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# AMERICAN BOND OPTIONS CLOSE TO EXPIRY

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Abstract. We address the pricing of American-style options on zero coupon bonds under the assumption that interest rates obey a mean-reverting random walk as given by the Vasicek model. We use a technique due to Kolodner (1956) and Kim (1990) to derive an expression involving integrals for the price of such an option close to expiry. We then evaluate this expression on the optimal exercise boundary to obtain a pair of integral equations for the location of this exercise boundary, and solve these equations close to expiry. As with American equity options, as we approach expiry, there are three possible behaviors for the optimal exercise boundary.

## 1. Introduction

Financial engineering as a field dates back at least as far as 1900, when Louis Bachelier [**5**] derived an analytical option pricing formula in his doctoral dissertation at the Sorbonne, but came of age in the early 1970's when the more well-known Nobel prize winning Black-Scholes-Merton option pricing formula [**7, 27**] was published and the Chicago Board Options Exchange opened its doors as the first organized options exchange. In the years since then, there has been a revolution in quantitative finance and the weaponry of both numerical analysis and classical applied mathematics has been used to model countless diverse assets such as equity options, interest rate swaps, and electricity futures. A large part of the efforts of financial engineers has been directed at the pricing and hedging of derivative securities, whose values are based on some other underlying asset, with options garnering the lion's share of the attention. Options are derivatives which grant the holder the right but not the obligation to carry out a specified transaction on the underlying security, and they come in two main flavors, European and American.

European options can be exercised only at expiry, which is specified in the contract. A European call option on a stock will pay the holder the amount $\max(S-E,0)$ at expiry, where $S$ is the price of the underlying stock and $E$ is the strike price of the option, also specified in the contract, while a European put option will pay an amount $\max(E-S,0)$ at expiry. Although many options are cash-settled, a call on a stock essentially gives the right to sell the stock at

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the price $E$ while a put gives the right to buy the stock at that price. Closed form expressions have been found for many European-style options, the most well-known of these expressions being of course the Black-Scholes-Merton option pricing formula [**7, 27**] for equity options mentioned above, but a number of such solutions are also known for interest rate options, such as European bond options [**17**].

One drawback of European options to the holder is that the payoff is based solely on the price of the underlying at expiry, so that if a European is deep in-the-money before expiry, the holder is powerless to act and must wait until expiry and hope that the moneyness does not decrease. American options in part address this drawback as they can be exercised at any time at or before expiry, with an American equity call paying $\max(S-E,0)$ when exercised, regardless of when exercise occurs, and an American equity put paying $\max(E-S,0)$. A third class of option, known as Bermudan or semi-American, allows early exercise but only on a finite number of discrete dates. Clearly an American option allows the holder the opportunity to lock in the profits at any time, but in doing so, he must forfeit the right to benefit from any further upside. The early exercise feature means that the holder of an American option must constantly decide whether to exercise the option or retain it, with the holder aiming to maximize the present value of the payoff from the option. This in turn leads to a free boundary, known as the optimal exercise boundary, on which exercise will take place, and in order to price American-style options, it is necessary to first locate the free boundary. Because of this, closed form pricing formulas for American-style options have remained elusive.

Before we address the location of the free boundary, we would mention some theorems that have been proved about the nature of the boundary. For vanilla equity options, Karatzas [**10, 19**] was able to prove the existence of an optimal exercise policy for American options and show that there was an optimal stopping time, while Van Moerbeke [**34**] showed that the free boundary for American options was continuously differentiable. In addition, there are also studies [**13, 32**] on the analyticity of the free boundary in Stefan problems, a class of physical problems involving melting and solidification which are formulated in a manner very similar to American options. It is reasonable to assume that these theorems can be carried over to the American bond options considered here.

In addition to the theorems mentioned above, there has been a considerable body of research aimed both at pricing American equity options and also locating the free boundary. For the numerical aspect of the problem, the reader is referred to [**36**]. On the theoretical side, two popular approaches have been Tao's method [**33**] and the integral equation approach. Tao's method involves applying asymptotics to the underlying partial differential equation using the time remaining until expiry as a small parameter, and for vanilla Americans, studies by [**1, 2, 9, 25**] have yielded the first few terms in series for the value of the option and the location of the free boundary close to expiry. The integral equation approach [**12, 16, 20, 21, 23, 24, 26, 31**] involves decoupling the location of the free boundary from the pricing of the option, leading to an integral equation for the location of the free boundary, which can subsequently be solved asymptotically or numerically.

One of these integral equation methods is of particular interest to us as we will be applying that method to American interest rate options in the present study, and that is the method used by Kim [**20**] and Jacka [**16**]. This approach was originally developed for physical Stefan problems [**22**] and later applied to economics by McKean [**26**], and to vanilla Americans with great success by Kim [**20**] and Jacka [**16**], who independently derived the same results, Kim both by using McKean's formula and by taking the continuous limit of the Geske-Johnson formula [**14**] which is a discrete approximation for American options, and thereby demonstrating that those two approaches led to the same result, and Jacka by applying probability theory to the optimal stopping problem. [**8**] later used these results to show how to decompose the value of an American into intrinsic value and time value. The approach in [**16, 20, 26**] leads to an integral equation for the location of the free boundary, which was solved numerically by [**15**] and by approximating the free boundary as a multipiece exponential function by [**18**].

While both Tao's method and the integral equation approach have been used successfully for American-style equity options, very few studies have looked at American-style interest rate options. In the present study, we will consider calls and puts on zero coupon bonds, which carry the right to buy or sell a zero coupon bond rather than a stock, so that at exercise a call pays the amount $\max\left[V_B\left(r,t,\hat{\tau}\right)-E,0\right)$ while a put pays $\max\left[E-V_B\left(r,t,\hat{\tau}\right),0\right)$, where $E$ is again the strike price, $V_B\left(r,t,\hat{\tau}\right)$ is the price at time $t$ of a zero coupon bond with tenor $\hat{\tau}$, which is assumed to pay a fixed amount \$ 1 a time $\hat{\tau}$ in the future, and $r$ is the interest rate at the time of exercise. It is straightforward to price a European bond on a zero coupon bond [**17**], but somewhat less straightforward to price an American option, largely because of the presence of the optimal exercise boundary.

We should mention that there are several different flavors of American bond option. In the present study, we are concerned with a *trombone* option, wherein the holder receives a zero coupon bond with a fixed tenor regardless of when exercise occurs, so that the maturity date of the bond will depend on the date of exercise. Another flavor is the *wasting* option, wherein the holder receives a zero coupon bond with a fixed maturity date regardless of when exercise occurs, so that the tenor of the bond will depend on the date of exercise. Because the tenor of the bond varies for a wasting option, the strike price will also depend upon the exercise date, and because of this, there does not appear to be a standard contract for the wasting bond option, which makes analysis of that option problematic.

We would mention that in addition to bond options, which as their name implies are interest rates derivatives based on bonds, there are also interest rate derivatives based directly on an interest rate, such as LIBOR, the London InterBank Offer Rate: caplets and floorlets are calls and puts on a specified interest rate, and in a previous study [**3**], we used Tao's method to find series solutions both for the prices of American caplets and floorlets close to expiry, and also for the location of the associated free boundaries. Both bond options and options on an interest rate can be used to hedge against interest rate movements, and while bond options offer an indirect interest rate hedge, they themselves can be hedged with the underlying bonds which is one reason why many dealers prefer them to options on an interest rate. In addition to their utility for interest rate hedging, bond options have obvious uses in conjunction with a bond portfolio: covered calls can be used to generate income and protective puts to protect a portfolio against catastrophic losses.

In the present study, we will use the integral equation approach and extend the analysis of [**16, 20, 26**] to bond options. The key to both our earlier study on interest rate caplets and floorlets [**3**] and the present work is using a change of variables to transform the governing equation into the diffusion equation. This transformation is straightforward for equity options, where the price obeys the Black-Scholes-Merton partial differential equation [**7, 27**], and is discussed in standard texts such as [**36**]. However, while the Black-Scholes-Merton partial differential equation is widely accepted for equity options, a variety of different models are used for interest rate derivatives. In the present work, as in [**3**], we will use the Vasicek model, which is a mean reverting model popular amongst academic practitioners. The main reason for choosing the Vasicek model is precisely because it is also straightforward to transform the governing equation for this model into the nonhomogeneous diffusion equation [**4**]. The details of this model will be given in the next section, where we will use the techniques developed by [**16, 20, 22, 26**] to arrive at expressions involving integrals for the value of American bond call and put options. We will then evaluate these expressions on the free boundary to arrive at integral equations for the location of the free boundary for these options, and find series solutions for the free boundary close to expiry. The final section contains a discussion of our results.

## 2. Analysis

To find the price $V(r,t)$ of a security dependent on a stochastic spot interest rate $r(t)$, it is necessary to model the behavior of that interest rate, and to do so, it is usual to assume that $r$ obeys the stochastic differential equation

$$\mathrm{d}r = u(r,t)\,\mathrm{d}t + w(r,t)\,\mathrm{d}X, \tag{1}$$

where $dX$ is normally distributed with zero mean and variance $dt$ and $w$ is the volatility. By constructing a risk neutral portfolio, it can be shown that the price of the security obeys the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0, \tag{2}$$

where $\lambda(r,t)$ is the market price of interest rate risk and $u - \lambda w$ is the risk adjusted drift. This equation is valid for times $t \leq T$, where $T$ is the maturity of the security. The derivation of (2) can be found in, for example, [**36**], with a more detailed discussion in [**11**], and this equation governs the behavior of all interest rate securities: the boundary and initial conditions rather than the PDE differentiate amongst them [**28**]. As noted in [**36**], we can interpret the solution of (2) as the expected present value of all cashflows, but this expectation is not with respect to the real random variable given by (1) but rather with respect to the risk-neutral variable. The risk-neutral spot rate obeys

$$\mathrm{d}r = [u(r,t) - \lambda(r,t)\,w(r,t)]\,\mathrm{d}t + w(r,t)\mathrm{d}X \tag{3}$$

with $u - \lambda w$ the risk adjusted drift, as mentioned above.

There is a number of popular interest rate models, and several of these are special cases of the general affine model, for which $u - \lambda w = a(t) - b(t)r$ and $w = (c(t)r - d(t))^{1/2}$; a table of these special cases can be found in §46.2 of [**36**]. For these models, the equation for the risk-neutral spot rate (3) becomes

$$\mathrm{d}r = [a(t) - b(t)r]\,\mathrm{d}t + [c(t)r - d(t)]^{1/2}\,\mathrm{d}X, \tag{4}$$

One popular model is the Vasicek model [**35**], which was one of the first interest rate models to incorporate a stochastic interest rate. For this model, $u-\lambda w = a-br$ and $w = \sigma$, with $a$, $b$ and $\sigma$ constants rather than functions of time, so that the risk-neutral spot rate obeys

$$\mathrm{d}r = (a - br)\,\mathrm{d}t + \sigma\mathrm{d}X, \tag{5}$$

where $a$, $b$ and $\sigma$ are constants. This model is mean-reverting, with the interest rate pulled to a level $a/b$ at a rate $b$, together with a normally distributed stochastic term $\sigma dX$. The pricing equation (2) becomes

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial r^2} + (a - br)\frac{\partial V}{\partial r} - rV = 0\,. \tag{6}$$

This model is popular amongst academic practitioners because it is highly tractable and it is possible to find closed form expressions for many interest rate derivatives.

There is actually a reason why the Vasicek is so tractable: it is possible to transform (6) into the heat conduction (or diffusion) equation. This is a property which the Vasicek PDE shares with the much more well-known Black-Scholes-Merton PDE [**7, 27**] which governs the price of equity options. To transform (6) into the diffusion equation, we make the transformation

$$V(r,t) = \exp\left[\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\tau - \frac{r}{b} + \frac{a}{b^2}\right] v(x,\zeta)\,, \tag{7}$$

where $\tau = T - t$ is the remaining tenor of the option and we have introduced the new variables

$$\begin{aligned} \zeta &= 1 - \mathrm{e}^{-2b\tau} \\ x &= \frac{2\sqrt{b}}{\sigma}\left[r - \frac{a}{b} + \frac{\sigma^2}{b^2}\right]\mathrm{e}^{-b\tau}, \end{aligned} \tag{8}$$

which we can invert

$$\begin{aligned} r &= \frac{a}{b} - \frac{\sigma^2}{b^2} + \frac{\sigma x}{2\sqrt{b(1-\zeta)}} \\ \tau &= -\frac{\ln(1-\zeta)}{2b}. \end{aligned} \tag{9}$$

It is worth noting that the new spatial coordinate $x$ depends on time as well as on $r$. Using this transformation in (6), we arrive at the diffusion equation

$$\mathcal{L}v = \left[\frac{\partial}{\partial\zeta} - \frac{\partial^2}{\partial x^2}\right] v = 0, \tag{10}$$

which governs one-dimensional heat conduction. If $v\,(x, 0)$ is known at $\zeta = 0$, we can write down an expression for $v\,(x, \zeta)$ for $\zeta > 0$, using a Green's function

$$\begin{aligned} v\,(x,\zeta) &= \int_{-\infty}^{\infty} v(z,0) g(x-z,\zeta) \mathrm{d}z, \\ g(x,\zeta) &= \frac{\mathrm{e}^{-x^2/(4\zeta)}}{\sqrt{4\pi\zeta}}. \end{aligned} \tag{11}$$

Using this, we can write a solution to (6) in the original variables

$$\begin{aligned} &V\left(r + \frac{a}{b} - \frac{\sigma^2}{b^2}, t\right) \\ &= \sqrt{\frac{b}{\pi\sigma^2(1-\mathrm{e}^{-2b(T-t)})}} \exp\left[\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)(T-t)\right] \\ &\quad\times \int_{-\infty}^{\infty} V\left(\tilde{r} + \frac{a}{b} - \frac{\sigma^2}{b^2}, T\right) \exp\left[\frac{\tilde{r}-r}{b} - \frac{b(\tilde{r} - r\,\mathrm{e}^{-b(T-t)})^2}{\sigma^2(1-\mathrm{e}^{-2b(T-t)})}\right] \mathrm{d}\tilde{r}. \end{aligned} \tag{12}$$

It is straightforward to use the formula (12) to price bonds under the Vasicek model.

If we apply (12) to a zero coupon bond, for which the payoff at maturity is $V_B(r, t + \hat{\tau}, 0) = 1$, we arrive at the well-known expression for the value of a zero coupon bond with time to maturity of $\hat{\tau}$

$$\begin{aligned} V_B(r,t,\hat{\tau}) = \exp\Bigg[&\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\left(\hat{\tau} - \frac{1-\mathrm{e}^{-b\hat{\tau}}}{b}\right) \\ &- \frac{\sigma^2\left(1-\mathrm{e}^{-b\hat{\tau}}\right)^2}{4b^3} - \frac{r\left(1-\mathrm{e}^{-b\hat{\tau}}\right)}{b}\Bigg], \end{aligned} \tag{13}$$

or in the transformed coordinates,

$$\begin{aligned} v_B\,(x,\zeta,\hat{\tau}) = \exp\Bigg[&\frac{\sigma x\,\mathrm{e}^{-b\hat{\tau}}}{2b^{3/2}\sqrt{1-\zeta}} \\ &+ \left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\left(\frac{\ln(1-\zeta)}{2b} + \hat{\tau}\right) - \frac{\sigma^2\left(3+\mathrm{e}^{-2b\hat{\tau}}\right)}{4b^3}\Bigg]. \end{aligned} \tag{14}$$

We can use (13) in (12), to recover the price of a European option on a zero coupon bond, which was found by [**17**]. At time $t = T$, the holder of a call option will have the option of purchasing a zero coupon bond with tenor $\hat{\tau}$ for price $E$, so that we can use (12) with $V\,(\tilde{r}, T) = H\,(-\hat{r})\,[V_B(r, T, \hat{\tau}) - E)$ for a European call and $V(\tilde{r}, T) = H(\hat{r})\,[E - V_B(r, T, \hat{\tau}))$ for a European put, where $\hat{r}$ is the root of $V_B\left(\hat{r} + \frac{a}{b} - \frac{\sigma^2}{b^2}, T, \hat{\tau}\right) = E$, so that the price of a European call is given by

$$
\begin{aligned}
& V^{(\mathrm{e})}\left(r + \frac{a}{b} - \frac{\sigma^2}{b^2}, t\right) \\
&= \frac{1}{2}\exp\left[\frac{\sigma^2\left(1 - \mathrm{e}^{-2b\tau}\right)}{4b^3} - \frac{r\left(1 - \mathrm{e}^{-b(\tau+\hat{\tau})}\right)}{b} + \left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)(\tau + \hat{\tau})\right] \\
&\quad \times \operatorname{erfc}\left[\frac{\sigma^2\,\mathrm{e}^{-b\hat{\tau}}\left(1 - \mathrm{e}^{-2b\tau}\right) + 2b^2\left(r\,\mathrm{e}^{-b\tau} - \hat{r}\right)}{2b^{3/2}\sigma\left(1 - \mathrm{e}^{-2b\tau}\right)^{1/2}}\right] \\
&\quad - \frac{E}{2}\exp\left[\frac{\sigma^2\left(1 - \mathrm{e}^{-2b\tau}\right)}{4b^3} - \frac{r\left(1 - \mathrm{e}^{-b\tau}\right)}{b} + \left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\tau\right] \\
&\quad \times \operatorname{erfc}\left[\frac{\sigma^2\left(1 - \mathrm{e}^{-2b\tau}\right) + 2b^2\left(r\,\mathrm{e}^{-b\tau} - \hat{r}\right)}{2b^{3/2}\sigma\left(1 - \mathrm{e}^{-2b\tau}\right)^{1/2}}\right],
\end{aligned}
\tag{15}
$$

or in the transformed variables,

$$
\begin{aligned}
v^{(\mathrm{e})}(x, \zeta) &= \frac{1}{2}\exp\left[\frac{\sigma\,\mathrm{e}^{-b\hat{\tau}}\,x}{2b^{3/2}} + \left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\hat{\tau} + \frac{\sigma^2\left(\mathrm{e}^{-2b\hat{\tau}}(\zeta - 1) - 3\right)}{4b^3}\right] \\
&\quad \times \operatorname{erfc}\left[\frac{\sigma\zeta^{1/2}\,\mathrm{e}^{-b\hat{\tau}}}{2b^{3/2}} + \frac{x - \hat{x}}{2\zeta^{1/2}}\right] \\
&\quad - \frac{E}{2}\exp\left[\frac{\sigma x}{2b^{3/2}} + \frac{\sigma^2(\zeta - 4)}{4b^3}\right]\operatorname{erfc}\left[\frac{\sigma\zeta^{1/2}}{2b^{3/2}} + \frac{x - \hat{x}}{2\zeta^{1/2}}\right],
\end{aligned}
\tag{16}
$$

where

$$
\hat{r} = -\frac{b\ln E}{1 - \mathrm{e}^{-b\hat{\tau}}} + \frac{\hat{\tau}\left(\sigma^2 - 2ab\right)}{2b\left(1 - \mathrm{e}^{-b\hat{\tau}}\right)} + \frac{\sigma^2\left(\mathrm{e}^{-b\hat{\tau}} - 3\right)}{4b^2} + \frac{a}{b}
$$

is the value of $r$ at which $V_B(r, T, \hat{\tau}) = E$ and

$$
\hat{x} = -\frac{2b^{3/2}\ln E}{\sigma\left(1 - \mathrm{e}^{-b\hat{\tau}}\right)} + \frac{\left(\sigma^2 - 2ab\right)\hat{\tau}}{\sigma b^{1/2}\left(1 - \mathrm{e}^{-b\hat{\tau}}\right)} + \frac{\sigma\left(1 + \mathrm{e}^{-b\hat{\tau}}\right)}{2b^{3/2}}
$$

is the corresponding value of $x$. The price of a European put is given by (15), (16) with the sign of the terms involving erfc changed and the sign of the argument of erfc also changed.

For American-style options with early exercise features, it follows from the work of [**16, 20, 22, 26**] that if such an option obeys equation (10), where it is optimal to hold the option and the payoff at expiry is $v(x, 0)$ while that from immediate exercise is $p(x, \zeta)$, then we can write the value of the option as the sum of the value of the corresponding European option $v^{(e)}(x,\zeta)$ together with another term representing both the premium from early exercise,

$$v(x,\zeta) = v^{(e)}(x,\zeta) + \int_0^\zeta \int_0^\infty f(z,\eta) g(x-z,\zeta-\eta) dz d\eta. \tag{17}$$

In this equation, we define $f(x,\zeta)$ to be equal to 0 where it is optimal to hold the option while where exercise is optimal, $f(x,\zeta)$ is the result of substituting the early exercise payoff $p(x,\zeta)$ into the heat conduction equation partial differential equation, $f(x,\zeta) = \mathcal{L}p$, where the operator $\mathcal{L}$ was defined in (10). In other words, if $x_f(\zeta)$ is the location of the free boundary, which is located at $r_f(t)$ in the original variables, then $f(x,\zeta) = [\mathcal{L}p] \times \begin{cases} H\left(x_f(\zeta) - x\right) & \text{call} \\ H\left(x - x_f(\zeta)\right) & \text{put} \end{cases}$, so that (17) becomes

$$v(x,\zeta) = v^{(e)}(x,\zeta) + \begin{cases} \displaystyle\int_0^\zeta \int_{-\infty}^{x_f(\eta)} f(z,\eta) g(x-z,\zeta-\eta) \mathrm{d}z \mathrm{d}\eta & \text{call} \\ \displaystyle\int_0^\zeta \int_{x_f(\eta)}^{\infty} f(z,\eta) g(x-z,\zeta-\eta) \mathrm{d}z \mathrm{d}\eta & \text{put.} \end{cases} \tag{18}$$

The payoff from early exercise for the call is

$$P(r,t) = V_B(r,t,\hat{\tau}) - E, \tag{19}$$

so that for the call in the transformed variables,

$$\begin{aligned} p = {} & \exp\left[\frac{\sigma x\, \mathrm{e}^{-b\hat{\tau}}}{2b^{3/2}\sqrt{1-\zeta}} + \left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\left(\frac{\ln(1-\zeta)}{2b} + \hat{\tau}\right) \right. \\ & \left. - \frac{\sigma^2\left(3 + \mathrm{e}^{-2b\hat{\tau}}\right)}{4b^3}\right] \\ & - E \exp\left[\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\frac{\ln(1-\zeta)}{2b} + \frac{\sigma x}{2b^{3/2}\sqrt{1-\zeta}} - \frac{\sigma^2}{b^3}\right], \end{aligned}$$

$$\begin{aligned} \mathcal{L}p = {} & \exp\left[\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\frac{\ln(1-\zeta)}{2b} - \frac{\sigma^2}{b^3}\right] \\ & \times \left(\left[\frac{2ab - \left(1 + \mathrm{e}^{-2b\hat{\tau}}\right)\sigma^2}{4b^3(1-\zeta)} + \frac{\sigma x\, \mathrm{e}^{-b\hat{\tau}}}{4b^{3/2}(1-\zeta)^{3/2}}\right] \right. \\ & \times \exp\left[\frac{\sigma x\, \mathrm{e}^{-b\hat{\tau}}}{2b^{3/2}\sqrt{1-\zeta}} + \left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\hat{\tau} + \frac{\sigma^2\left(1 - \mathrm{e}^{-2b\hat{\tau}}\right)}{4b^3}\right] \\ & \left. + E\left[\frac{\sigma^2 - ab}{2b^3(1-\zeta)} - \frac{\sigma x}{4b^{3/2}(1-\zeta)^{3/2}}\right] \exp\left[\frac{\sigma x}{2b^{3/2}\sqrt{1-\zeta}}\right]\right). \end{aligned} \tag{20}$$

For the put, the payoff from early exercise is $P(r,t) = E - V_B(r,t,\hat{\tau})$, so that $p$ and $\mathcal{L}p$ are minus the expressions given in (20) for the call.

Using (16,18,20), it follows that for an American zero coupon call option,

$$
\begin{aligned}
v = {}& \frac{1}{2}\exp\left[\frac{\sigma\,\mathrm{e}^{-b\hat{\tau}}\,x}{2b^{3/2}} + \left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\hat{\tau} + \frac{\sigma^2\left(\mathrm{e}^{-2b\hat{\tau}}\,(\zeta-1) - 3\right)}{4b^3}\right] \\
& \times \operatorname{erfc}\left[\frac{\sigma\zeta^{1/2}\,\mathrm{e}^{-b\hat{\tau}}}{2b^{3/2}} + \frac{x-\hat{x}}{2\zeta^{1/2}}\right] \\
& - \frac{E}{2}\exp\left[\frac{\sigma x}{2b^{3/2}} + \frac{\sigma^2\,(\zeta-4)}{4b^3}\right]\operatorname{erfc}\left[\frac{\sigma\zeta^{1/2}}{2b^{3/2}} + \frac{x-\hat{x}}{2\zeta^{1/2}}\right] \\
& + \int_0^{\zeta} \frac{1}{4b^{3/2}}\exp\left[\left(\frac{\sigma^2}{4b^3} - \frac{a}{2b^2}\right)\ln(1-\eta)\right] \\
& \times \left\{ E\,\mathrm{e}^{-\sigma^2/b^3}\left(\frac{\sigma\,(\zeta-\eta)^{1/2}}{\pi^{1/2}\,(1-\eta)^{3/2}}\,\exp\left[\frac{\sigma x_f(\eta)}{2b^{3/2}\,(1-\eta)^{1/2}} - \frac{(x-x_f(\eta))^2}{4\,(\zeta-\eta)}\right]\right.\right. \\
& + \exp\left[\frac{\sigma x}{2b^{3/2}\,(1-\eta)^{1/2}} + \frac{\sigma^2(\zeta-\eta)}{4b^3\,(1-\eta)}\right] \\
& \times \operatorname{erfc}\left[\frac{\sigma(\zeta-\eta)^{1/2}}{2b^{3/2}\,(1-\eta)^{1/2}} + \frac{x-x_f(\eta)}{2(\zeta-\eta)^{1/2}}\right] \qquad (21) \\
& \times \left.\left[\frac{\sigma^2(2-\zeta-\eta)}{2b^{3/2}\,(1-\eta)^2} - \frac{a}{b^{1/2}\,(1-\eta)} - \frac{\sigma x}{2\,(1-\eta)^{3/2}}\right]\right) \\
& + \exp\left[\frac{\hat{\tau}\left(\sigma^2 - 2ab - 2b^3\right)}{2b^2} - \frac{3\sigma^2}{4b^3} - \frac{\sigma^2\,\mathrm{e}^{-2b\hat{\tau}}}{4b^3}\right] \\
& \times \left(\frac{1}{2}\left[\frac{\sigma\,\mathrm{e}^{b\hat{\tau}}\,x}{(1-\eta)^{3/2}} + \frac{\left(2ab-\sigma^2\right)\mathrm{e}^{2b\hat{\tau}}\,x}{b^{3/2}(1-\eta)} - \frac{\sigma^2(1-\zeta)}{b^{3/2}(1-\eta)^2}\right]\right. \\
& \times \exp\left[\frac{\sigma\,\mathrm{e}^{-b\hat{\tau}}\,x}{2b^{3/2}(1-\eta)^{1/2}} + \frac{\sigma^2\,\mathrm{e}^{-2b\hat{\tau}}(\zeta-\eta)}{4b^3(1-\eta)}\right] \\
& \times \operatorname{erfc}\left[\frac{\sigma\,\mathrm{e}^{-b\hat{\tau}}(\zeta-\eta)^{1/2}}{2b^{3/2}\,(1-\eta)^{1/2}} + \frac{x-x_f(\eta)}{2(\zeta-\eta)^{1/2}}\right] \\
& \left.\left. - \frac{\sigma(\zeta-\eta)^{1/2}}{\pi^{1/2}\,(1-\eta)^{3/2}}\exp\left[\frac{\sigma e^{-b\hat{\tau}}x_f(\eta)}{2b^{3/2}(1-\eta)^{1/2}} - \frac{(x-x_f(\eta))^2}{4\,(\zeta-\eta)}\right]\right)\right\}\mathrm{d}\eta.
\end{aligned}
$$

The expression for the American put is given by (21) but with the sign of the term involving erfc changed as well as the sign of the argument of erfc, while terms not involving erfc are unchanged, so that the expressions for the American options have the same symmetry as their European counterparts. The expression

(21) and its counterpart for the put are expressions in the transformed variables for the values of an American call and put option, respectively, on a zero coupon bond, under the assumption that interest rates are governed by the mean-reverting Vasicek model. Expressions in the original variables can be recovered using (8,9).

## 3. Integral Equations for the Free Boundary

In this section, we will obtain integral equations for the location of the free boundary $x = x_f(\tau)$. These equations are obtained by substituting the expression for the American call (21) and its counterpart for the put into the conditions at the free boundary. The conditions at the free boundary are that the option price and the rho, or derivative with respect to $r$, of the option are continuous there, so that $V = P$ and $(\partial V/\partial r) = (\partial P/\partial r)$ at $r = r_f(t)$. The condition on the rho is known as the high contact or smooth pasting condition [**30**]; for equity options the corresponding condition is, that the delta or derivative with respect to stock price is continuous. In the transformed variables, this means that we require $v = p$ and $(\partial v/\partial x) = (\partial p/\partial x)$ at $x = x_f(\zeta)$, so that the condition on the option price is

$$p\left(x_f(\zeta),\zeta\right) = v^{(e)}\left(x_f(\zeta),\zeta\right) + \begin{cases} \displaystyle\int_0^{\zeta}\int_{-\infty}^{x_f(\eta)} f(z,\eta)g(x_f(\zeta)-z,\zeta-\eta)\mathrm{d}z\mathrm{d}\eta & \text{call} \\ \displaystyle\int_0^{\zeta}\int_{x_f(\eta)}^{\infty} f(z,\eta)g(x_f(\zeta)-z,\zeta-\eta)\mathrm{d}z\mathrm{d}\eta & \text{put} \end{cases} \tag{22}$$

and similarly, the condition on the rho is

$$\left.\frac{\partial p}{\partial x}\right|_{(x_f(\zeta),\zeta)} = \left.\frac{\partial v}{\partial x}\right|_{(x_f(\zeta),\zeta)} + \begin{cases} \displaystyle\int_0^{\zeta}\int_{-\infty}^{x_f(\eta)} f(z,\eta)\left.\frac{\partial g}{\partial x}\right|_{(x_f(\zeta)-z,\zeta-\eta)}\mathrm{d}z\mathrm{d}\eta & \text{call} \\ \displaystyle\int_0^{\zeta}\int_{x_f(\eta)}^{\infty} f(z,\eta)\left.\frac{\partial g}{\partial x}\right|_{(x_f(\zeta)-z,\zeta-\eta)}\mathrm{d}z\mathrm{d}\eta & \text{put} \end{cases}. \tag{23}$$

If we use (21,20), then (22) for the call gives

$$\exp\left[\frac{\sigma x_f(\zeta)e^{-b\hat{\tau}}}{2b^{3/2}\sqrt{1-\zeta}} + \left(\frac{\sigma^2}{2b^2}-\frac{a}{b}\right)\left(\frac{\ln(1-\zeta)}{2b}+\hat{\tau}\right) - \frac{\sigma^2\left(3+e^{-2b\hat{\tau}}\right)}{4b^3}\right]$$
$$- E\exp\left[\left(\frac{\sigma^2}{2b^2}-\frac{a}{b}\right)\frac{\ln(1-\zeta)}{2b} + \frac{\sigma x_f(\zeta)}{2b^{3/2}\sqrt{1-\zeta}} - \frac{\sigma^2}{b^3}\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\exp\left[\frac{\sigma\,\mathrm{e}^{-b\hat{\tau}}\,x_f(\zeta)}{2b^{3/2}} + \left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\hat{\tau} + \frac{\sigma^2\left(\mathrm{e}^{-2b\hat{\tau}}\,(\zeta-1)-3\right)}{4b^3}\right] \\
&\quad\times \operatorname{erfc}\left[\frac{\sigma\zeta^{1/2}\,\mathrm{e}^{-b\hat{\tau}}}{2b^{3/2}} + \frac{x_f(\zeta)-\hat{x}}{2\zeta^{1/2}}\right] \\
&\quad - \frac{E}{2}\exp\left[\frac{\sigma x_f(\zeta)}{2b^{3/2}} + \frac{\sigma^2\,(\zeta-4)}{4b^3}\right]\operatorname{erfc}\left[\frac{\sigma\zeta^{1/2}}{2b^{3/2}} + \frac{x_f(\zeta)-\hat{x}}{2\zeta^{1/2}}\right] \\
&\quad + \int_0^\zeta \frac{1}{4b^{3/2}}\exp\left[\left(\frac{\sigma^2}{4b^3} - \frac{a}{2b^2}\right)\ln(1-\eta)\right] \\
&\quad\times \Bigg\{ E\,\mathrm{e}^{-\sigma^2/b^3}\Bigg(\frac{\sigma\,(\zeta-\eta)^{1/2}}{\pi^{1/2}\,(1-\eta)^{3/2}}\,\exp\left[\frac{\sigma x_f(\eta)}{2b^{3/2}\,(1-\eta)^{1/2}} - \frac{\left(x_f(\zeta)-x_f(\eta)\right)^2}{4\,(\zeta-\eta)}\right] \\
&\quad + \left[\frac{\sigma^2(2-\zeta-\eta)}{2b^{3/2}\,(1-\eta)^2} - \frac{a}{b^{1/2}\,(1-\eta)} - \frac{\sigma x_f(\zeta)}{2\,(1-\eta)^{3/2}}\right] \\
&\quad\times \exp\left[\frac{\sigma x_f(\zeta)}{2b^{3/2}\,(1-\eta)^{1/2}} + \frac{\sigma^2(\zeta-\eta)}{4b^3\,(1-\eta)}\right] \\
&\quad\times \operatorname{erfc}\left[\frac{\sigma(\zeta-\eta)^{1/2}}{2b^{3/2}\,(1-\eta)^{1/2}} + \frac{x_f(\zeta)-x_f(\eta)}{2(\zeta-\eta)^{1/2}}\right]\Bigg) \\
&\quad + \exp\left[\frac{\hat{\tau}\left(\sigma^2-2ab-2b^3\right)}{2b^2} - \frac{3\sigma^2}{4b^3} - \frac{\sigma^2\,\mathrm{e}^{-2b\hat{\tau}}}{4b^3}\right] \\
&\quad\times \Bigg(\frac{1}{2}\left[\frac{\sigma\,\mathrm{e}^{b\hat{\tau}}\,x_f(\zeta)}{(1-\eta)^{3/2}} + \frac{\left(2ab-\sigma^2\right)e^{2b\hat{\tau}}x_f(\zeta)}{b^{3/2}(1-\eta)} - \frac{\sigma^2(1-\zeta)}{b^{3/2}(1-\eta)^2}\right] \\
&\quad\times \exp\left[\frac{\sigma e^{-b\hat{\tau}}x_f(\zeta)}{2b^{3/2}(1-\eta)^{1/2}} + \frac{\sigma^2\,\mathrm{e}^{-2b\hat{\tau}}(\zeta-\eta)}{4b^3(1-\eta)}\right] \\
&\quad\times \operatorname{erfc}\left[\frac{\sigma\,\mathrm{e}^{-b\hat{\tau}}(\zeta-\eta)^{1/2}}{2b^{3/2}\,(1-\eta)^{1/2}} + \frac{x_f(\zeta)-x_f(\eta)}{2(\zeta-\eta)^{1/2}}\right] \\
&\quad - \frac{\sigma(\zeta-\eta)^{1/2}}{\pi^{1/2}\,(1-\eta)^{3/2}}\exp\left[\frac{\sigma\,\mathrm{e}^{-b\hat{\tau}}\,x_f(\eta)}{2b^{3/2}(1-\eta)^{1/2}} - \frac{\left(x_f(\zeta)-x_f(\eta)\right)^2}{4\,(\zeta-\eta)}\right]\Bigg)\Bigg\}\,\mathrm{d}\eta
\end{aligned} \tag{24}
$$

while in the corresponding equation for the put, the left hand side will have the opposite sign to the left hand side of (24), because the payoff from early exercise for the put is minus that for the call, while on the right hand side, the terms involving erfc will have the opposite sign to those in (24), as will the arguments of erfc.

If we substitute (21), (20) into (23), the condition on the rho at the free boundary gives us a second integral equation for the free boundary which is similar to (24), but longer and somewhat more complicated than original equation, with the equation for the put having the same symmetry as above.

For each of the call and the put, the equations coming from (22, 23) constitute a pair of integral equations for the location of the free boundary $x_f(\zeta)$ for an American option on a zero coupon bond under the Vasicek model.

## 4. Solution of the Integral Equations Close to Expiry

In this section, we will solve the above integral equations stemming from (22, 23) close to expiry, to find expressions for the location of the free boundary $x_f(\zeta)$ in the limit $\zeta \to 0$. We will seek a solution of the form

$$x_f(\zeta) \sim \sum_{n=0}^{\infty} x_n \zeta^{n/2}, \tag{25}$$

which is motivated by work on American equity options [**6, 9, 25**] and by the classic work of Tao [**33**] on Stefan problems in general.

Before we substitute the expansion (25) into the integral equations, we must find the coefficient $x_f(0) = x_0$, which is the location of the free boundary at expiry. For American bond options, there are two key values of $x$ at $t = T$, and the free boundary will start from one or the other of these two values. We have already discussed one of these values, $\hat{x}$, which is the value of $x$ at which the payoff at expiry becomes equal to zero, and the price of the zero coupon bond is equal to the strike price of the option. The other key value is $x^\dagger$, the value of $x$ at which $(\partial V/\partial t)$ changes sign at expiry. The procedure for calculating this value is outlined in [**36**] for American equity options, and involves substituting the payoff at expiry into the PDE (6) to calculate $(\partial V/\partial t)$ at expiry: if $(\partial V/\partial t) > 0$, then the value of the option will drop below the payoff from immediate exercise as we move backwards in time from expiry, which means that the option must already have been exercised. For bond options, $x^\dagger$ is the root of

$$\begin{aligned}
&\left[\frac{a}{b} - \frac{\sigma^2}{b^2} + \frac{\sigma x^\dagger}{2b^{1/2}}\right] E \\
&+ \left[\frac{\left(1 + \mathrm{e}^{-2b\hat{\tau}}\right)\sigma^2}{2b^2} - \frac{\sigma\, \mathrm{e}^{-b\hat{\tau}}\, x^\dagger}{2b^{1/2}} - \frac{a}{b}\right] \\
&\times \exp\left[\left(\frac{\sigma^2}{2b^2} - \frac{a}{b}\right)\hat{\tau} + \frac{\left(1 - \mathrm{e}^{-2b\hat{\tau}}\right)\sigma^2}{4b^3} - \frac{\left(1 - \mathrm{e}^{-b\hat{\tau}}\right)\sigma x^\dagger}{2b^{3/2}}\right] = 0.
\end{aligned} \tag{26}$$

The free boundary at expiry will start at $x_f(0) = \min\left(x^\dagger, \hat{x}\right)$ for the call and $x_f(0) = \max\left(x^\dagger, \hat{x}\right)$ for the put.

As with American equity options, and also American interest rate caplets and floorlets under the Vasicek model [**3**], there are three distinct behaviors for the free boundary, in this instance depending on the ratio of $\hat{x}$ to $x^\dagger$ and thereby on whether there is a discontinuity in rho $= (\partial V/\partial r)$ at the free boundary at expiry. Along the free boundary, we have the high contact condition that $(\partial V/\partial r) = (\partial P/\partial r)$, and this same condition holds at expiry as we approach the free boundary if $x_f(0) < \hat{x}$ for the call or $x_f(0) > \hat{x}$ for the put, and therefore, rho is continuous at the free boundary at expiry. However, if $x_f(0) \geq \hat{x}$ for the call or $x_f(0) \leq \hat{x}$ for the put, at expiry we will have $(\partial V/\partial r) = 0$ as we approach the free boundary so that rho will be discontinuous at the free boundary at expiry, and it appears to be this discontinuity which leads to the logarithmic and Lambert W behavior of the free boundary [**2, 6, 21, 23, 25, 31**].

### 4.1. Call with $x^\dagger < \hat{x}$

The free boundary starts at $x_0 = x^\dagger < \hat{x}$, with the behavior of the free boundary given by (25) close to expiry. From the condition on the boundary at $O(\zeta)$, we get

$$\int_0^1 \left( \frac{x_1}{2} \operatorname{erfc}\left[ \frac{x_1}{2} \sqrt{\frac{1-\xi^{1/2}}{1+\xi^{1/2}}} \right] - \frac{\sqrt{1-\xi}}{\sqrt{\pi}} \exp\left[ -\frac{x_1^2}{4} \left( \frac{1-\xi^{1/2}}{1+\xi^{1/2}} \right) \right] \right) \mathrm{d}\xi = 0, \tag{27}$$

while from rho at $O\left(\zeta^{3/2}\right)$, we get

$$\int_0^1 \left( \operatorname{erfc}\left[ \frac{x_1}{2} \sqrt{\frac{1-\xi^{1/2}}{1+\xi^{1/2}}} \right] - \frac{x_1}{\sqrt{\pi}} \sqrt{\frac{\xi}{1-\xi}} \exp\left[ -\frac{x_1^2}{4} \left( \frac{1-\xi^{1/2}}{1+\xi^{1/2}} \right) \right] \right) \mathrm{d}\xi = 0, \tag{28}$$

where we have made the change of variable $\eta = \zeta\xi$. In this pair of equations (27, 28), each has a numerical root $x_1 = 0.903446598$ which is the value reported for the vanilla American equity call and also the interest rate caplet under Vasicek.

In a sense, it is surprising that the same value would be found for options with very different payoffs, but it should be recalled that for each of these options it was possible to reformulate the option pricing equation as the heat conduction equation (10), which suggests that this value of $x_1$ is perhaps a property of the heat conduction equation. We would note however that the subsequent coefficients $x_2, x_3, \cdots$ for the option considered here differ from both the interest caplet and the equity call. As with equity American options, one of the equations (27), (28) appears to be redundant, and once we have found the root using either one of them, the second equation does not yield any additional information; the same is true at higher orders.

### 4.2. Put with $x^\dagger > \hat{x}$

The free boundary starts at $x_0 = x^\dagger > \hat{x}$, with the behavior of the free boundary given by (25) close to expiry. This case is very similar to the call with $x^\dagger < \hat{x}$.

From the condition on the boundary at $O(\zeta)$, we get

$$\int_0^1 \left( \frac{x_1}{2} \operatorname{erfc}\left[ -\frac{x_1}{2}\sqrt{\frac{1-\xi^{1/2}}{1+\xi^{1/2}}}\right] + \frac{\sqrt{1-\xi}}{\sqrt{\pi}} \exp\left[ -\frac{x_1^2}{4}\left(\frac{1-\xi^{1/2}}{1+\xi^{1/2}}\right)\right]\right) \mathrm{d}\xi = 0, \tag{29}$$

while from rho at $O\left(\zeta^{3/2}\right)$, we get

$$\int_0^1 \left( \operatorname{erfc}\left[ -\frac{x_1}{2}\sqrt{\frac{1-\xi^{1/2}}{1+\xi^{1/2}}}\right] + \frac{x_1}{\sqrt{\pi}}\sqrt{\frac{\xi}{1-\xi}} \exp\left[ -\frac{x_1^2}{4}\left(\frac{1-\xi^{1/2}}{1+\xi^{1/2}}\right)\right]\right) \mathrm{d}\xi = 0. \tag{30}$$

This pair of equations (29, 30) is simply (27, 28) with $x_1$ replaced by $-x_1$, and therefore, (29, 30) have a numerical root $x_1 = -0.903446598$, or minus the value for the call, and this was also the value reported for the vanilla American equity put and also the interest rate floorlet under Vasicek.

### 4.3. Call with $x^\dagger > \hat{x}$

The free boundary starts at $x_0 = \hat{x} < x^\dagger$. If we try a series of the form (25), from the condition on the boundary at $O\left(\zeta^{1/2}\right)$, we get

$$\frac{x_1}{2} \operatorname{erfc}\left(-\frac{x_1}{2}\right) + \pi^{-1/2}\,\mathrm{e}^{-x_1^2/4} = 0, \tag{31}$$

while from rho at $O(\zeta)$, we get

$$\operatorname{erfc}\left(-\frac{x_1}{2}\right) = 0. \tag{32}$$

We note that these equations appear a power of $\tau$ earlier than (27, 28) did when $x^\dagger < \hat{x}$. For (31, 32), to have a solution requires that $x_1 = -\infty$, which of course, is not possible, and rather suggests that the series (25) is inappropriate for $x_f(\zeta)$, and we will instead suppose that

$$x_f(\zeta) \sim x_0 + \sum_{n=1}^{\infty} x_n(\tau)\zeta^{n/2}, \tag{33}$$

so that the coefficients in the series are functions of $\zeta$, and in turn expand the $x_n(\zeta)$ themselves as series in an unknown function $f(\zeta)$, which is assumed to be small,

$$x_1(\zeta) \sim \sqrt{f(\zeta)} \sum_{m=0}^{\infty} x_1^{(m)} \left(f(\zeta)\right)^{-m}. \tag{34}$$

We need to solve $f(\zeta)$ as part of the solution process. Using this new series (33), (34), we need to balance the leading order terms in (24), so that

$$\begin{aligned}&\zeta^{1/2}\left[x_1(\zeta)\operatorname{erfc}\left(-\frac{x_1(\zeta)}{2}\right)+\frac{2}{\pi^{1/2}}\,\mathrm{e}^{-x_1^2(\zeta)/4}\right]\\&\sim\frac{\zeta}{2}\left[\frac{\sigma\left(1+\mathrm{e}^{-b\hat{\tau}}\right)}{b^{3/2}}-\hat{x}\right]\int_0^1\operatorname{erfc}\left(-\frac{x_1(\zeta)-\sqrt{\xi}x_1(\zeta\xi)}{2\sqrt{1-\xi}}\right)\mathrm{d}\xi,\end{aligned}\tag{35}$$

where once again we have written $\eta=\zeta\xi$. To evaluate the right hand side of (35), we make the change of variable $\xi=1-\nu/f(\zeta)$ to enable us to strip the $\zeta$ dependence out of the integral, and we note that $\int_0^1 \mathrm{d}\xi$ becomes $\int_0^{1/f(\zeta)}\mathrm{d}\nu/f(\zeta)\to\int_0^\infty \mathrm{d}\nu/f(\zeta)$. In the limit,

$$\begin{aligned}&\int_0^1\operatorname{erfc}\left(-\frac{x_1(\zeta)-\sqrt{\xi}x_1(\zeta\xi)}{2\sqrt{1-\xi}}\right)\mathrm{d}\xi\\&\sim\frac{1}{f(\zeta)}\int_0^\infty\operatorname{erfc}\left(-\frac{\sqrt{f(\zeta)}}{2\sqrt{\nu}}\left[x_1(\zeta)-\sqrt{1-\frac{\nu}{f(\zeta)}}x_1\left(\zeta\left[1-\frac{\nu}{f(\zeta)}\right]\right)\right]\right)\mathrm{d}\nu\\&\sim\frac{1}{f(\zeta)}\int_0^\infty\operatorname{erfc}\left(-\frac{x_1^{(0)}\sqrt{\nu}}{4}\right)\mathrm{d}\nu=\frac{8}{x_1^{(0)2}f(\zeta)},\end{aligned}\tag{36}$$

so that (35) becomes

$$\begin{aligned}&x_1(\zeta)\operatorname{erfc}\left(-\frac{x_1(\zeta)}{2}\right)+\frac{2}{\pi^{1/2}}\,\mathrm{e}^{-x_1(\zeta)^2/4}\\&\sim\left[\frac{\sigma\left(1+\mathrm{e}^{-b\hat{\tau}}\right)}{b^{3/2}}-\hat{x}\right]\frac{4\zeta^{1/2}}{x_1^{(0)2}f(\zeta)}.\end{aligned}\tag{37}$$

As $z\to\infty$, we know that $\pi^{1/2}z\,\mathrm{e}^{z^2}\operatorname{erfc}z\sim1+\sum_{m=1}^{\infty}(-1)^m\frac{(2m-1)!!}{(2z^2)^m}$, so that (37) becomes

$$\left[\frac{\sigma\left(1+\mathrm{e}^{-b\hat{\tau}}\right)}{b^{3/2}}-\hat{x}\right]\frac{\zeta^{1/2}}{x_1^{(0)2}f(\zeta)}\sim\frac{\mathrm{e}^{-x_1^2/4}}{x_1^2\sqrt{\pi}},\tag{38}$$

or

$$\pi^{1/2}\left[\frac{\sigma\left(1+\mathrm{e}^{-b\hat{\tau}}\right)}{b^{3/2}}-\hat{x}\right]\zeta^{1/2}\sim e^{-x_1^{(0)2}f(\zeta)/4-x_1^{(0)}x_1^{(1)}/2},\tag{39}$$

so that $\mathrm{e}^{-x_1^{(0)2}f(\zeta)/4}=\zeta^{1/2}$ or $f(\zeta)=-\frac{2}{x_1^{(0)2}}\ln\zeta$ and

$$\mathrm{e}^{-x_1^{(0)}x_1^{(1)}/2}=\pi^{1/2}\left[\frac{\sigma\left(1+\mathrm{e}^{-b\hat{\tau}}\right)}{b^{3/2}}-\hat{x}\right].\tag{40}$$

If we pick $x_1^{(0)}=-\sqrt{2}$, then $f(\zeta)=-\ln\zeta$ and

$$x_1^{(1)}=2^{1/2}\ln\left(\pi^{1/2}\left[\frac{\sigma\left(1+e^{-b\hat{\tau}}\right)}{b^{3/2}}-\hat{x}\right]\right).\tag{41}$$

We get the same result from the equation for rho at the free boundary.

### 4.4. Put with $x^\dagger < \hat{x}$

The free boundary starts at $x_0 = \hat{x} > x^\dagger$, with the behavior of the free boundary given by (33, 34) close to expiry. This case is very similar to the call with $x^\dagger > \hat{x}$, and the counterpart of (35) is

$$\begin{aligned}\zeta^{1/2}&\left[x_1(\zeta)\operatorname{erfc}\left(\frac{x_1(\zeta)}{2}\right) - \frac{2}{\pi^{1/2}}\,\mathrm{e}^{-x_1^2(\zeta)/4}\right]\\ &\sim \frac{\zeta}{2}\left[\frac{\sigma\left(1+\mathrm{e}^{-b\hat{\tau}}\right)}{b^{3/2}} - \hat{x}\right]\int_0^1 \operatorname{erfc}\left(\frac{x_1(\zeta)-\sqrt{\xi}x_1(\zeta\xi)}{2\sqrt{1-\xi}}\right)\mathrm{d}\xi.\end{aligned} \tag{42}$$

If we follow a similar procedure to that used for the call with $x^\dagger > \hat{x}$, we find that once again $f(\zeta) = -\ln\zeta$, with the sign of $x_1^{(0)}$ reversed, $x_1^{(0)} = \sqrt{2}$, and

$$x_1^{(1)} = -2^{1/2}\ln\left(-\pi^{1/2}\left[\frac{\sigma\left(1+\mathrm{e}^{-b\hat{\tau}}\right)}{b^{3/2}} - \hat{x}\right]\right). \tag{43}$$

### 4.5. Call with $x^\dagger = \hat{x}$

The free boundary starts at $x_0 = \hat{x} = x^\dagger = \frac{\sigma}{b^{3/2}}\left(1+e^{-b\hat{\tau}}\right)$. If we try a series of the form (25), we arrive at the same pair of equations (31, 32) as for the call with $x^\dagger > \hat{x}$, which once again have the solution $x_1 = -\infty$, which again means that we need to use a series for $x_f(\zeta)$ of the form (33), (34) rather than (25). Using the series (33), (34), we need to balance the leading order terms in (24), so that

$$\begin{aligned}\zeta^{1/2}&\left[x_1(\zeta)\operatorname{erfc}\left(-\frac{x_1(\zeta)}{2}\right) + \frac{2}{\pi^{1/2}}\,\mathrm{e}^{-x_1^2(\zeta)/4}\right]\\ &\sim \frac{\zeta^{3/2}}{2}\left[\frac{\sigma^2\left(1-\mathrm{e}^{-b\hat{\tau}}\right)}{2b^3} - 1 - \frac{a}{b^2}\right]\\ &\quad\times\int_0^1\left[x_1(\zeta)\operatorname{erfc}\left(-\frac{x_1(\zeta)-\sqrt{\xi}x_1(\zeta\xi)}{2\sqrt{1-\xi}}\right)\right.\\ &\quad\left.+\frac{2\sqrt{1-\xi}}{\sqrt{\pi}}\exp\left(-\left[\frac{x_1(\zeta)-\sqrt{\xi}x_1(\zeta\xi)}{2\sqrt{1-\xi}}\right]^2\right)\right]\mathrm{d}\xi,\end{aligned} \tag{44}$$

where again we have written $\eta = \zeta\xi$. The left hand side of (44) is the same as (35), but the right hand side is different and involves $\zeta^{3/2}$ rather than $\zeta$. This is because the $\mathcal{O}(\zeta)$ term on the right hand side vanishes when the free boundary starts from $x^\dagger$, as was the case for the call with $x^\dagger < \hat{x}$, and indeed if we set $x_1(\zeta\xi) = x_1(\zeta) = x_1$, then we recover (27) from the case $x^\dagger < \hat{x}$. To evaluate the right hand side of (35), we make the same change of variable $\xi = 1 - \nu/f(\zeta)$ as for the call with $x^\dagger > \hat{x}$, which will again enable us to strip the $\zeta$ dependence out of the integral, and in the limit, (44) becomes

$$\begin{aligned}&x_1(\zeta)\operatorname{erfc}\left(-\frac{x_1(\zeta)}{2}\right)+\frac{2}{\pi^{1/2}}\,\mathrm{e}^{-x_1^2(\zeta)/4}\\&\sim\left[\frac{\sigma^2\left(1-\mathrm{e}^{-b\hat{\tau}}\right)}{2b^3}-1-\frac{a}{b^2}\right]\frac{4\zeta}{x_1^{(0)}\sqrt{f(\zeta)}}.\end{aligned}\tag{45}$$

It is worth mentioning that the right hand side of (45) involves $\zeta/f^{1/2}(\zeta)$ while that of (37) involve $\zeta^{1/2}/f(\zeta)$, which will lead to a different, $f(\zeta)$. If we again use the behavior of $\operatorname{erfc} z$ as $z\to\infty$, (45) becomes

$$\left[\frac{\sigma^2\left(1-\mathrm{e}^{-b\hat{\tau}}\right)}{2b^3}-1-\frac{a}{b^2}\right]\frac{\zeta}{x_1^{(0)}\sqrt{f(\zeta)}}\sim\frac{\mathrm{e}^{-x_1^2/4}}{x_1^2\sqrt{\pi}},\tag{46}$$

or

$$\pi^{1/2}x_1^{(0)}\left[\frac{\sigma^2\left(1-\mathrm{e}^{-b\hat{\tau}}\right)}{2b^3}-1-\frac{a}{b^2}\right]\zeta\sqrt{f(\zeta)}\sim\mathrm{e}^{-x_1^{(0)2}f(\zeta)/4-x_1^{(0)}x_1^{(1)}/2},\tag{47}$$

which has a solution $x_1^{(0)}=-\sqrt{2}$, $f(\zeta)=W_L\left(\zeta^{-2}/2\right)$, where $W_L$ is the Lambert W function which obeys $W_L(x)\,\mathrm{e}^{W_L(x)}=x$, and

$$x_1^{(1)}=2^{1/2}\ln\left[\pi^{1/2}\left(1+\frac{a}{b^2}-\frac{\sigma^2\left(1-\mathrm{e}^{-b\hat{\tau}}\right)}{2b^3}\right)\right].\tag{48}$$

Again, this Lambert W behavior is what we might expect in this case because similar behavior occurs for American equity options when the dividend yield is equal to the risk-free rate [**25**]. We get the same result from the equation for rho at the free boundary.

### 4.6. Put with $x^\dagger=\hat{x}$

The free boundary starts at $x_0=\hat{x}=x^\dagger=\frac{\sigma}{b^{3/2}}\left(1+\mathrm{e}^{-b\hat{\tau}}\right)$, with the behavior of the free boundary given by (33, 34) close to expiry. This case is very similar to the call with $x^\dagger=\hat{x}$, and the counterpart of (44) is

$$\begin{aligned}&\zeta^{1/2}\left[x_1(\zeta)\operatorname{erfc}\left(\frac{x_1(\zeta)}{2}\right)-\frac{2}{\pi^{1/2}}\,\mathrm{e}^{-x_1^2(\zeta)/4}\right]\\&\sim\frac{\zeta^{3/2}}{2}\left[\frac{\sigma^2\left(1-\mathrm{e}^{-b\hat{\tau}}\right)}{2b^3}-1-\frac{a}{b^2}\right]\\&\quad\times\int_0^1\left[x_1(\zeta)\operatorname{erfc}\left(\frac{x_1(\zeta)-\sqrt{\xi}x_1(\zeta\xi)}{2\sqrt{1-\xi}}\right)\right.\\&\quad\left.-\frac{2\sqrt{1-\xi}}{\sqrt{\pi}}\exp\left(-\left[\frac{x_1(\zeta)-\sqrt{\xi}x_1(\zeta\xi)}{2\sqrt{1-\xi}}\right]^2\right)\right]\mathrm{d}\xi.\end{aligned}\tag{49}$$

If we follow a similar procedure to that used for the call with $x^\dagger = \hat{x}$, we find that once again $f(\zeta) = W_L\left(\zeta^{-2}/2\right)$, with the sign of $x_1^{(0)}$ reversed, $x_1^{(0)} = \sqrt{2}$, and

$$x_1^{(1)} = -2^{1/2} \ln \left[ -\pi^{1/2} \left( 1 + \frac{a}{b^2} - \frac{\sigma^2 \left(1 - \mathrm{e}^{-b\hat{\tau}}\right)}{2b^3} \right) \right]. \tag{50}$$

## 5. Discussion

In this paper, we have used an integral equation approach due to [**16, 20, 22, 26**] to study American trombone options on a zero coupon bond, studying both call and put options. In our analysis, we assumed that the spot interest rate $r$ obeyed a mean-reverting random walk described by the Vasicek model [**35**], and we used a change of variables [**4**] to transform the governing PDE into the diffusion equation, which enabled us to use the Green's function for the diffusion equation, and thereby write expressions involving integrals for the values of the American bond options. These expressions are similar in form to those for American equity options [**16, 20**], and as in that study, our expressions are the sum of the corresponding European together with a term representing the value of early exercise.

We then applied the conditions that the value of the option and that of its rho, or derivative with respect to interest rate, must be continuous across the free boundary to these expressions to obtain integral equations for the location of the free boundary for the options, and once again, these equations, which are Volterra equations of the second kind [**20, 22, 29**], are similar to those for American equity options [**16, 20**].

These integral equations were then solved close to expiry, and we found that there were three possible behaviors for the free boundary $x_f(\zeta)$ in this limit. In terms of $\zeta$, the transformed time remaining until expiry, these behaviors were

$$x_f \sim x_0 + \begin{cases} x_1 \zeta^{1/2} + \mathcal{O}(\zeta) \\ \sqrt{-\zeta \ln \zeta} \left[ \pm\sqrt{2} + x_1^{(1)} \left(-\ln \zeta\right)^{-1} + \cdots \right] + \mathcal{O}(\zeta) \\ \sqrt{2\zeta W_L\left(\frac{\zeta^{-2}}{2}\right)} \left[ \pm 1 + x_1^{(1)} \left[ 2 W_L\left(\frac{\zeta^{-2}}{2}\right) \right]^{-1} \right] + \mathcal{O}(\zeta). \end{cases} \tag{51}$$

These same three behaviors occur for American equity put and call options [**1, 2, 6, 9, 12, 21, 23, 25, 31**]. In one sense, this similarity between American bond and equity options is surprising because interest rates obey a rather different random walk to equity prices, and indeed are mean-reverting. In another sense, it is not surprising as it is possible to transform the Vasicek PDE into the diffusion equation [**4**] and subsequently into the Black-Scholes-Merton PDE governing equity derivatives, which implies American bond options under the Vasicek model are equivalent to some sort of exotic American equity option, and the behavior of that exotic might be expected to behave similarly to other exotic American equity options as well as vanilla calls and puts, and in studies of such options it appears that the first of three behaviors in (51), namely the $\zeta^{1/2}$ behavior, prevails when both $V$ and $(\partial V/\partial r)$ (or $V$ and $(\partial V/\partial S)$ for equity options) are continuous at the free boundary at expiry, while the second, the $\sqrt{\zeta \ln \zeta}$ behavior, prevails when $(\partial V/\partial r)$ or $(\partial V/\partial S)$ is discontinuous there, and the third, the $\sqrt{\zeta W_L(\zeta^{-2}/2)}$ behavior, occurs on the boundary between the other two cases.

Although the behaviors in (51) occur both here for American bond options and elsewhere for equity options with American-style features, it should be recalled that we used a a change of variables to transform the Vasicek PDE into the diffusion equation using (9), so that in the original variables, the free boundaries for American bond options will of course look somewhat different to those for American call and put equity options.

In closing, we would recall that the results presented here assume that interest rates are governed by the Vasicek model, and it would be interesting to see whether similar results hold using other models.

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