ON EXTENDED GENERALIZED $\phi$-RECURRENT TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study the extended generalized $\phi$-recurrent trans-Sasakian manifold and its various geometric properties.

1. Introduction

A new class of almost contact manifolds was introduced by Oubina [8], called trans-Sasakian manifolds, which are of the type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ and called the cosympletic, $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds, respectively. Here $\alpha$, $\beta$, are scalar smooth functions. In particular, if $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 0$ then the trans-Sasakian manifold becomes Kenmotsu and Sasakian manifold, respectively.

The idea of local symmetry of a Riemannian manifold started with the study of Cartan [1]. This idea has been worked up by many authors in different directions such as recurrent manifold by Walker [17], semi-symmetric manifold by Szabo [14], pseudo-symmetric manifold by Chaki [2], pseudo-symmetric spaces by Deszcz [6], weakly symmetric manifold by Tamassy and Binh [16], weakly symmetric Riemannian spaces by Selberg [11]. Despite, the idea of pseudo-symmetry by Chaki and Deszcz and weak symmetry by Selberge, Tamassy and Binh are not the same. As a mild version of local symmetry, Takahashi [15] introduced the notion of $\phi$-symmetry on a Sasakian manifold. For generalizing the idea of $\phi$-symmetry, De et al. [4] introduced the concept of $\phi$-recurrent Sasakian manifold. The notion of generalized recurrent manifolds was initiated by Dubey [7]. Extending the notion of generalized $\phi$-recurrent, Shaikh and Hui [12] introduced the concept of extended generalized $\phi$-recurrent manifolds. The extended generalized $\phi$-recurrent property in Sasakian manifold was considered by Prakasha [9]. The aim of this paper is to study the extended generalized $\phi$-recurrent trans-Sasakian manifolds.

In this paper, we study an extended generalized $\phi$-recurrent trans-Sasakian manifold. The paper is organized as follows: In section 2, we give the brief introduction of trans-Sasakian manifold and some relevant definitions. In the next section, we discuss a generalized $\phi$-recurrent trans-Sasakian manifold and obtain a sufficient condition for such a manifold to be Einstein, super generalized Ricci-recurrent and Ricci-symmetric.

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2. Preliminaries

In this section, we recall some basic definitions and basic formulae which we use later.

Let $M$ be an $m = (2n + 1)$ dimensional almost contact metric manifold [5, 10] equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a $(1, 1)$ tensor field $\phi$, a characteristic vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$. Then

\begin{align*}
\phi^2 X &= -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \quad \phi \xi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\
g(\xi, \xi) &= 1, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0
\end{align*}

(2.1)  

for any $X, Y$ in $TM$. From (2.1) and (2.2), it can be easily seen that

\begin{align*}
g(X, \phi Y) &= -g(\phi X, Y), \\
g(X, \xi) &= \eta(X).
\end{align*}

(2.2)

For an almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$, we put

\begin{equation}
\Phi(X, Y) = g(X, \phi Y).
\end{equation}

(2.3)

In the almost contact metric structure, $\Phi$ is the fundamental 2-form and has rank $2n$, we have $\eta \wedge \Phi \neq 0$.

An almost contact metric structure with $\Phi = d\eta$ is a contact metric structure and a manifold with such a structure is called a contact metric manifold.

Let $M$ be almost contact manifold and consider the structure $(M \times \mathbb{R}, J, G')$ belongs to the class $W_4$ of the Hermitian manifolds, we denote a vector field on $M \times \mathbb{R}$ by $(X, f \frac{d}{dt})$, where $X$ is tangent to $M$, $t$ is the co-ordinates of $\mathbb{R}$ and $f$ is $c^\infty$ function on $M \times \mathbb{R}$. Define an almost complex structure

\begin{equation}
J\left(X, f \frac{d}{dt}\right) = (\phi X - f\xi, \eta(X)\frac{d}{dt})
\end{equation}

for any vector field $X$ on $M \times \mathbb{R}$, and $G'$ as Hermitian metric on the product $M \times \mathbb{R}$. This may be expressed by the condition

\begin{equation}
(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),
\end{equation}

(2.6)

where $\nabla$ is a Levi-Civita connection and $\alpha, \beta$ are some smooth functions on $M$. We say that trans-Sasakian structure is type $(\alpha, \beta)$. From the above it follows that

\begin{align*}
(\nabla_X \eta)Y &= -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \\
(\nabla_X \xi) &= -\alpha \phi X + \beta (X - \eta(X)\xi).
\end{align*}

(2.7)  

(2.8)

On trans-Sasakian manifold $M$ with structure $(\phi, \xi, \eta, g)$, the following relations hold [5, 10]

\begin{align*}
R(X, Y, \xi) &= (\alpha^2 - \beta^2)\eta(Y)X - \eta(X)Y + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\
&\quad + (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \\
R(\xi, X, \xi) &= (\alpha^2 - \beta^2 - \xi\beta)\eta(X)\xi - X, \\
2\alpha\beta &= \xi\alpha = 0,
\end{align*}

(2.9)  

(2.10)  

(2.11)
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(2.12) \[ \eta(R(X, Y, \xi)) = \eta(R(\xi, Y, \xi)) = 0, \]
\[ R(\xi, Y, Z) = (\alpha^2 - \beta^2)[g(Z, Y)\xi - \eta(Z)Y] + 2\alpha\beta[g(\phi Z, Y)\xi \]
\[ - \eta(Z)\phi Y] + (Z\alpha)\phi Y + g(\phi Z, Y)\text{grad} \alpha \]
\[ + (X\beta)[Y - \eta(Y)\xi] - g(\phi Z, \phi Y), \]
(2.13) \[ R(\xi, Y, Z) = (\alpha^2 - \beta^2)\left[g(Z, Y)\xi - \eta(Z)Y\right] + 2\alpha\beta\left[g(\phi Z, Y)\xi \]
\[ - \eta(Z)\phi Y\right] + (Z\alpha)\phi Y + g(\phi Z, Y)\text{grad} \alpha \]
\[ + (X\beta)[Y - \eta(Y)\xi] - g(\phi Z, \phi Y), \]
(2.14) \[ S(X, \xi) = 2n(\alpha^2 - \beta^2) - \xi\beta\eta(X) - (2n - 1)X\beta - (\phi X)\alpha, \]
(2.15) \[ S(\xi, \xi) = 2n(\alpha^2 - \beta^2 - \xi\beta), \]
(2.16) \[ S(\phi X, \phi Y) = S(X, Y) - 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(Y), \]
(2.17) \[ Q\xi = (2n(\alpha^2 - \beta^2 - \xi\beta) - (2n - 1)\text{grad} \beta + \phi(\text{grad} \alpha), \]
(2.18) \[ S(X, Y) = g(QX, Y), \]

where $R$ is the curvature tensor, $S$ is the Ricci tensor, $r$ is scalar curvature and $Q$ is the symmetric endomorphism of the tangent space at each point corresponding to Ricci-tensor $S$.

If we assume
(2.19) \[ \phi(\text{grad} \alpha) = (2n - 1)\text{grad} \beta, \]
then [5, 10]
(2.20) \[ S(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X), \]
(2.21) \[ S(\phi X, \phi Y) = S(X, Y) - 2n(\alpha^2 - \beta^2)\eta(X)\eta(Y), \]
(2.22) \[ Q\xi = 2n(\alpha^2 - \beta^2)\xi. \]

Now mention some definitions are be considered later.

**Definition 2.1** ([3]). A Riemannian manifold $M$ is said to be $\phi$-symmetric if the curvature tensor $R$ satisfies the relation
(2.23) \[ \phi^2((\nabla_W R)(X, Y, Z)) = 0 \]
for all $X, Y$ and $Z \in TM$.

**Definition 2.2** ([3]). A Riemannian manifold $M$ is said to be Ricci-symmetric if the Ricci-tensor $S$ satisfies the relation
(2.24) \[ (\nabla_W S)(X, Y) = 0 \]
for all $X, Y$ and $W \in TM$.

**Definition 2.3** ([13]). A Riemannian manifold $M$ is said to be generalized Ricci-recurrent if the Ricci-tensor $S$ satisfies the relation
(2.25) \[ (\nabla_W S)(X, Y) = A'(W)S(X, Y) + B'(W)g(X, Y) \]
for all $X, Y$ and $W \in TM$, $A'$ and $B'$ are two 1-forms.

Now, mention the following Lemma.

**Lemma 2.1.** In a trans-Sasakian manifold if $\phi(\text{grad} \alpha) = (2n - 1)\text{grad} \beta$, then
(2.26) \[ (\nabla_W S)(Y, \xi) = 2n(\alpha^2 - \beta^2)[-\alpha g(Y, \phi W) + \beta g(Y, W)] \]
\[ + \alpha S(Y, \phi W) - \beta S(Y, W). \]
Proof. We know that
\[
(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).
\]
Using the relations (2.2), (2.7) and (2.19), we can easily derive expression (2.26). 

3. Extended Generalized ϕ-Recurrent Trans-Sasakian Manifold

Definition 3.1. A trans-Sasakian manifold M is said to be an extended generalized ϕ-recurrent if the curvature tensor R satisfies the relation
\[
\phi^2((\nabla_W R)(X, Y, Z)) = A(W)\phi^2(R(X, Y, Z)) + B(W)\phi^2(G(X, Y, Z)),
\]
where A and B are two 1-forms, B is non zero, defined by
\[
g(W, \rho_1) = A(W), \quad g(W, \rho_2) = B(W),
\]
\[
G(X, Y, Z) = g(Y, Z)X - g(X, Z)Y
\]
for all X, Y, Z, W ∈ TM and ρ₁, ρ₂ being vector fields associated with the 1-form A and B, respectively.

In this section, we start with the lemma which statement is as follows:

Lemma 3.1. If an extended generalized ϕ-recurrent trans-Sasakian manifold M satisfies \(\phi(\text{grad } \alpha) = (2n - 1) \text{grad } \beta\), then the following relation holds
\[
(\alpha^2 - \beta^2)A(W) + B(W) = 0.
\]

Proof. Let us consider that M is an extended generalized ϕ-recurrent trans-Sasakian manifold. Then by virtue of relation (2.1), the equation (3.1) becomes
\[
(\nabla_W R)(X, Y, Z) - \eta((\nabla_W R)(X, Y, Z))\xi
\]
\[
= A(W)[R(X, Y, Z) - \eta(R(X, Y, Z))]\xi
\]
\[
+ B(W)[G(X, Y, Z) - \eta(G(X, Y, Z))]\xi.
\]
From which is follows that
\[
g((\nabla_W R)(X, Y, Z), U) - g((\nabla_W R)(X, Y, Z), \xi)g(U, \xi)
\]
\[
= A(W)[g(R(X, Y, Z), U) - g(R(X, Y, Z), \xi)g(U, \xi)]
\]
\[
+ B(W)[g(G(X, Y, Z), U) - g(G(X, Y, Z), \xi)g(U, \xi)].
\]
Let us suppose \(\{e_1, e_2, \ldots, e_{2n+1}\}\) is an orthonormal basis of the tangent space at any point of the manifold. Setting X = U = eᵢ in the relation (3.4) and taking summation over i, 1 ≤ i ≤ 2n + 1, we obtain
\[
(\nabla_W S)(Y, Z) - \eta((\nabla_W R)(\xi, Y, Z))
\]
\[
= A(W)[S(Y, Z) - \eta(R(\xi, Y, Z))]
\]
\[
+ B(W)[((2n - 1)g(Y, Z) + \eta(Y)\eta(Z)].
\]
Putting Z = \xi in the above equation, we find
\[
(\nabla_W S)(Y, \xi) - \eta((\nabla_W R)(\xi, Y, \xi))
\]
\[
= A(W)[S(Y, \xi) - \eta(R(\xi, Y, \xi))]
\]
\[
+ 2nB(W)\eta(Y).
\]
By virtue of the Lemma 2.1 and then using equations (2.12) and (2.20), we have
\[2n(\alpha^2 - \beta^2)[-\alpha g(Y,\phi W) + \beta g(Y, W)] + \alpha S(Y, \phi W) - \beta S(Y, W) = 2n(\alpha^2 - \beta^2)A(W) + 2nB(W)]\eta(Y).
\]
Replacing \(Y\) by \(\xi\) in equation (3.7) and using the relations (2.3), (2.4), (2.19) and (2.22), we can get the expression (3.2).

**Theorem 3.1.** If an extended generalized \(\phi\)-recurrent trans-Sasakian manifold \(M\) satisfies \(\phi(\text{grad} \alpha) = (2n-1)\text{grad} \beta\), then it is an Einstein manifold.

**Proof.** Let us consider \(M\) is an extended generalized \(\phi\)-recurrent trans-Sasakian manifold. By virtue of the Lemma 3.1 and the equation (3.7), we have
\[2n(\alpha^2 - \beta^2)[-\alpha g(Y,\phi W) + \beta g(Y, W)] + \alpha S(Y, \phi W) - \beta S(Y, W) = 0.
\]
Now replacing \(Y\) and \(W\) by \(\phi Y\) and \(\phi W\), respectively, in the above relation and then using relations (2.1), (2.4), (2.18), (2.19) and (2.22), we obtain
\[S(Y, W) = 2n(\alpha^2 - \beta^2)g(Y, W)
\]
and
\[S(\phi Y, W) = 2n(\alpha^2 - \beta^2)g(\phi Y, W),
\]
which proves the result.

**Theorem 3.2.** If an extended generalized \(\phi\)-recurrent trans-Sasakian manifold \(M\) satisfies \(\phi(\text{grad} \alpha) = (2n-1)\text{grad} \beta\), then \(2n(\alpha^2 - \beta^2)\) is the eigen value of the Ricci-tensor corresponding to the eigen vector \(\rho_1\), where \(\rho_1\) is the associated vector field of the 1-form \(A\).

**Proof.** By virtue of the Bianchi’s identity and then using equation (3.3), we get
\[A(W)[R(X,Y,Z) - \eta(R(X,Y,Z))\xi] + B(W)[G(X,Y,Z) - \eta(G(X,Y,Z))\xi]\]
\[+A(X)[R(Y,W,Z) - \eta(R(Y,W,Z))\xi] + B(Y)[G(Y,W,Z) - \eta(G(Y,W,Z))\xi] = 0.
\]
Applying inner product to both side by \(U\), then setting \(Y = Z = \epsilon_i\) in the above equation, taking summation over \(i\), \(1 \leq i \leq 2n+1\), and using the Lemma 3.1, we obtain
\[-A(R(W,X,U)) + A(R(W,X,\xi))g(U,\xi)
\]
\[-(\alpha^2 - \beta^2)[A(X)g(W,U) - A(W)g(X,U)]
\]
\[-(A(X)g(W,\xi) - A(W)g(X,\xi))g(U,\xi) = 0.
\]
Again contracting over \(X\) and \(U\) and using relation (2.10), we get
\[S(W,\rho_1) = 2n(\alpha^2 - \beta^2)A(W),
\]
which proves the theorem.

**Theorem 3.3.** An extended generalized \(\phi\)-recurrent trans-Sasakian manifold \(M\) with \(\phi(\text{grad} \alpha) = (2n-1)\text{grad} \beta\) is a generalized Ricci-recurrent manifold.
Proof. Since $M$ is an extended generalized $\phi$-recurrent trans-Sasakian manifold, hence the relation (3.4) holds. Taking contraction of the equation (3.4) over $Y$ and $Z$, we obtain
\[
(\nabla_W S)(X, U) = [(\nabla_W S)(X, \xi) - A(W)S(X, \xi)]\eta(U)
+ A(W)S(X, U) + 2nB(W)g(X, U)
- 2nB(W)\eta(X)\eta(U).
\]
(3.13)
By virtue of the Lemma 2.1, equation (2.9), Theorem 3.1 and Lemma 3.1, the above expression becomes
\[
(\nabla_W S)(X, U) = A(W)S(X, U) + 2nB(W)g(X, U),
\]
which finishes the proof. □

**Theorem 3.4.** If an extended generalized $\phi$-recurrent trans-Sasakian manifold $M$ is $\phi$-symmetric and satisfying $\phi(\text{grad} \alpha) = (2n - 1) \text{grad} \beta$, then it is Ricci-symmetric.

**Proof.** By virtue of the equation (3.1) and Definition 2.1, we take
\[
\phi^2(\nabla_W R)(X, Y, Z) = 0.
\]
(3.14)
From the relation (2.1), it follows that
\[
(\nabla_W R)(X, Y, Z) - \eta(\nabla_W R)(X, Y, Z))\xi = 0.
\]
(3.15)
Taking inner product and contracting over $Y$ and $Z$, we get
\[
(\nabla_W S)(X, U) = (\nabla_W S)(X, \xi)\eta(U).
\]
(3.16)
By virtue of the Theorem 3.1 and using the Lemma 2.1, we obtain
\[
(\nabla_W S)(X, U) = 0.
\]
(3.17)
Thus theorem is proved. □

**Theorem 3.5.** If an extended generalized $\phi$-recurrent trans-Sasakian manifold $M$ is locally $\phi$-symmetric, then the following relation holds
\[
A(W)(\alpha^2 - \beta^2 - \xi\beta) + B(W) = 0
\]
(3.18)
for all $X, Y, W$ orthogonal to $\xi$.

**Proof.** Suppose $M$ is locally $\phi$-symmetric, then
\[
\phi^2((\nabla_W R)(X, Y, Z)) = 0.
\]
By virtue of the equation (3.1) and then using the relation (2.1), we get
\[
A(W)[R(X, Y, Z) - \eta(R(X, Y, Z))\xi]
+ B(W)[G(X, Y, Z) - \eta(G(X, Y, Z))\xi] = 0.
\]
Putting $X = Z = \xi$ and using the equation (2.10), we obtain
\[
A(W)(\alpha^2 - \beta^2 - \xi\beta) + B(W)(\eta(Y)\xi - Y) = 0
\]
(3.19)
which implies that
\[
A(W)(\alpha^2 - \beta^2 - \xi\beta) + B(W) = 0.
\]
(3.20)
And thus we have the result. □
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