ *or* ℓ_∞.

4. On the solvability of five (SSE) of the form $E_a + F_x = F_b$ where E, F are any of the sets c_0, c, or ℓ_∞

The solvability of the (SSE) $E_a + F_x = F_b$ consists in determining the set of all positive sequences x that satisfy the statement $y/b \in F$ if and only if there are two sequences α, β such that $y = \alpha + \beta$ and

$$\frac{\alpha}{a} \in E \quad \text{and} \quad \frac{\beta}{x} \in F.$$

For instance, the solvability of the equation $s_a + s_x^{(c)} = s_b^{(c)}$ for $a,\ b \in U^+$, consists in determining the set of all $x \in U^+$ that satisfy the next statement. For every sequence y, the condition $y_n/b_n \to l\ (n \to \infty)$ if and only if there are two sequences α, β such that $y = \alpha + \beta$ for which $\sup_{n \geq 1}(|\alpha_n|/a_n) < \infty$ and $\beta_n/x_n \to l'\ (n \to \infty)$ for some scalars l, l'. For any given linear spaces of sequences E and F, we put $\mathcal{I}(E, F) = \{x \in U^+ :\ F_b \subset E_a + F_x\}$ and

$$\mathcal{S}(E, F) = \left\{x \in U^+ :\ E_a + F_x = F_b\right\}.$$

For $b \in U^+$ and any subset F of ω, by $\mathrm{cl}^F(b)$, we denote the equivalent class for the equivalence relation R_F defined by xR_Fy if $D_xF = D_yF$ for $x,\ y \in U^+$. It can be easily seen that $\mathrm{cl}^F(b)$ is the set of all $x \in U^+$ such that $x/b \in M(F, F)$ and $b/x \in M(F, F)$, (cf. [**17**]). We then have $\mathrm{cl}^F(b) = \mathrm{cl}^{M(F,F)}(b)$. For instance $\mathrm{cl}^c(b)$, is the set of all $x \in U^+$ such that $D_xc = D_bc$, that is, $s_x^{(c)} = s_b^{(c)}$. This is the set of all sequences $x \in U^+$ such that $x_n \sim Cb_n\ (n \to \infty)$ for some $C > 0$. In the following, we write $\mathrm{cl}^\infty(b)$ for $\mathrm{cl}^{\ell_\infty}(b)$. For $b = (r^n)_{n \geq 1}$, we write $\mathrm{cl}^F(r)$ instead of $\mathrm{cl}^F(b)$ to simplify.

Now recall the next elementary result on the sum of linear spaces of sequences. Let E, F, and G be linear subspaces of ω, then we have $E + F \subset G$ if and only if $E \subset G$ and $F \subset G$. For instance, we have $c_0 + s_x \subset s_1$ if and only if $x \in s_1 \cap U^+$ and there is no positive sequence x for which $s_1 + s_x^0 \subset c_0$. Then we let $s_b^\bullet = \{x \in U^+ : x_n \geq Kb_n$ for some $K > 0$ and for all $n\}$ and $s_b^{\bullet(c)} = \{x \in U^+ : \lim_{n\to\infty}(x_n/b_n) = l$ for some $l \in]0, +\infty]\}$, ([**17**]). To simplify, we write $s^\bullet_{(r^n)_{n\geq1}} = s_r^\bullet$ and $s^{\bullet(c)}_{(r^n)_{n\geq1}} = s_r^{\bullet(c)}$, (or $c_r^\bullet$) for $r > 0$. Notice that $\mathrm{cl}^{(c)}(b) = s_b^{(c)} \setminus s_b^0$. It can be easily seen that $s_b^{\bullet(c)} = c_b^\bullet = \left\{x \in U^+ : s_b^{(c)} \subset s_x^{(c)}\right\}$, (cf. [**17**]). We need to recall the next results.

Lemma 6. ([**17**, Theorem 4.4, p. 7]) *Let a, $b \in U^{+}$.*

i) *We have $\mathcal{S}(c, c_0) = \mathcal{S}(s_1, c_0)$ and*

$$\mathcal{S}(c_0, s_1) = \begin{cases} \mathrm{cl}^{\infty}(b) & \text{if } a/b \in \ell_\infty, \\ \emptyset & \text{if } a/b \notin \ell_\infty. \end{cases} \qquad \mathcal{S}(s_1, c_0) = \begin{cases} \mathrm{cl}^{\infty}(b) & \text{if } a/b \in c_0, \\ \emptyset & \text{if } a/b \notin c_0. \end{cases}$$

ii)

$$\mathcal{S}(c_0, c) = \begin{cases} \mathrm{cl}^{(c)}(b) & \text{if } a/b \in \ell_\infty, \\ \emptyset & \text{if } a/b \notin \ell_\infty. \end{cases} \qquad \mathcal{S}(s_1, c) = \begin{cases} \mathrm{cl}^{(c)}(b) & \text{if } a/b \in c_0, \\ \emptyset & \text{if } a/b \notin c_0. \end{cases}$$

5. The sets $\widehat{\Gamma}$, $\widehat{C}$, Γ, $\widehat{C_1}$, and G_1.

To solve the next equations, we recall some definitions and results. Now let U be the set of all sequences $(u_n)_{n\geq 1} \in \omega$ with $u_n \neq 0$ for all n. The infinite matrix $C(a)$ with $a = (a_n)_n \in U$ is the triangle defined by $[C(a)]_{nk} = 1/a_n$ for $k \leq n$. It can be shown that the triangle $\Delta(a)$ whose the nonzero entries are defined by $[\Delta(a)]_{nn} = a_n$, and $[\Delta(a)]_{n,n-1} = -a_{n-1}$ for all $n \geq 2$, is the inverse of $C(a)$, that is, $C(a)(\Delta(a)y) = \Delta(a)(C(a)y)$ for all $y \in \omega$. If $a = e$, we obtain the well known operator of the first difference represented by $\Delta(e) = \Delta$. We then have $\Delta_n y = y_n - y_{n-1}$ for all sequences y, for all $n \geq 1$, with the convention $y_0 = 0$. It is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and Δ, $\Sigma \in S_R$ for any $R > 1$. By $\widehat{C_1}$ and $\widehat{C}$, we define the sets of all positive sequences a that satisfy the conditions $C(a)a \in \ell_\infty$, and $C(a)a \in c$, respectively. Then by $\widehat{\Gamma}$ and Γ, we define the sets of all positive sequences a that satisfy the conditions $\lim_{n\to\infty}(a_{n-1}/a_n) < 1$, and $\overline{\lim}_{n\to\infty}(a_{n-1}/a_n) < 1$, respectively, (cf. [**3**]). The set G_1 is defined by $G_1 = \{x \in U^{+} : x_n \geq K\gamma^n \text{ for all } n \text{ and for some } K > 0 \text{ and } \gamma > 1\}$. We obtain the next lemmas.

Lemma 7. *We have $\widehat{\Gamma} = \widehat{C} \subset \Gamma \subset \widehat{C}_1 \subset G_1$.*

Proof. The identity $\widehat{\Gamma} = \widehat{C}$ follows from [**12**, Proposition 2.2 p. 88] and the inclusions $\Gamma \subset \widehat{C}_1 \subset G_1$ follow from [**3**, Proposition 2.1, p. 1786]. □

We will use the next lemma.

Lemma 8. *Let $a \in U^{+}$. Then we have:*

i) *The following statements are equivalent:*

α) $a \in \widehat{C}_1$, $\quad$ β) $(s_a)_\Delta = s_a$, $\quad$ γ) $(s_a^0)_\Delta = s_a^0$.

ii) $a \in \widehat{\Gamma}$ *if and only if* $(s_a^{(c)})_\Delta = s_a^{(c)}$.

iii) $a \in \Gamma$ *implies* $(W_a)_\Delta = W_a$.

Proof. The statement in i) follows from [**3**, Theorem 2.6, pp. 1789–1790]. ii) follows from [**3**, Theorem 2.6, pp. 1789–1790] and [**12**, Proposition 2.2, p. 88]. The statement in iii) was shown in [**16**, Proposition 3.1, pp. 122–123]. □

6. Some properties of the (SSE) with operator $(E_a)_\Delta + F_x = F_b$.

In this section, we give some properties of the (SSIE) $F_b \subset (E_a)_\Delta + F_x$, where E and F are two linear subspaces of ω that satisfy E, $F \subset s_1$ and $F \supset c_0$. Then we deal with the (SSE) $(E_a)_\Delta + F_x = F_b$, where $c_0 \subset E \subset s_1$ and $F \in \{c_0, c, s_1\}$.

6.1. On the solvability of the (SSIE) $F_b \subset (E_a)_\Delta + F_x$.

We need some lemmas, where by $\mathcal{I}((E_a)_\Delta, F)$, we define the set of all $x \in U^+$ such that $F_b \subset (E_a)_\Delta + F_x$. We write $\mathcal{I}_E^F = \mathcal{I}((E_a)_\Delta, F)$, and more precisely we let $\mathcal{I}_E^\infty = \mathcal{I}_E^{\ell_\infty}$ and $\mathcal{I}_E^0 = \mathcal{I}_E^{c_0}$ to simplify.

In the following, we use the sequence $\sigma = (\sigma_n)_{n \geq 1}$ defined for a, $b \in U^+$ by $\sigma_n = (\sum_{k=1}^n a_k)/b_n$. First we state the next lemma.

Lemma 9. ([**11**, Lemma 16, pp. 116–117]) *Let a, $b \in U^+$, and let E and F be two linear subspaces of ω that satisfy E, $F \subset s_1$ and $F \supset c_0$. Then we have:*

i) *Assume $\sigma \in c_0$.*
Then a) $\mathcal{I}_E^F \subset \mathcal{I}_{s_1}^{s_1}$, b) $\mathcal{I}_E^F \subset s_b^\bullet$.

ii) *Assume $a \in c_0$. Then we have $\mathcal{I}_E^F \subset s_1^\bullet$ for $b = e$.*

As a direct consequence of Lemma 9, we obtain the following.

Lemma 10. *Let E be a linear subspaces of ω, that satisfies $E \subset s_1$.*

i) *If $\sigma \in c_0$, then*
  a) $\mathcal{I}_E^\chi \subset \mathcal{I}_{s_1}^\infty$, where χ is any of the symbols 0, c, or ∞.
  b) $\mathcal{I}_E^\chi \subset s_b^\bullet$, where χ is any of the symbols 0, c, or ∞.

ii) *If $a \in c_0$ and $b = e$, then $\mathcal{I}_E^\chi \subset s_1^\bullet$, where χ is any of the symbols 0, c, or ∞.*

Now, we state an elementary result used further.

Lemma 11. *Let E be a linear subspaces of ω, that satisfies $c_0 \subset E \subset s_1$. Then $(E, s_1) = S_1$.*

Proof. The proof follows from the fact that $(s_1, s_1) = (c_0, s_1) = S_1$ and $(s_1, s_1) \subset (E, s_1) \subset (c_0, s_1)$. □

6.2. On the sets S_E^0, S_E^c and S_E^∞

By $\mathcal{S}((E_a)_\Delta, F)$, we denote the set of all $x \in U^+$ such that $(E_a)_\Delta + F_x = F_b$. To simplify, we write $S_E^0 = \mathcal{S}((E_a)_\Delta, c_0)$, $S_E^c = \mathcal{S}((E_a)_\Delta, c)$ and $S_E^\infty = S((E_a)_\Delta, \ell_\infty) = \mathcal{S}((E_a)_\Delta, s_1)$. In all that follows, we must have in mind that the infinite matrix $D_{1/b}\Sigma D_a$ is the triangle whose the nonzero entries are defined by $\left[D_{1/b}\Sigma D_a\right]_{nk} = a_k/b_n$ for all $k \leq n$.

Theorem 12. *Let E be a linear subspace of ω that satisfies $c_0 \subset E \subset s_1$.*

i) *If $\sigma \in c_0$, then $S_E^0 = S_E^\infty = \mathrm{cl}^\infty(b)$ and $S_E^c = \mathrm{cl}^c(b)$.*

ii) *If $\sigma \notin \ell_\infty$, then $S_E^0 = S_E^\infty = S_E^c = \emptyset$.*

Proof. i) First, let $\sigma \in c_0$ and show $S_E^\infty = \mathrm{cl}^\infty(b)$. For this, let $x \in S_E^\infty$. Then, we have $(E_a)_\Delta + s_x = s_b$ which implies $x \in s_b$, and since $\sigma \in c_0$ by Lemma 9, we have $S_E^\infty \subset \mathcal{I}_E^\infty \subset s_b^\bullet$. We conclude $x \in s_b \cap s_b^\bullet = \mathrm{cl}^\infty(b)$ and $S_E^\infty \subset \mathrm{cl}^\infty(b)$. Conversely, let $x \in \mathrm{cl}^\infty(b)$. By Lemma 11, we have $\sigma \in c_0$ implies $D_{1/b}\Sigma D_a \in (s_1, s_1) = (E, s_1)$ and $(E_a)_\Delta \subset s_b$. Then we have $(E_a)_\Delta + s_x = (E_a)_\Delta + s_b = s_b$ and $x \in S_E^\infty$. So we have shown $\mathrm{cl}^\infty(b) \subset S_E^\infty$. We conclude $S_E^\infty = \mathrm{cl}^\infty(b)$.

Now we show $S_E^0 = \mathrm{cl}^\infty(b)$ for $\sigma \in c_0$. First, we show $S_E^0 \subset \mathrm{cl}^\infty(b)$. Let $x \in S_E^0$. Then we have $(E_a)_\Delta + s_x^0 = s_b^0$. This implies $s_x^0 \subset s_b^0$, implies $x/b \in M(c_0, c_0) = s_1$ and $x \in s_b$. As above, by Lemma 9, we have $S_E^0 \subset \mathcal{I}_E^0 \subset s_b^\bullet$ and conclude $x \in s_b \cap s_b^\bullet = \mathrm{cl}^\infty(b)$ and $S_E^0 \subset \mathrm{cl}^\infty(b)$. Conversely, let $x \in \mathrm{cl}^\infty(b)$. We have $\sigma \in c_0$ implies $D_{1/b}\Sigma D_a \in (s_1, c_0)$, and since $E \subset s_1$, we successively obtain $(s_1, c_0) \subset (E, c_0)$, $D_{1/b}\Sigma D_a \in (E, c_0)$, and $(E_a)_\Delta \subset s_b^0$. Then, we have $(E_a)_\Delta + s_x^0 = (E_a)_\Delta + s_b^0 = s_b^0$ and $x \in S_E^0$. So we have shown $S_E^0 = \mathrm{cl}^\infty(b)$. It remains to show $S_E^c = \mathrm{cl}^c(b)$. Let $x \in S_E^c$. Then we have $(E_a)_\Delta + c_x = c_b$ which implies $x \in c_b$. Again by Lemma 9, we have $x \in S_E^c$ implies $x \in \mathcal{I}_E^c \subset s_b^\bullet$, and $S_E^c \subset s_b^\bullet$. So we have shown $S_E^c \subset c_b \cap s_b^\bullet = \mathrm{cl}^c(b)$ if $\sigma \in c_0$. Conversely, let $x \in \mathrm{cl}^c(b)$. We have $\sigma \in c_0$ implies $D_{1/b}\Sigma D_a \in (s_1, c_0) \subset (E, c)$ since $E \subset s_1$. We conclude $(E_a)_\Delta \subset c_b$. Then $(E_a)_\Delta + c_x = (E_a)_\Delta + c_b = c_b$ and $x \in S_E^c$. So we have shown $\sigma \in c_0$ implies $S_E^c \subset c_b \cap s_b^\bullet = \mathrm{cl}^c(b)$ and $S_E^c = \mathrm{cl}^c(b)$. This concludes the proof of i).

ii) It is trivial that $\sigma \notin \ell_\infty$ implies $S_E^0 = S_E^\infty = S_E^c = \emptyset$. Indeed, assume there is $x \in S_E^\chi$, where χ is any of the symbols 0, ∞, or c. Then we have $(E_a)_\Delta + F_x = F_b$ which implies $(E_a)_\Delta \subset F_b$ and $D_{1/b}\Sigma D_a \in (E, F)$ for $F \in \{c_0, s_1, c\}$. But since we have $(E, F) \subset (s_1, s_1)$, we conclude $D_{1/b}\Sigma D_a \in S_1$ and $\sigma \in \ell_\infty$. This is contradictory, and $S_E^0 = S_E^\infty = S_E^c = \emptyset$. □

Example 13. Let $\mathtt{a}$, $\mathtt{b} > 0$. We consider the set S_c^c of all $x \in U^+$ such that $y_n/n^{\mathtt{b}} \to l_1$ $(n \to \infty)$ if and only if there are α, $\beta \in \omega$ such that $y = \alpha + \beta$, $n^{-\mathtt{a}}\Delta_n\alpha \to l_2$, and $\beta_n/x_n \to l_3$ $(n \to \infty)$ for some scalars l_1, l_2, l_3 and for all y. We are led to deal with the (SSE) $(c_{(n^{\mathtt{a}})_{n\geq 1}})_\Delta + c_x = c_{(n^{\mathtt{b}})_{n\geq 1}}$. We have $\sum_{k=1}^n k^{\mathtt{a}} \sim n^{\mathtt{a}+1}/(\mathtt{a}+1)$ $(n \to \infty)$, and $\sigma_n \sim n^{\mathtt{a}-\mathtt{b}+1}/(\mathtt{a}+1)$ $(n \to \infty)$. By Theorem *12*, we obtain $S_c^c = \mathrm{cl}^c((n^{\mathtt{b}})_{n\geq 1})$ for $\mathtt{b} > \mathtt{a}+1$; and $S_c^c = \emptyset$ for $\mathtt{b} < \mathtt{a}+1$. For instance, for $\mathtt{b} = 2$ and $\mathtt{a} = 1/2$, we obtain $S_c^c = \mathrm{cl}^c((n^2)_{n\geq 1})$ and $x \in S_c^c$ if and only if $x_n \sim Kn^2$ $(n \to \infty)$ for some $K > 0$.

Proposition 14. *Let r, $u > 0$ and let $E = s_1$, c_0, or c.*

i) $S_E^\infty = S_E^0 = \mathrm{cl}^\infty(u)$ *and* $S_E^c = \mathrm{cl}^c(u)$ *in each of the next cases:*
a) $r \leq 1 < u$ *and* *b)* $1 < r < u$.

ii) $S_E^\infty = S_E^0 = S_E^c = \emptyset$ *in each of the next cases:*
a) r, $u < 1$, *b)* $u \leq r = 1$, *c)* $r > u$ *if* $r > 1$.

Proof. i) We successively have $\sigma_n \sim r(1-r)^{-1}u^{-n}$ $(n \to \infty)$ for $r < 1$, $\sigma_n \sim nu^{-n}$ $(n \to \infty)$ for $r = 1$, and $\sigma_n \sim r(r-1)^{-1}(r/u)^n$ $(n \to \infty)$ for $r > 1$. So we have $\sigma_n \to 0$ $(n \to \infty)$ in each of the cases a) and b).

ii) It can be easily seen that $\sigma \notin \ell_\infty$ in each of the cases a), b), and c). □

Remark 15. Let $\mathbb{R}^{+2}$ be the set of all (r, u) with r, $u > 0$. Consider the subsets I_1, I_2, and I_3 of $\mathbb{R}^{+2}$, respectively, defined by i) a), and i) b) in Proposition 14 for

I_1; by ii) a), ii) b) and ii) c) in Proposition 14 for I_2; and by α) $r < u = 1$, and β) $r = u > 1$ for I_3. We easily see that $\{I_1, I_2, I_3\}$ constitute a partition of $\mathbb{R}^{+2}$.

7. Application to the solvability of (SSE) of the form $(E_r)_\Delta + F_x = F_u$ for r, $u > 0$

In this section, we deal with each of the (SSE) $(E_r)_\Delta + s_x = s_u$, $(E_r)_\Delta + s_x^0 = s_u^0$, and $(E_r)_\Delta + s_x^{(c)} = s_u^{(c)}$ for r, $u > 0$ with $E = c_0$, c, or ℓ_∞.

7.1. On the (SSE) $(E_r)_\Delta + s_x = s_u$ for r, $u > 0$ with $E = c_0$, c, or ℓ_∞.

Let r, $u > 0$ and $E = c_0$, c, or $\ell_\infty = s_1$, and consider the (SSE) defined by $(E_r)_\Delta + s_x = s_u$. For instance, for $E = s_1$, it can be easily seen that $x \in S_{s_1}^\infty$ means that the condition $\sup_{n\geq 1}(|y_n|/u^n) < \infty$ holds if and only if there are α and $\beta \in \omega$ such that $y = \alpha + \beta$ for which $\sup_{n\geq 1}(|\alpha_n - \alpha_{n-1}|r^{-n}) < \infty$ and $\sup_{n\geq 1}(|\beta_n|/x_n) < \infty$ for all y. In all that follows, we write $E \cap U^+ = E^+$ for any subset E of ω. We obtain the next theorem.

Theorem 16. *Let r, $u > 0$ and let $E = s_1$, or c_0.*

i) *If $r < 1$, then*

$$S_E^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } u \geq 1, \\ \emptyset & \text{if } u < 1. \end{cases}$$

ii) *If $r = 1$, then*

$$S_E^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1. \end{cases}$$

iii) *If $r > 1$, then*

a)

$$S_{s_1}^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } r < u, \\ s_u^+ & \text{if } r = u, \\ \emptyset & \text{if } r > u. \end{cases} \tag{1}$$

b)

$$S_{c_0}^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } r \leq u, \\ \emptyset & \text{if } r > u. \end{cases}$$

Proof. By Proposition 14 and Remark 15, it is enough to deal with the cases α) $r < u = 1$, and β) $r = u > 1$. Consider

Case α): We have $r < u = 1$, so $x \in S_E^\infty$ implies $(E_r)_\Delta + s_x = s_1$. Then we have $x \in s_1$ and

$$s_1 \subset (E_r)_\Delta + s_x. \tag{2}$$

Then by Lemma 9 ii), where $a = (r^n)_{n\geq 1} \in c_0$, the condition in (2) implies $x \in \mathcal{I}_E^\infty \subset s_1^\bullet$. We conclude $x \in s_1 \cap s_1^\bullet = \mathrm{cl}^\infty(1)$. Conversely, assume $x \in \mathrm{cl}^\infty(1)$, that is, $s_x = s_1$. It can be easily seen that $\Sigma D_r \in (s_1, s_1)$ since $\sup_{n\geq 1}(\sum_{k=1}^n r^k) < \infty$, and the inclusion $(s_1, s_1) \subset (E, s_1)$ implies $\Sigma D_r \in (E, s_1)$, which is equivalent to

$(E_r)_\Delta \subset s_1$. We conclude $(E_r)_\Delta + s_x = (E_r)_\Delta + s_1 = s_1$ and $x \in S_E^\infty$. So we have shown $S_E^\infty = \mathrm{cl}^\infty(1)$.

Case β): By Lemma 8, we have $(E_r)_\Delta = E_r$ since $r > 1$, and $(E_r)_\Delta + s_x = E_r + s_x = s_r$. Then the (SSE) $E_r + s_x = s_r$ is equivalent to $x \in s_r$ for $E = s_1$, and by Lemma 6, it is equivalent to $x \in \mathrm{cl}^\infty(r)$ for $E = c_0$. This concludes the proof. □

Remark 17. Notice that the statement ii) in Theorem 16 was extended in [**13**, Proposition 7.1, pp. 95–96] in the following way. For any given $b \in U^+$, the (SSE) defined by $(c_0)_\Delta + s_x = s_b$ has solutions if and only if $(n/b_n)_{n\geq 1} \in \ell_\infty$ which are defined by $x \in \mathrm{cl}^\infty(b)$.

We immediately obtain the following corollary.

Corollary 18. *Let r, $u > 0$ and let $E \in \{c_0, s_1\}$. Then $S_E^\infty \neq \emptyset$ if and only if $r \leq 1 < u$, or $1 < r \leq u$.*

Corollary 19. *If $u = 1$, we obtain*

$$S_{s_1}^\infty = \begin{cases} \mathrm{cl}^\infty(1) & \text{if } r < 1, \\ \emptyset & \text{if } r \geq 1. \end{cases}$$

We deduce from Corollary 19 that the equation $(s_r)_\Delta + s_x = \ell_\infty$ has solutions if and only if $r < 1$, that are determined by $K_1 \leq x_n \leq K_2$ for all n and for some K_1, $K_2 > 0$.

Concerning the space S_c^∞, we state the following proposition which can be obtained by similar arguments as those used in Theorem 16, except in the case $r = u > 1$ for which we have no response until now.

Proposition 20. *Let r, $u > 0$.*

i) *If $r \leq 1$, then $S_c^\infty = S_E^\infty$ for $E = c_0$, or ℓ_∞.*

ii) *If $r > 1$, then $S_c^\infty = \mathrm{cl}^\infty(u)$ for $r < u$, and $S_c^\infty = \emptyset$ if $r > u$.*

Example 21. The set of all $x \in U^+$ that satisfy $(c_{1/2})_\Delta + s_x = \ell_\infty$, is equal to $\mathrm{cl}^\infty(e)$. The solutions of the (SSE) $(c_{1/2})_\Delta + s_x = s_2$ are determined by $K_1 2^n \leq x_n \leq K_2 2^n$ for all n and for some K_1, $K_2 > 0$. The (SSE) defined by $(c_{1/2})_\Delta + s_x = s_{1/4}$ has no solution.

7.2. The solvability of the (SSE) $(E_r)_\Delta + s_x^0 = s_u^0$

Here, we consider the (SSE) $(E_r)_\Delta + s_x^0 = s_u^0$ for r, $u > 0$ and $E = s_1$, c_0, or c. For instance, for $E = s_1$, it can be easily seen that x is a solution of the (SSE) $(s_r)_\Delta + s_x^0 = s_u^0$, that is, $x \in S_{s_1}^0$ if the next statement holds. For every $y \in \omega$, the condition $y_n/u^n \to 0$ $(n \to \infty)$ holds if and only if there are α and $\beta \in \omega$ such that $y = \alpha + \beta$ for which $|\Delta_n \alpha|/r^n \leq K$ for all n, and $\beta_n/x_n \to 0$ $(n \to \infty)$ for some scalar $K > 0$. The next result was extended in [**11**, Theorem 2, pp. 127–128] with the solvability of the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ for a, $b \in U^+$ and for $E \in \{c, s_1\}$. In this part, we give another proof based on Proposition 14, and Lemma 6 and we deal with the case $E = c_0$.

We can state the next theorem.

Theorem 22. *Let r, $u > 0$ and let $E \in \{c_0, c, s_1\}$.*
i) *If $r < 1$, then*

$$S_E^0 = \begin{cases} cl^{\infty}(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1. \end{cases}$$

ii) *If $r = 1$, then*

$$S_E^0 = S_E^{\infty} = \begin{cases} \mathrm{cl}^{\infty}(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1. \end{cases}$$

iii) *If $r > 1$, then*
a) $S_{c_0}^0 = S_{s_1}^{\infty}$, where $S_{s_1}^{\infty}$ is determined by (1) *in Theorem 16.*
b) $S_c^0 = S_{s_1}^0$ and

$$S_{s_1}^0 = \begin{cases} \mathrm{cl}^{\infty}(u) & \text{if } r < u, \\ \emptyset & \text{if } r \geq u. \end{cases}$$

Proof. As above by Proposition 14, we only need to deal with the cases α) $r < u = 1$, and β) $r = u > 1$.

Case α). Let $E \in \{c_0, c, s_1\}$. We have $r < u = 1$ so $x \in S_E^0$ implies $(E_r)_\Delta + s_x^0 = c_0$. Then we have

$$(E_r)_\Delta \subset c_0, \tag{3}$$

and since $[\Sigma D_r]_{nk} = r^k \nrightarrow 0$ $(n \to \infty)$ for all k, we deduce $\Sigma D_r \notin (c_0, c_0)$ and $\Sigma D_r \notin (E, c_0)$. So the condition in (3) cannot hold and $S_E^0 = \emptyset$.

Case β). We have $(E_r)_\Delta = E_r$ since $r > 1$. Then we have $(E_r)_\Delta + s_x^0 = E_r + s_x^0 = s_r^0$. Then the (SSE) $E_r + s_x^0 = s_r^0$ is equivalent to $x \in s_r$ for $E = c_0$, and by Lemma 6 this (SSE) has no solution for $E = c$, or s_1. This concludes the proof. □

Remark 23. Theorem 22 was extended in [**11**, Theorem 2, p. 127] in the following way. For any given a, $b \in U^+$, the (SSE) defined by $(E_a)_\Delta + s_x^0 = s_b^0$ has solutions if and only if $\sigma_n = b_n^{-1} \sum_{k=1}^n a_k \to 0$ $(n \to \infty)$ and they are defined by $x \in \mathrm{cl}^{\infty}(b)$. We obtain the results stated above noticing that there are k_1, $k_2 > 0$ such that $\sigma_n \sim k_1 u^{-n}$ $(n \to \infty)$ for $r < 1$, $\sigma_n \sim n u^{-n}$ $(n \to \infty)$ for $r = 1$, and $\sigma_n \sim k_2 (r/u)^n$ $(n \to \infty)$ for $r < 1$.

7.3. Solvability of the (SSE) $(E_r)_\Delta + c_x = c_u$.

Here we consider the set S_E^c of all $x \in U^+$ that satisfy the (SSE) $(E_r)_\Delta + c_x = c_u$ for r, $u > 0$ and $E \in \{c_0, c, \ell_\infty\}$. It can be easily seen that $x \in S_c^c$ means that for every $y \in \omega$, the condition $y_n/u^n \to l$ $(n \to \infty)$ holds if and only if there are α and $\beta \in \omega$ such that $y = \alpha + \beta$ for which $r^{-n} \Delta_n \alpha \to l_1$ and $\beta_n / x_n \to l_2$ $(n \to \infty)$ for some scalars l, l_1, and l_2. We obtain the next theorem.

Theorem 24. *Let r, $u > 0$ and let $E = c_0$, c, or ℓ_∞.*

i) *If $r < 1$, then*

$$S_E^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } u \geq 1, \\ \emptyset & \text{if } u < 1. \end{cases}$$

ii) *If $r = 1$, then*

$$S_E^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1. \end{cases}$$

iii) *If $r > 1$, then*

$$S_{s_1}^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } r < u, \\ \emptyset & \text{if } r \geq u; \end{cases}$$

$$S_{c_0}^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } r \leq u, \\ \emptyset & \text{if } r > u; \end{cases}$$

and

$$S_c^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } r < u, \\ c_u^+ & \text{if } r = u, \\ \emptyset & \text{if } r > u. \end{cases}$$

Proof. Again by Proposition 14 and Remark 15, we only need to deal with the cases α) $r < u = 1$, and β) $r = u > 1$. Consider

Case α): We have $r < u = 1$ so $x \in S_E^c$ implies $(E_r)_\Delta + c_x = c$. Then we have $x \in c$ and

$$c \subset (E_r)_\Delta + c_x. \tag{4}$$

Then by Lemma 9 ii), where $a = (r^n)_{n \geq 1} \in c_0$, we have condition (4) implies $x \in s_1^\bullet$. We conclude $x \in c \cap s_1^\bullet = \mathrm{cl}^c(1)$. Conversely, $x \in \mathrm{cl}^c(1)$ implies $c_x = c$ and since $(E_r)_\Delta \subset c$, we obtain $(E_r)_\Delta + c_x = (E_r)_\Delta + c = c$.

Case β): We have $(E_r)_\Delta = E_r$ since $r > 1$. Then we have $(E_r)_\Delta + c_x = E_r + c_x = c_r$. Trivially the (SSE) $E_r + c_x = c_r$ is equivalent to $x \in c_r$ for $E = c$, by Lemma 6, it is equivalent to $x \in \mathrm{cl}^c(r)$ for $E = c_0$, and the (SSE) has no solutions for $E = s_1$. This concludes the proof. □

Remark 25. For $u \neq 1$, we have $S_E^c \neq \emptyset$ if and only if $u > \max(r, 1)$, or $r = u > 1$. More precisely, we have $S_E^c = \mathrm{cl}^c(u)$ for $u > \max(r, 1)$, and $S_E^c = c_u$ for $r = u > 1$.

Remark 26. The statement ii) in Theorem 24 was extended in [**10**, Proposition 7.1, pp. 95–96], in the following way. For any given $b \in U^+$, the (SSE) defined by $(c_0)_\Delta + c_x = c_b$ has solutions if and only if $1/b \in s_{(1/n)_{n \geq 1}}$ defined by $x \in \mathrm{cl}^c(b)$.

Remark 27. Theorem 24 with $E = c_0$ was extended in [**11**, Theorem 1, pp. 117–118] in the following way. For any given $a, b \in U^+$, the (SSE) defined by $(s_a^0)_\Delta + c_x = c_b$ has solutions if and only if $b_n^{-1} \sum_{k=1}^n a_k \to 0$ $(n \to \infty)$ that are defined by $x \in \mathrm{cl}^c(b)$.

8. The solvability of the (SSE) $(W_r)_\Delta + c_x = c_u$.

We will consider the set of $a-$*strongly bounded sequences* defined for $a \in U^+$ by $W_a = \left\{ y \in \omega : \|y\|_{W_a} = \sup_{n\geq 1}(n^{-1}\sum_{k=1}^n |y_k|/a_k) < \infty \right\}$, (cf. [**16**]). If $a = (r^n)_{n\geq 1}$, the set W_a is denoted by W_r. For $r = 1$, we obtain the well known set w_∞ defined by $w_\infty = \left\{ y \in \omega : \|y\|_{w_\infty} = \sup_{n\geq 1}(n^{-1}\sum_{k=1}^n |y_k|) < \infty \right\}$ ([**19**]). In [**10**, Proposition 7.3, p. 98], was given a solvability of each of the (SSE) $(w_0)_\Delta + c_x = c_b$ and $(w_0)_\Delta + s_x = s_b$, where $w_0 = \left\{ y \in \omega : \lim_{n\to\infty}(n^{-1}\sum_{k=1}^n |y_k|) = 0 \right\}$. It was shown that $\mathcal{S}((w_0)_\Delta, c)$ is nonempty and equal to $\mathrm{cl}^c(b)$ if and only if $1/b \in s_{(1/n)_{n\geq 1}}$. So if $b = (u^n)_{n\geq 1}$, we have $\mathcal{S}((w_0)_\Delta, c) = \mathrm{cl}^c(b)$ if and only if $u > 1$. In the following, we consider the set $\mathcal{S}((W_r)_\Delta, c)$ of all $x \in U^+$ that satisfy the (SSE)

$$(W_r)_\Delta + c_x = c_u \tag{5}$$

for r, $u > 0$. We write $S_w^c = \mathcal{S}((W_r)_\Delta, c)$ to simplify. It can be easily seen that $x \in S_w^c$, which means that for every $y \in \omega$, the condition $y_n/u^n \to l$ $(n \to \infty)$ holds if and only if there are α and $\beta \in \omega$ such that $y = \alpha + \beta$ for which

$$\sup_{n\geq 1}\left(\frac{1}{n}\sum_{k=1}^n \frac{|\Delta_k \alpha|}{r^k}\right) < \infty \quad \text{and} \quad \frac{\beta_n}{x_n} \to l' \ (n \to \infty)$$

for some scalars l, l'. In the following, we use next elementary statement, whose the proof is elementary and left to the reader.

$$s_1 \subset w_\infty \subset s_{(n)_{n\geq 1}}. \tag{6}$$

Now we may state the next theorem.

Theorem 28. *Let r, $u > 0$. Then*

i) *If $r \leq 1$, then $S_w^c = S_E^c$ for $E = c_0$, c, or s_1, and S_E^c is determined by i) and ii) in Theorem 24.*

ii) *If $r > 1$, then $S_w^c = S_{s_1}^c$, where $S_{s_1}^c$ is determined by iii) in Theorem 24.*

Proof. i) *Case $r < 1$.*
Inside the case $r < 1$, we deal with each of the cases a) $u < 1$, b) $u > 1$, and c) $u = 1$.

a) Case $u < 1$. Assume $S_w^c \neq \emptyset$ and let $x \in S_w^c$. Then we have $D_{1/u}\Sigma D_r \in (w_\infty, c)$, and since $s_1 \subset w_\infty$, we obtain $D_{1/u}\Sigma D_r \in (s_1, c)$. But we have $\sigma_n = u^{-n}\sum_{k=1}^n r^k \to \infty$ $(n \to \infty)$ and $D_{1/u}\Sigma D_r \notin (w_\infty, c)$. This leads to a contradiction and we conclude $S_w^c = \emptyset$.

b) Case $u > 1$. Let $x \in S_w^c$. Then we have $x \in c_u$, and $c_u \subset (W_r)_\Delta + c_x$. Then there are $\nu = (\nu_k)_{k\geq 1} \in w_\infty$ and $\varphi = (\varphi_n)_{n\geq 1} \in c$ such that

$$u^n = \sum_{k=1}^n r^k \nu_k + x_n \varphi_n$$

and

$$\frac{u^n}{x_n}\Big(1-\frac{1}{u^n}\sum_{k=1}^{n} r^k \nu_k\Big)=\varphi_n \qquad \text{for all } n.$$

By the condition in (6), there is $K>0$ such that $\nu_k \le Kk$ for all k, and we have

$$u^{-n}\Big|\sum_{k=1}^{n} r^k\nu_k\Big| \le Ku^{-n}\sum_{k=1}^{n} kr^k = o(1) \qquad (n\to\infty).$$

Since we have

$$\frac{u^n}{x_n}=\frac{\varphi_n}{1-\frac{1}{u^n}\sum_{k=1}^{n} r^k\nu_k},$$

we conclude $(u^n/x_n)_{n\ge 1}\in c$ and $x\in c_u^{\bullet}$. So we have shown the inclusion $S_w^c\subset \mathrm{cl}^c(u)$. Conversely, assume $x\in \mathrm{cl}^c(u)$. We need to show

$$(W_r)_\Delta \subset c_u. \tag{7}$$

Since $r<1<u$, we have $u^{-n}\sum_{k=1}^{n} kr^k\to 0$ $(n\to\infty)$ which implies

$$D_{1/u}\Sigma D_{(nr^n)_{n\ge 1}}\in (s_1,c)$$

and $D_{1/u}\Sigma D_r\in (s_{(n)_{n\ge 1}},c)$. Then the condition in (7) holds since $(s_{(n)_{n\ge 1}},c)\subset (w_\infty,c)$. We conclude $(W_r)_\Delta + c_x=(W_r)_\Delta + c_u = c_u$, and $x\in S_w^c$. So we have shown $S_w^c=\mathrm{cl}^c(u)$ for $u>1$.

c) Case $u=1$. Let $x\in S_w^c$. Then we have $x\in c$. We may apply Lemma 9 ii) with $E=W_{r^{1/2}}$. Indeed, we have $(W_r)_\Delta=(E_{r^{1/2}})_\Delta$, and since by [**15**, Lemma 4.2 pp. 598–599], get $M(w_\infty,s_1)=s_{(1/n)_{n\ge 1}}$, we deduce $(r^{n/2})_n\in M(w_\infty,s_1)$ and $W_{r^{1/2}}\subset s_1$. So we may apply Lemma 9 and we have $S_w^c\subset \mathcal{I}^c_{W_{r^{1/2}}}\subset s_1^{\bullet}$. We conclude $S_w^c\subset c\cap s_1^{\bullet}$ and $S_w^c\subset \mathrm{cl}^c(1)$. Conversely, assume $x\in\mathrm{cl}^c(1)$, that is, $c_x=c$. Then we have $\Sigma D_r\in (s_{(n)_{n\ge 1}},c)$ since $\sum_{k=1}^{n} kr^k\to L$ $(n\to\infty)$ for some scalar L, and the inclusion

$$(s_{(n)_{n\ge 1}},c)\subset (w_\infty,c)$$

implies $\Sigma D_r\in (w_\infty,c)$ and $(W_r)_\Delta\subset c$. Finally, we obtain

$$(W_r)_\Delta + c_x = (W_r)_\Delta + c = c$$

and $x\in S_w^c$. We conclude $\mathrm{cl}^c(1)\subset S_w^c$ and since, we have shown $S_w^c\subset \mathrm{cl}^c(1)$ we conclude $S_w^c=\mathrm{cl}^c(1)$.

Case $r=1$. Here we show $x\in S_w^c$ if and only if $x\in \mathrm{cl}^c(e)$ and $u>1$. For this, let $x\in S_w^c$. Since we have $w_\infty\supset s_1$, the inclusion $(w_\infty)_\Delta\subset c_u$ implies $D_{1/u}\Sigma\in (w_\infty,c)$ and $D_{1/u}\Sigma\in (s_1,c)$. So we obtain $(n/u^n)_{n\ge 1}\in c$ and $u>1$. Then $x\in c_u$, since $(w_\infty)_\Delta + c_x\subset c_u$, and using similar arguments that in the case i) b), we also have $x\in c_u^{\bullet}$. We conclude $x\in S_w^c$ implies $x\in \mathrm{cl}^c(e)$ and $u>1$. Conversely, assume $x\in \mathrm{cl}^c(e)$, that is, $c_x=c_u$ and $u>1$. We need to show

$$(w_\infty)_\Delta\subset c_u. \tag{8}$$

The condition $u>1$ implies $D_{1/u}\Sigma\in (s_{(n)_{n\ge 1}},c_0)$, and since $(s_{(n)_{n\ge 1}},c_0)\subset (w_\infty,c)$, we conclude (8) holds. Then we have $(w_\infty)_\Delta + c_x=(w_\infty)_\Delta + c_u = c_u$, and we have shown $x \in S_w^c$ if and only if $x \in \mathrm{cl}^c(e)$ and $u > 1$. We conclude $S_w^c = \mathrm{cl}^c(e)$ for $u > 1$ and $S_w^c = \emptyset$ for $u \leq 1$.

ii) *Case* $r > 1$. By Lemma 8 iii) we have $(W_r)_\Delta = W_r$, so we are led to solve the (SSE)

$$W_r + c_x = c_u. \tag{9}$$

Let $x \in S_w^c$. Then we have $W_r \subset c_u$, which implies $((r/u)^n)_{n\geq 1} \in M(w_\infty, c)$. But by [**15**, Lemma 4.2, p. 598], we have $M(w_\infty, s_1) = s_{(1/n)_{n\geq 1}}$, and since $M(w_\infty, c) \subset M(w_\infty, s_1)$, we obtain $(n(r/u)^n)_{n\geq 1} \in s_1$ and $r < u$. Then the identity in (9) implies $c_u \subset W_r + c_x$. So there are two sequences $\nu \in w_\infty$ and $\varphi \in c$ such that $u^n = r^n \nu_n + x_n \varphi_n$ and $u^n x_n^{-1} [1 - (r/u)^n \nu_n] = \varphi_n$ for all n. Again the condition in (6) implies $(r/u)^n |\nu_n| \leq Cn(r/u)^n$ for all n and for some $C > 0$. But since

$$n\left(\frac{r}{u}\right)^n \to 0 \qquad (n \to \infty), \tag{10}$$

we deduce $1 - (r/u)^n \nu_n \to 1$ $(n \to \infty)$, and since $u^n / x_n = \varphi_n / [1 - (r/u)^n \nu_n]$, we conclude $(u^n / x_n)_{n\geq 1} \in c$ and $x \in c_u^\bullet$. Now, by the identity in (9) we have $c_x \subset c_u$ and $x \in c_u$. We conclude $x \in c_u^\bullet \bigcap c_u = \mathrm{cl}^c(u)$. So we have shown $x \in S_w^c$ implies $r < u$ and $x \in \mathrm{cl}^c(u)$. Conversely, assume $r < u$ and $x \in \mathrm{cl}^c(u)$. Again by [**15**, Lemma 4.2, p. 598], we have $M(w_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$ and the condition $r < u$ implies that condition (10) holds and $((r/u)^n)_{n\geq 1} \in M(w_\infty, c_0)$. So we have shown $W_r \subset c_u$. Finally, since we have $c_x = c_u$, we obtain $W_r + c_u = W_r + c_x = c_u$ and $x \in S_w^c$. This concludes the proof. □

Corollary 29. *Let r, $u > 0$. Then $S_w^c = \mathrm{cl}^c(u)$ if and only if $r < 1 \leq u$, or $r = 1 < u$, or $1 < r < u$.*

Remark 30. From Theorem 28 and [**10**, Proposition 7.3, p. 98], we obtain $\mathcal{S}((w_\infty)_\Delta, c) = \mathcal{S}((w_0)_\Delta, c) = \mathrm{cl}^c(u)$ if and only if $u > 1$. So the (SSE) $(w_\infty)_\Delta + c_x = c_u$ and $(w_0)_\Delta + c_x = c_u$ are equivalent for all $u > 1$.

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