A NOTE ON CERTAIN MATRICES WITH h(x)-FIBONACCI POLYNOMIALS

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ABSTRACT. In this paper, it is considered a g-circulant, right circulant, left circulant and a special kind of tridiagonal matrices whose entries are h(x)-Fibonacci polynomials. The determinant of these matrices is established and with the tridiagonal matrices we show that the determinant is equal to the *n*th term of the h(x)-Fibonacci polynomials.

1. INTRODUCTION

For a natural number n we consider a *g*-circulant matrix as square matrix of order n with the following form

(1)
$$A_{g,n} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g} \\ a_{n-2g+1} & a_{n-2g+2} & \cdots & a_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g+1} & a_{g+2} & \cdots & a_g \end{pmatrix},$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n. The first row of $A_{g,n}$ is (a_1, a_2, \dots, a_n) and its (j + 1)th row is obtained by giving its *j*th row a right circular shift by g positions.

Note that g = 1 or g = n + 1 yields the standard *right circulant* matrix, or simply, *circulant* matrix. Thus a right circulant matrix is written as

(2)
$$\operatorname{RCirc}(a_1, a_2, \cdots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.$$

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If g = n - 1, we obtain the so called *left circulant* matrix or *reverse circulant* matrix. In this case we write a left circulant matrix as

(3)
$$\operatorname{LCirc}(a_1, a_2, \cdots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \vdots & \vdots & & \vdots \\ a_n & a_1 & \cdots & a_{n-1} \end{pmatrix}$$

The history of circulant matrices is a long one (see, for example, [6], [10], [12], [19] and [24]). All types of circulant matrices arise in the study of periodic or multiply symmetric dynamical systems and play a crucial role for solving various differential equations (see, for example, [1], [7], [16] and [23]). These matrices were exploited to obtain the transient solution in closed form for fractional order differential equations (see, for example, [1]). Wu and Zou in [23] discussed the existence and approximation of solutions of asymptotic or periodic boundary value problems of mixed functional differential equations. In the recent years, there have been several papers on several types of circulant matrices (see, for example, [3], [11], [17], [18] and [20]). Some authors study these type of matrices whose entries are integers sequences defined recursively. This is, for example, the case of [11], [17], [18] and [22] where the authors considered circulant matrices with the Fibonacci and Lucas numbers, the case of [3] and [9] where the entries of the circulant matrices are Jacobsthal and Jacobsthal-Lucas numbers, and the case of [20] and [21] where the k-Horadam numbers are considered as entries of circulant matrices.

In this paper, we consider a g-circulant, right and left circulant matrices whose entries are kind of polynomials instead of numbers. The Fibonacci polynomials are polynomials that can be defined by Fibonacci-like recursion relations were studied in 1883 by E. C. Catalan and E. Jacobsthal. For example, E. C. Catalan studied the polynomials $F_n(x)$ defined by the recurrence relation

(4)
$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \ge 3,$$

where $F_1(x) = 1$ and $F_2(x) = x$. This is an example of several polynomial sequences that can be defined by recurrence relations of order two. Many mathematicians were involved in the study of Fibonacci polynomials such as P. F. Byrd, M. Bicknell-Johnson among others.

Let h(x) be a polynomial with real coefficients. Nalli and Haukkanen [15] introduced h(x)-Fibonacci polynomials that generalize both Catalan's Fibonacci polynomials and Byrd's Fibonacci polynomials. In their paper, the h(x)-Fibonacci polynomials $\{F_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

(5)
$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \ge 1,$$

with the initial conditions $F_{h,0}(x) = 0$ and $F_{h,1}(x) = 1$.

For h(x) = x, we obtain Catalan's Fibonacci polynomials (4) and for h(x) = 2x, we obtain Byrd's Fibonacci polynomials. Moreover, for h(x) = k, k any real number, we obtain the k-Fibonacci numbers studied by several researchers (see, for example, [2], [4], [5] and [8] among others) and in particular, for k = 1 and k = 2, we obtain the sequences of Fibonacci numbers and Pell numbers, respectively.

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This paper is organized as follows: In Section 2, we consider the g-circulant, right circulant and left circulant matrices whose entries are h(x)-Fibonacci polynomials and present the determinant of these matrices. In Section 3, we consider a special kind of tridiagonal matrices whose entries are also h(x)-Fibonacci polynomials and show that the determinant is equal to the *n*th term of the h(x)-Fibonacci polynomial sequence. We end this paper with some conclusions and plans for further investigation.

2. Circulant type matrices with h(x)-Fibonacci polynomials

Let $A_n(x) = \operatorname{RCirc}(F_{h,1}(x), F_{h,2}(x), \dots, F_{h,n}(x))$ be a right circulant matrix. We give a new expression for det $A_n(x)$ following the idea of Gong, Jiang and Gao in [9].

Theorem 2.1. For $n \ge 1$, let $A_n(x) = \operatorname{RCirc}(F_{h,1}(x), F_{h,2}(x), \cdots, F_{h,n}(x))$ be a right circulant matrix. Then we have

(6) det
$$A_n(x) = (1 - F_{h,n+1}(x))^{n-1} + F_{h,n}(x)^{n-2} \sum_{k=1}^{n-1} (\frac{1 - F_{h,n+1}(x)}{F_{h,n}(x)})^{k-1} F_{h,k}(x).$$

Proof. For n = 1, det $A_1(x) = 1$ satisfies the formula (6). In the case $n \ge 2$, we consider the following square matrices of order n of common use in the theory of circulant matrices

$$\Gamma(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -h(x) & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -h(x) \\ 0 & 0 & 0 & 0 & \cdots & 1 & -h(x) & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -h(x) & \cdots & 0 & 0 & 0 \\ 0 & 1 & -h(x) & -1 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and

$$\Pi(x) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{F_{h,n}(x)}{1 - F_{h,n+1}(x)}\right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left(\frac{F_{h,n}(x)}{1 - F_{h,n+1}(x)}\right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{F_{h,n}(x)}{1 - F_{h,n+1}(x)} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that

$$\det \Gamma(x) = \det \Pi(x) = (-1)^{\frac{(n-1)(n-2)}{2}}.$$

(7)

Considering the product $\Gamma(x)A_n(x)\Pi(x)$ of matrices, we obtain the following matrix

$$C = \begin{pmatrix} F_{h,1}(x) & \alpha_n & F_{h,n-1}(x) & \cdots & F_{h,3}(x) & F_{h,2}(x) \\ 0 & \beta_n & F_{h,n-2}(x) & \cdots & F_{h,2}(x) & F_{h,1}(x) \\ 0 & 0 & F_{h,1}(x) - F_{h,n+1}(x) & & & \\ 0 & 0 & -F_{h,n}(x) & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & & & -F_{h,n}(x) & F_{h,1}(x) - F_{h,n+1}(x) \end{pmatrix},$$

where

$$\alpha_n = \sum_{k=1}^{n-1} \left(\frac{F_{h,n}(x)}{F_{h,1}(x) - F_{h,n+1}(x)}\right)^{n-(k+1)} F_{h,k+1}(x)$$

and

(8)
$$\beta_n = (F_{h,1}(x) - F_{h,n+1}(x)) + \sum_{k=1}^{n-1} (\frac{F_{h,n}(x)}{F_{h,1}(x) - F_{h,n+1}(x)})^{n-(k+1)} F_{h,k}(x).$$

Next we calculate the determinant of the matrix $C = \Gamma(x)A_n(x)\Pi(x)$ and obtain

(9)
$$\det C = F_{h,1}(x)\beta_n(F_{h,1}(x) - F_{h,n+1}(x))^{n-2}.$$

Now, using the identity (8), the recurrence relation (5) with the initial conditions, and doing some calculations, a new expression for det C is obtained as follows

(10) det
$$C = (1 - F_{h,n+1}(x))^{n-1} + F_{h,n}(x)^{n-2} \sum_{k=1}^{n-1} (\frac{1 - F_{h,n+1}(x)}{F_{h,n}(x)})^{k-1} F_{h,k}(x).$$

Using the property of the determinant of a product of matrices and the identity (7), we conclude that

$$\det A_n = \det C$$

and the result follows.

Let $B_n(x) = \text{LCirc}(F_{h,1}(x), F_{h,2}(x), \dots, F_{h,n}(x))$ be a *left circulant* matrix which entries are h(x)-Fibonacci polynomials. Next we give a new expression for det $B_n(x)$ following once more the idea used by Gong, Jiang and Gao in [9] and [13] that helps us to obtain the determinant of $B_n(x)$. In [13, Lemma 5], the authors define the following matrix

(11)
$$\Delta := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

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that is an orthogonal cyclic shift matrix (and a left circulant matrix) of order n. The main result is

(12)
$$\operatorname{LCirc}(a_1, a_2, \cdots, a_n) = \Delta \operatorname{RCirc}(a_1, a_2, \cdots, a_n).$$

Using the fact of det $\Delta = (-1)^{\frac{(n-1)(n-2)}{2}}$, calculating the determinant in both sides of the identity (12) and according the result obtained in Theorem 2.1, the following result is easily proved.

Theorem 2.2. For $n \ge 1$, let $B_n(x) = \operatorname{LCirc}(F_{h,1}(x), F_{h,2}(x), \cdots, F_{h,n}(x))$ be a left circulant matrix. Then we have

(13)
$$\det B_n(x) = (-1)^{\frac{(n-1)(n-2)}{2}} (1 - F_{h,n+1}(x))^{n-1} + F_{h,n}(x)^{n-2} \sum_{k=1}^{n-1} (\frac{1 - F_{h,n+1}(x)}{F_{h,n}(x)})^{k-1} F_{h,k}(x)$$

Now let $C_n(x) = A_{g,n}$ be a *g*-circulant matrix as in (1), which entries are h(x)-Fibonacci polynomials. In order to obtain a new expression for det $C_n(x)$, we use the following results of [13].

Lemma 2.3. ([13, Lemma 6]) The $n \times n$ matrix \mathbb{Q}_g is unitary if and only if (n,g) = 1, where \mathbb{Q}_g is a g-circulant matrix with the first row $e^* = [1, 0, \dots, 0]$.

Lemma 2.4. ([13, Lemma 7]) $A_{g,n}$ is a g-circulant matrix with the first row $[a_1, a_2, \dots, a_n]$ if and only if $A_{g,n} = \mathbb{Q}_g \operatorname{RCirc}(a_1, a_2, \dots, a_n)$.

From these Lemmas and Theorem 2.1, we deduce the following result.

Theorem 2.5. Let $C_n(x) = A_{g,n}$ be a g-circulant matrix as in (1), which entries are h(x)-Fibonacci polynomials. Then one has

(14)
$$\det C_n(x) = \det \mathbb{Q}_g[(1 - F_{h,n+1}(x))^{n-1} + F_{h,n}(x)^{n-2} \sum_{k=1}^{n-1} (\frac{1 - F_{h,n+1}(x)}{F_{h,n}(x)})^{k-1} F_{h,k}(x)].$$

3. TRIDIAGONAL MATRICES WITH h(x)-FIBONACCI POLYNOMIALS

Following the ideas of Falcón in [8], we have that the determinant of a special kind of tridiagonal matrices is related to a special *n*-th order polynomial. If we consider the $(n \times n)$ tridiagonal matrices M_n , defined as

(15)
$$\begin{pmatrix} a & b & & & & \\ c & d & e & & & \\ & c & d & e & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & c & d & e \\ & & & & & c & d \end{pmatrix}$$

and compute the sequence of determinants, we obtain:

$$|M_{1}| = a$$

$$|M_{2}| = d |M_{1}| - bc$$

$$|M_{3}| = d |M_{2}| - ce |M_{1}|$$

$$|M_{4}| = d |M_{3}| - ce |M_{2}|$$

$$\vdots$$

$$|M_{n+1}| = d |M_{n}| - ce |M_{n-1}|$$

and therefore we can easily obtain the following result

Proposition 1. The $(n \times n)$ tridiagonal matrices

$$F_n^h(x) = \begin{pmatrix} h(x) & -1 & & & \\ 1 & h(x) & -1 & & & \\ & 1 & h(x) & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & h(x) & -1 \\ & & & & & 1 & h(x) \end{pmatrix}$$

satisfy

$$\left|F_{n-1}^{h}(x)\right| = F_{h,n}(x),$$

that is, the n-th h(x)-Fibonacci polynomial may be obtained through the computation of the determinant of the $((n-1) \times (n-1))$ tridiagonal matrix $F_{n-1}^h(x)$.

Proof. If we consider a = d = h(x), b = e = -1 and c = 1, it is straightforwardly seen that the sequence of determinants becomes:

$$|M_{1}| = |F_{1}^{h}(x)| = h(x) = F_{h,2}(x)$$

$$|M_{2}| = |F_{2}^{h}(x)| = 1 + (h(x))^{2} = F_{h,3}(x)$$

$$|M_{3}| = |F_{3}^{h}(x)| = (h(x))^{3} + 2h(x) = F_{h,4}(x)$$

$$\vdots$$

$$|M_{n-1}| = |F_{n-1}^{h}(x)| = F_{h,n}(x),$$

as required.

Another way of relating the *n*th order h(x)-Fibonacci polynomial as the computation of a tridiagonal matrix may be obtained using the ideas of [14], where the following result was presented

Theorem 3.1. Let $\{x_n\}_n$ be any second order linear sequence defined recursively as

$$x_{n+1} = Ax_n + Bx_{n-1}, \quad n \ge 1,$$

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with $x_0 = C$, $x_1 = D$. Then, for all $n \ge 0$

$$x_n = \begin{vmatrix} C & D & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & B & 0 & \cdots & 0 & 0 \\ 0 & -1 & A & B & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A & B \\ 0 & 0 & 0 & 0 & \cdots & -1 & A \end{vmatrix}_{(n+1)\times(n+1)}$$

In the case of the h(x)-Fibonacci polynomials sequence, we have A = h(x), B = D = 1 and C = 0, and then, a direct application of Theorem 3.1 leads to the following proposition

Proposition 2. For $n \ge 0$, we have

$$F_{h,n}(x) = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & h(x) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & h(x) & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & h(x) \end{vmatrix}_{(n+1)\times(n+1)}$$

4. Conclusions

Several studies involving all types of circulant matrices and tridiagonal matrices can be easily found in the literature. Here we have considered the *g*-circulant, right and left circulant matrices whose entries are h(x)-Fibonacci polynomials. For these cases we have provided the determinant of these matrices. Some kind of tridiagonal matrices whose entries are h(x)-Fibonacci polynomials have been considered and we presented a different way, to obtain the *n*-th term of the h(x)-Fibonacci polynomial sequence.

In the future, we intend to discuss the invertibility of these circulant type matrices associated with these polynomials, such as, for example, the work of Shen [17] in the case of Fibonacci and Lucas numbers, Yazlik [21] with Generalized k-Horadam numbers, and Bozkurt [3] with Jacobsthal and Jacobsthal-Lucas numbers, among others.

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