ON EXTENDED GENERALIZED $\phi$-RECURRENT TRANS-SASAKIAN MANIFOLDS

A. S. YADAV AND J. P. JAISWAL

Abstract. The object of the present paper is to study the extended generalized $\phi$-recurrent trans-Sasakian manifold and its various geometric properties.

1. Introduction

A new class of almost contact manifolds was introduced by Oubina [8], called trans-Sasakian manifolds, which are of the type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ and called the cosympletic, $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds, respectively. Here $\alpha$, $\beta$, are scalar smooth functions. In particular, if $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 0$ then the trans-Sasakian manifold becomes Kenmotsu and Sasakian manifold, respectively.

The idea of local symmetry of a Riemannian manifold started with the study of Cartan [1]. This idea has been worked up by many authors in different directions such as recurrent manifold by Walker [17], semi-symmetric manifold by Szabo [14], pseudo-symmetric manifold by Chaki [2], pseudo-symmetric spaces by Deszcz [6], weakly symmetric manifold by Tamassy and Binh [16], weakly symmetric Riemannian spaces by Selberg [11]. Despite, the idea of pseudo-symmetry by Chaki and Deszcz and weak symmetry by Selberge, Tamassy and Binh are not the same. As a mild version of local symmetry, Takahashi [15] introduced the notion of $\phi$-symmetry on a Sasakian manifold. For generalizing the idea of $\phi$-symmetry, De et al. [4] introduced the concept of $\phi$-recurrent Sasakian manifold. The notion of generalized recurrent manifolds was initiated by Dubey [7]. Extending the notion of generalized $\phi$-recurrent, Shaikh and Hui [12] introduced the concept of extended generalized $\phi$-recurrent manifolds. The extended generalized $\phi$-recurrent property in Sasakian manifold was considered by Prakasha [9]. The aim of this paper is to study the extended generalized $\phi$-recurrent trans-Sasakian manifolds.

In this paper, we study an extended generalized $\phi$-recurrent trans-Sasakian manifold. The paper is organized as follows: In section 2, we give the brief introduction of trans-Sasakian manifold and some relevant definitions. In the next section, we discuss a generalized $\phi$-recurrent trans-Sasakian manifold and obtain a sufficient condition for such a manifold to be Einstein, super generalized Ricci-recurrent and Ricci-symmetric.

Received March 8, 2016.

2010 Mathematics Subject Classification. Primary 53C15, 53B05.

Key words and phrases. Trans-Sasakian manifold; extended generalized $\phi$-recurrent; Einstein manifold; $\phi$-symmetry; generalized Ricci-recurrent.
2. Preliminaries

In this section, we recall some basic definitions and basic formulae which we use later.

Let $M$ be an $m = (2n + 1)$ dimensional almost contact metric manifold [5, 10] equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a $(1, 1)$ tensor field $\phi$, a characteristic vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$. Then

\begin{align*}
\phi^2 X = & -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \quad \phi\xi = 0, \\
g(\phi X, \phi Y) = & g(X, Y) - \eta(X)\eta(Y), \\
g(\xi, \xi) = & 1, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0
\end{align*}

for any $X, Y$ in $TM$. From (2.1) and (2.2), it can be easily seen that

\begin{align*}
g(X, \phi Y) = & -g(\phi X, Y), \\
g(X, \xi) = & \eta(X).
\end{align*}

For an almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$, we put

\begin{equation}
\Phi(X, Y) = g(X, \phi Y).
\end{equation}

In the almost contact metric structure, $\Phi$ is the fundamental 2-form and has rank $2n$, we have $\eta \wedge \Phi^n \neq 0$.

An almost contact metric structure with $\Phi = d\eta$ is a contact metric structure and a manifold with such a structure is called a contact metric manifold.

Let $M$ be almost contact manifold and consider the structure $(M \times \mathbb{R}, J, G')$ belongs to the class $W_4$ of the Hermitian manifolds, we denote a vector field on $M \times \mathbb{R}$ by $(X, f \frac{d}{dt})$, where $X$ is tangent to $M$, $t$ is the co-ordinates of $\mathbb{R}$ and $f$ is $c^\infty$ function on $M \times \mathbb{R}$. Define an almost complex structure [3]

\begin{equation}
J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})
\end{equation}

for any vector field $X$ on $M \times \mathbb{R}$, and $G'$ as Hermitian metric on the product $M \times \mathbb{R}$. This may be expressed by the condition

\begin{equation}
(\nabla_X \phi) Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),
\end{equation}

where $\nabla$ is a Levi-Civita connection and $\alpha, \beta$ are some smooth functions on $M$. We say that trans-Sasakian structure is type $(\alpha, \beta)$. From the above it follows that

\begin{align*}
(\nabla_X \eta) Y = & -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \\
(\nabla_X \xi) = & -\alpha \phi X + \beta (X - \eta(X)\xi).
\end{align*}

On trans-Sasakian manifold $M$ with structure $(\phi, \xi, \eta, g)$, the following relations hold [5, 10]

\begin{equation}
R(X, Y, \xi) = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y]
+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y;
\end{equation}

\begin{align*}
R(\xi, X, \xi) = & (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X], \\
2\alpha\beta + \xi\alpha = & 0,
\end{align*}

In the almost contact metric structure, $\Phi$ is the fundamental 2-form and has rank $2n$, we have $\eta \wedge \Phi^n \neq 0$. An almost contact metric structure with $\Phi = d\eta$

is a contact metric structure and a manifold with such a structure is called a contact metric manifold.

Let $M$ be almost contact manifold and consider the structure $(M \times \mathbb{R}, J, G')$ belongs to the class $W_4$ of the Hermitian manifolds, we denote a vector field on $M \times \mathbb{R}$ by $(X, f \frac{d}{dt})$, where $X$ is tangent to $M$, $t$ is the co-ordinates of $\mathbb{R}$ and $f$ is $c^\infty$ function on $M \times \mathbb{R}$. Define an almost complex structure [3]

\begin{equation}
J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})
\end{equation}

for any vector field $X$ on $M \times \mathbb{R}$, and $G'$ as Hermitian metric on the product $M \times \mathbb{R}$. This may be expressed by the condition

\begin{equation}
(\nabla_X \phi) Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),
\end{equation}

where $\nabla$ is a Levi-Civita connection and $\alpha, \beta$ are some smooth functions on $M$. We say that trans-Sasakian structure is type $(\alpha, \beta)$. From the above it follows that

\begin{align*}
(\nabla_X \eta) Y = & -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \\
(\nabla_X \xi) = & -\alpha \phi X + \beta (X - \eta(X)\xi).
\end{align*}

On trans-Sasakian manifold $M$ with structure $(\phi, \xi, \eta, g)$, the following relations hold [5, 10]

\begin{equation}
R(X, Y, \xi) = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y]
+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y;
\end{equation}

\begin{align*}
R(\xi, X, \xi) = & (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X], \\
2\alpha\beta + \xi\alpha = & 0,
\end{align*}
ON EXTENDED $\phi$-RECURRENT TRANS-SASAKIAN MANIFOLDS

In the context of a trans-Sasakian manifold, the following relations hold:

\begin{equation}
\eta(R(X,Y,\xi)) = \eta(R(\xi,Y,\xi)) = 0,
\end{equation}

\begin{equation}
R(\xi,Y,Z) = (\alpha^2 - \beta^2)[g(Z,Y)\xi - \eta(Z)Y] + 2\alpha\beta[g(\phi Z,Y)\xi
\end{equation}

\begin{equation}
- \eta(Z)\phi Y] + (Z\alpha)\phi Y + g(\phi Z,Y)\text{grad} \alpha
\end{equation}

\begin{equation}
+ (X\beta)[Y - \eta(Y)\xi] - g(\phi Z,\phi Y),
\end{equation}

\begin{equation}
S(X,\xi) = 2n(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (2n - 1)X\beta - (\phi X)\alpha,
\end{equation}

\begin{equation}
S(\xi,\xi) = 2n(\alpha^2 - \beta^2) - \xi\beta]
\end{equation}

\begin{equation}
S(\phi X,\phi Y) = S(X,Y) - 2n(\alpha^2 - \beta^2)\eta(X)\eta(Y),
\end{equation}

\begin{equation}
Q\xi = 2n(\alpha^2 - \beta^2) - \xi\beta] - (2n - 1)\text{grad} \beta + \phi(\text{grad} \alpha),
\end{equation}

where $R$ is the curvature tensor, $S$ is the Ricci tensor, $r$ is scalar curvature and $Q$ is the symmetric endomorphism of the tangent space at each point corresponding to Ricci-tensor $S$.

If we assume

\begin{equation}
\phi(\text{grad} \alpha) = (2n - 1)\text{grad} \beta,
\end{equation}

then \cite{5, 10}

\begin{equation}
S(X,\xi) = 2n(\alpha^2 - \beta^2)\eta(X),
\end{equation}

\begin{equation}
S(\phi X,\phi Y) = S(X,Y) - 2n(\alpha^2 - \beta^2)\eta(X)\eta(Y),
\end{equation}

\begin{equation}
Q\xi = 2n(\alpha^2 - \beta^2)\xi.
\end{equation}

Now mention some definitions are be considered later.

**Definition 2.1** \cite{3}. A Riemannian manifold $M$ is said to be $\phi$-symmetric if the curvature tensor $R$ satisfies the relation

\begin{equation}
\phi^2((\nabla_W R)(X,Y,Z)) = 0
\end{equation}

for all $X,Y$ and $Z \in TM$.

**Definition 2.2** \cite{3}. A Riemannian manifold $M$ is said to be Ricci-symmetric if the Ricci-tensor $S$ satisfies the relation

\begin{equation}
(\text{grad} \phi(\nabla_W S)(X,Y) = 0
\end{equation}

for all $X,Y$ and $W \in TM$.

**Definition 2.3** \cite{13}. A Riemannian manifold $M$ is said to be generalized Ricci-recurrent if the Ricci-tensor $S$ satisfies the relation

\begin{equation}
(\nabla_W S)(X,Y) = A'(W)S(X,Y) + B'(W)g(X,Y)
\end{equation}

for all $X,Y$ and $W \in TM$, $A'$ and $B'$ are two 1-forms.

Now, mention the following Lemma.

**Lemma 2.1.** In a trans-Sasakian manifold if $\phi(\text{grad} \alpha) = (2n - 1)\text{grad} \beta$, then

\begin{equation}
(\nabla_W S)(Y,\xi) = 2n(\alpha^2 - \beta^2)[-\alpha g(Y,\phi W) + \beta g(Y,\phi W)]
\end{equation}

\begin{equation}
+ \alpha S(Y,\phi W) - \beta S(Y,\phi W).
\end{equation}
Using the relations (2.2), (2.7) and (2.19), we can easily derive expression (2.26).

\[ \phi^2((\nabla_W S)(X, Y, Z)) = A(W)\phi^2(R(X, Y, Z)) + B(W)\phi^2(G(X, Y, Z)), \]

where \( A \) and \( B \) are two 1-forms, \( B \) is non zero, defined by

\[ g(W, \rho_1) = A(W), \quad g(W, \rho_2) = B(W), \]

\[ G(X, Y, Z) = g(Y, Z)X - g(X, Z)Y \]

for all \( X, Y, Z, W \in TM \) and \( \rho_1, \rho_2 \) being vector fields associated with the 1-form \( A \) and \( B \), respectively.

In this section, we start with the lemma which statement is as follows:

**Lemma 3.1.** If an extended generalized \( \phi \)-recurrent trans-Sasakian manifold \( M \) satisfies \( \phi(\text{grad}\,\alpha) = (2n - 1)\text{grad}\,\beta \), then the following relation holds

\[ (\alpha^2 - \beta^2)A(W) + B(W) = 0. \]

**Proof.** Let us consider that \( M \) is an extended generalized \( \phi \)-recurrent trans-Sasakian manifold. Then by virtue of relation (2.1), the equation (3.1) becomes

\[ (\nabla_W R)(X, Y, Z) - \eta((\nabla_W R)(X, Y, Z))\xi = A(W)[R(X, Y, Z) - \eta(R(X, Y, Z))\xi] + B(W)[G(X, Y, Z) - \eta(G(X, Y, Z))\xi]. \]

From which is follows that

\[ g((\nabla_W R)(X, Y, Z), U) - g((\nabla_W R)(X, Y, Z), \xi)g(U, \xi) = A(W)[g(R(X, Y, Z), U) - g(R(X, Y, Z), \xi)g(U, \xi)] + B(W)[g(G(X, Y, Z), U) - g(G(X, Y, Z), \xi)g(U, \xi)]. \]

Let us suppose \( \{e_1, e_2, \ldots, e_{2n+1}\} \) is an orthonormal basis of the tangent space at any point of the manifold. Setting \( X = U = e_i \) in the relation (3.4) and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we obtain

\[ (\nabla_W S)(Y, Z) - \eta((\nabla_W R)(\xi, Y, Z)) = A(W)[S(Y, Z) - \eta(R(\xi, Y, Z))] + B(W)[(2n - 1)g(Y, Z) + \eta(Y)\eta(Z)]. \]

Putting \( Z = \xi \) in the above equation, we find

\[ (\nabla_W S)(Y, \xi) - \eta((\nabla_W R)(\xi, Y, \xi)) = A(W)[S(Y, \xi) - \eta(R(\xi, Y, \xi))] + 2nB(W)\eta(Y). \]
By virtue of the Lemma 2.1 and then using equations (2.12) and (2.20), we have
\[ 2n(\alpha^2 - \beta^2)[-\alpha g(Y,\phi W) + \beta g(Y,W)] + \alpha S(Y,\phi W) - \beta S(Y,W) \]
(3.7)
\[ = [2n(\alpha^2 - \beta^2)\phi Y + 2nB(W)]\eta(Y). \]
Replacing Y by \( \xi \) in equation (3.7) and using the relations (2.3), (2.4), (2.19) and (2.22), we can get the expression (3.2).

**Theorem 3.1.** If an extended generalized \( \phi \)-recurrent trans-Sasakian manifold \( M \) satisfies \( \phi(\text{grad} \alpha) = (2n - 1) \text{grad} \beta \), then it is an Einstein manifold.

**Proof.** Let us consider \( M \) is an extended generalized \( \phi \)-recurrent trans-Sasakian manifold. By virtue of the Lemma 3.1 and the equation (3.7), we have
\[ 2n(\alpha^2 - \beta^2)[-\alpha g(Y,\phi W) + \beta g(Y,W)] + \alpha S(Y,\phi W) - \beta S(Y,W) = 0. \]
Now replacing \( Y \) and \( W \) by \( \phi Y \) and \( \phi W \), respectively, in the above relation and then using relations (2.1), (2.4), (2.18), (2.19) and (2.22), we obtain
\[ S(Y,W) = 2n(\alpha^2 - \beta^2)g(Y,W) \]
and
\[ S(\phi Y,W) = 2n(\alpha^2 - \beta^2)g(\phi Y,W), \]
which proves the result.

**Theorem 3.2.** If an extended generalized \( \phi \)-recurrent trans-Sasakian manifold \( M \) satisfies \( \phi(\text{grad} \alpha) = (2n - 1) \text{grad} \beta \), then \( 2n(\alpha^2 - \beta^2) \) is the eigen value of the Ricci-tensor corresponding to the eigen vector \( \rho \), where \( \rho \) is the associated vector field of the 1-form \( \phi \).

**Proof.** By virtue of the Bianchi's identity and then using equation (3.3), we get
\[ A(W)[R(X,Y,Z) - \eta(R(X,Y,Z))\xi] + B(W)[G(X,Y,Z) - \eta(G(X,Y,Z))\xi] \]
\[ + \eta(G(X,Y,Z))\xi] + A(X)[R(Y,W,Z) - \eta(R(Y,W,Z))\xi] \]
\[ = 0. \]
(3.10)
Applying inner product to both side by \( U \), then setting \( Y = Z = e_i \) in the above equation, taking summation over \( i \), \( 1 \leq i \leq 2n + 1 \), and using the Lemma 3.1, we obtain
\[ -A(R(W,X,U)) + A(R(W,X,\xi))g(U,\xi) \]
\[ -(\alpha^2 - \beta^2)[A(X)g(W,U) - A(W)g(X,U)] \]
\[ -\eta(R(W,X,\xi))(\xi) + B(Y)[G(W,X,Z) - \eta(G(W,X,Z))\xi] = 0. \]
(3.11)
Again contracting over \( X \) and \( U \) and using relation (2.10), we get
\[ S(W,\rho) = 2n(\alpha^2 - \beta^2)A(W), \]
which proves the theorem.

**Theorem 3.3.** An extended generalized \( \phi \)-recurrent trans-Sasakian manifold \( M \) with \( \phi(\text{grad} \alpha) = (2n - 1) \text{grad} \beta \) is a generalized Ricci-recurrent manifold.
Since $M$ is an extended generalized $\phi$-recurrent trans-Sasakian manifold, hence the relation (3.4) holds. Taking contraction of the equation (3.4) over $Y$ and $Z$, we obtain

\begin{equation}
(\nabla_W S)(X, U) = [(\nabla_W S)(X, \xi) - A(W)S(X, \xi)]\eta(U)
+ A(W)S(X, U) + 2nB(W)g(X, U)
- 2nB(W)\eta(X)\eta(U).
\end{equation}

By virtue of the Lemma 2.1, equation (2.9), Theorem 3.1 and Lemma 3.1, the above expression becomes

\begin{equation}
(\nabla_W S)(X, U) = A(W)S(X, U) + 2nB(W)g(X, U),
\end{equation}

which finishes the proof. □

**Theorem 3.4.** If an extended generalized $\phi$-recurrent trans-Sasakian manifold $M$ is $\phi$-symmetric and satisfying $\phi(\text{grad}\, \alpha) = (2n - 1)\text{grad}\, \beta$, then it is Ricci-symmetric.

**Proof.** By virtue of the equation (3.1) and Definition 2.1, we take

\begin{equation}
\phi^2(\nabla_W R)(X, Y, Z) = 0.
\end{equation}

From the relation (2.1), it follows that

\begin{equation}
(\nabla_W R)(X, Y, Z) - \eta((\nabla_W R)(X, Y, Z))\xi = 0.
\end{equation}

Taking inner product and contracting over $Y$ and $Z$, we get

\begin{equation}
(\nabla_W S)(X, U) = (\nabla_W S)(X, \xi)\eta(U).
\end{equation}

By virtue of the Theorem 3.1 and using the Lemma 2.1, we obtain

\begin{equation}
(\nabla_W S)(X, U) = 0.
\end{equation}

Thus theorem is proved. □

**Theorem 3.5.** If an extended generalized $\phi$-recurrent trans-Sasakian manifold $M$ is locally $\phi$-symmetric, then the following relation holds

\begin{equation}
A(W)(\alpha^2 - \beta^2 - \xi\beta) + B(W) = 0
\end{equation}

for all $X, Y, W$ orthogonal to $\xi$.

**Proof.** Suppose $M$ is locally $\phi$-symmetric, then

\begin{equation}
\phi^2((\nabla_W R)(X, Y, Z)) = 0.
\end{equation}

By virtue of the equation (3.1) and then using the relation (2.1), we get

\begin{equation}
A(W)[R(X, Y, Z) - \eta(R(X, Y, Z))\xi]
+ B(W)[G(X, Y, Z) - \eta(G(X, Y, Z))\xi] = 0.
\end{equation}

Putting $X = Z = \xi$ and using the equation (2.10), we obtain

\begin{equation}
A(W)(\alpha^2 - \beta^2 - \xi\beta) + B(W)(\eta(Y)\xi - Y) = 0
\end{equation}

which implies that

\begin{equation}
A(W)(\alpha^2 - \beta^2 - \xi\beta) + B(W) = 0.
\end{equation}

And thus we have the result. □
Acknowledgment. The authors are thankful to the reviewer for his/her useful comments. The authors would also like to express their gratitude to the National Board for Higher Mathematics (Department of Atomic Energy), Mumbai, India (No. NBHM/R. P. 48/2012/Fresh/364) for financial support in the form of research project.

References


A. S. Yadav, Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal, M.P. India-462051, e-mail: arjunsinghyadav7@yahoo.com

J. P. Jaiswal, Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal, M.P. India-462051, e-mail: asstprofjmanit@gmail.com