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APPLICATION OF LIMIT THEOREM TO SUM OF LEGENDRE SYMBOLS

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ABSTRACT. In this work, we make an extension of earlier results provided by Kubilus and Linnik who modelled specific class of Brownian motions using the sums [**1**], where p is the sequence of odd square free numbers. Our results are relevant and can be used in a range of applications, especially when applying Monte-Carlo methods in finance and econometrics.

1. INTRODUCTION

H. Davenport and P. Erdös in [**1**] proved several theorems on distribution of quadratic residues and non-residues in sets of consequent natural numbers. They considered moments of sums $S_h(x)$ of Legendre symbols modulo prime p

$$S_h(x) = \sum_{n=x+1}^{x+h} \left(\frac{n}{p}\right). \tag{1.1}$$

From this consideration the authors concluded that the distribution of these sums when $p \to \infty$ are asymptotically normal with a certain restriction imposed on h as a function of p.

In 1959, I. P. Kubilus and Yu. V. Linnik [**2**] gave a model of brownian motion with the help of the sums (1.1), where p is the sequence of odd square free numbers. This fact extended the range of application of Monte-Carlo method with the given manner for modelling a brownian motion. We note that $\frac{n}{p}$ in this case is the Jacobi symbol. In the present work, we consider the product of Legendre symbols over shifted sequences of natural numbers. Generally speaking, the expression involved won't be a *Dirichlet character* anymore. We also state the limiting distribution for the product of Legendre symbols referred above.

Now, we proceed to formulate basic results.

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2. Main results

Let $S_h(x)$ be a sum given by

$$S_h(x) = \sum_{n=h+1}^{x+h} \prod_{i=1}^{s} \Big(\frac{x+a_i}{p_i}\Big), \qquad 1 \leq x,\ h \leq Q, \tag{2.1}$$

where the Legendre Symbol is given by

$$\Big(\frac{a}{p}\Big) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p, \\ 0, & \text{if } a \text{ is a non-quadratic residue modulo } p, \\ -1, & \text{if } a = 0 \text{ modulo } p. \end{cases}$$

Let us consider $\xi = \xi_0(x) = \dfrac{S_h(x)}{\sqrt{h}}$ as a random variable with a probability $\dfrac{1}{Q}$ for each integer x.

Lemma 2.1. *Let X be a multiplicative character in the field F_q of order $m > 1$ and $f \in F_q[x]$ be an unitary polynomial of a positive degree, which is not an m^{th} degree of another polynomial. If d is the number of distinct roots of the polynomial f in its factorization field over F_q, then for any $a \in F_q$, the following inequality holds*

$$\Big| \sum_{c \in F_q} X(af(c)) \Big| \leq (d-1)\sqrt{q}.$$

Theorem 2.2. *Let $a > 0$ be a constant with $s \leq a$, let $p_1, \ldots, p_s$ be prime numbers and $Q = p_1, \ldots, p_s$, we denote $N_p\{\ldots\}$ the number of natural numbers satisfying the conditions indicated in braces. We define an incomplete sum S as following*

$$S = S_h(x) = S_h(x;\, p_1, \ldots, p_s) = \sum_{n=x+1}^{x+h} \Big(\frac{n+a_1}{p_1}\Big) \cdots \Big(\frac{n+a_s}{p_s}\Big).$$

And we also define a random variable ξ by $\xi = \xi_Q = h^{-1/2} S_h(x)$. Then ξ_Q has an asymptotic standard normal distribution, i.e.,

$$\lim_{Q \to \infty} \frac{1}{p} N_p\{\xi_Q < x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{\frac{-u^2}{2}}\, \mathrm{d}\, u,$$

where

$$h = h(Q) \to \infty, \qquad \frac{\log h}{\log Q} \to 0, \quad Q \to \infty.$$

Proof. Let $p_1, \ldots, p_s$ be prime numbers. Consider a sum $S = S_h(x)$

$$S = S_h(x) = S_h(x;\, p_1, \ldots, p_s) = \sum_{n=x+1}^{x+h} \Big(\frac{n+a_1}{p_1}\Big) \cdots \Big(\frac{n+a_s}{p_s}\Big).$$

First we calculate the $2r^{th}$ moment of $\xi = \xi_{p_1,\ldots,p_s} = \dfrac{S_h(x)}{\sqrt{h}}$. We obtain

$$\begin{aligned} A_{p_1,\ldots,p_s}(2r) &= \frac{1}{p_1,\ldots,p_s h^r} \sum_{x=1}^{p_1,\ldots,p_s} \Big(\sum_{n=x+1}^{x+h} \Big(\frac{n+a_1}{p_1}\Big) \ldots \Big(\frac{n+a_s}{p_s}\Big)\Big)^{2r} \\ &= \frac{1}{p_1,\ldots,p_s h^r} \sum_{x=1}^{p_1,\ldots,p_s} \Big(\sum_{n=1}^{h} \Big(\frac{n+x+a_1}{p_1}\Big) \ldots \Big(\frac{n+x+a_s}{p_s}\Big)\Big)^{2r} \\ &= \frac{1}{p_1,\ldots,p_s h^r} \sum_{n_1,\ldots,n_{2r}=1}^{h} \Big(\sum_{n=x+1}^{x+h} \Big(\frac{f_1(x;\overline{n})}{p_1}\Big) \ldots \Big(\frac{f_s(x;\overline{n})}{p_s}\Big)\Big)^{2r}, \end{aligned}$$

where $\overline{n} = (n_1, \ldots, n_{2r})$ and

$$\begin{array}{rclcl} f_1(x;\overline{n}) & = & (x+n_1+a_1) & \ldots & (x+n_{2r}+a_1) \\ \vdots & & \vdots & & \vdots \\ f_s(x;\overline{n}) & = & (x+n_1+a_s) & \ldots & (x+n_{2r}+a_s). \end{array}$$

By the Chinese Remainder Theorem (CRT), we have that if $x \equiv Qp_1^{-1}x_1 + \ldots + Qp_s^{-1}x_s \pmod Q$, $Q = p_1, \ldots, p_s$ and x ranges over the complete set of residues modulo Q, then x_1 ranges over the complete set of residues modulo p_1 and x_r over the complete set of residues modulo p_s.
Therefore, the latter sum equals to

$$\frac{1}{Qh^r} \sum_{n_1,\ldots,n_{2r}=1}^{h} \sum_{x_1=1}^{p_1} \Big(\frac{f(Qp_1^{-1}x_1;\overline{n}}{p_1}\Big) \cdots \sum_{x_r=1}^{p_r} \Big(\frac{f_s(Qp_1^{-1}x_s;\overline{n}}{p_s}\Big),$$

where

$$f_s(Qp_s^{-1}x;\overline{n}) = (n_1 + Qp_s^{-1}x_s + a_s)\ldots(n_{2r} + Qp_s^{-1}x_s + a_s).$$

Further, we obtain

$$A_{p_1,\ldots,p_s}(r) = \frac{1}{p_1,\ldots,p_s h^r} \sum_{n_1,\ldots,n_{2r}=1}^{h} \prod_{r=1}^{s} \sum_{x_r=1}^{p_r} \Big(\frac{f_s(Qp_r^{-1}x_r}{p_r}\Big).$$

Now we split tuples of natural numbers $(n_1, \ldots, n_{2r})$ into two classes.
First of all, we note the following: since $Q = p_1, \ldots, p_s$, then for some r, $1 \le r \le s$, there exists $p = p_r$ such that $p \ge Q^{1/s} \ge Q^{1/a}$. Since $\frac{\log h}{\log Q} \to 0$ as $Q \to \infty$, we can suppose that $Q^{1/a} > h$ whenever Q is sufficiently large. The tuples $(n_1, \ldots, n_{2r})$, consisting of no more than r distinct natural numbers, each of which occurs an even number of times, we put into the first class. Then, we put the rest tuples into the second class. In accordance with this partition, the sum $A_{p_1,\ldots,p_s}(r)$ can be represented as follows:

$$A_{p_1,\ldots,p_s}(r) = B_1 + B_2,$$

where the sum B_1 contains tuples from the first class, whereas the sum B_2 contains tuples from the second class. Then, for each tuple from the first class modulo p_r,

$1 \le r \le s$, polynomials $f_r(Qp_r^{-1}x_r)$ are exact squares modulo p_r.
Hence, for any r, $1 \le r \le s$, the sum

$$\sum_{x_r=1}^{p_r} \Big(\frac{f_r(Qp_r^{-1}x_r}{p_r}\Big)$$

is equal to $p_r - \theta_1 r$ for sum θ_1 with $0 \le \theta_1 \le 1$.
By lemma, for any tuple from the second class and p chosen above, we have

$$\Big|\sum_{x_r=1}^{p_r} \Big(\frac{f_r(Qp_r^{-1}x_r}{p_r}\Big)\Big| \le r\sqrt{p},$$

hence

$$B_1 = \frac{1}{Qh^r}((2r-1)!!h^r(Q-\theta r) + O(ph^{r-1})).$$

In the case of the sum $A_{p_1,\ldots,p_s}(2r-1)$, the first class is empty, so $B_1 = 0$, whereas for the second class the same estimation of B_2 is valid. Thus, depending on the parity of the order of the moment of the variable ξ, we obtain the following estimation

$$\begin{cases} A_{p_1,\ldots,p_s}(2r) = 1\cdot 3\ldots(2r-1) + O(h^{-1}) \\ A_{p_1,\ldots,p_s}(2r-1) \ll h'Q^{1/2a}, \end{cases}$$

Consequently, as $Q \to \infty$, we have

$$\begin{cases} A_Q(2r) \to (2r-1)!! \\ A_Q(2r-1) \to 0, \end{cases}$$

where $(2r-1)!! = 1\cdot 3\ldots(2r-)$. So, the random variable $\xi_0 = \frac{S_h(x)}{\sqrt{x}}$ as $Q \to \infty$ has an asymptotic standard normal distribution $N(0, 1)$. This completely proves the theorem. □

3. Conclusion

In the present paper, we consider the product of Legendre symbols over shifted sequences of natural numbers. Generally speaking, the expression involved will not be a *Dirichlet character* anymore. We state and prove limiting distribution for the product of Legendre symbols. Our results are a clear extension of earlier results provided by Kubilus and Linnik who modelled specific class of Brownian motions using the sums[**1**], where p is the sequence of odd square free numbers. Our results are relevant and can be used in a range of applications, especially when applying Monte-Carlo methods in finance and econometrics.

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